第46卷 第1期　　宁夏大学学报(自然科学版中英文)　　2025年3月
Vol. 46 No. 1　　Journal of Ningxia University (Natural Science Edition in Chinese and English)　　Mar. 2025

文章编号：0253-2328(2025)01-0007-09

Riemann-Liouville分数阶微分方程边值问题的积分形式解

史新杰[1,2]，　汪文帅[1,2*]，　龙品红[1,2]

(1.宁夏大学 数学统计学院，宁夏 银川 750021；2.宁夏数学基础学科研究中心，宁夏 银川 750021)

摘　要：研究了一类Riemann-Liouville分数阶微分方程Robin边值问题解的存在唯一性，并以积分形式给出了边值问题的解．在该Robin边值问题满足特定条件时，证明了解的存在唯一性，然后通过几个Riemann-Liouville分数阶微分方程边值问题数值解的算例说明结论的有效性．

关键词：积分解；分数阶微分方程；边值问题

分类号：(中图)O175.8　　**文献标志码：**A

在经典微分方程理论中，处理的通常是整数阶微分[1]．然而，自然界和社会经济系统中的许多现象表现出记忆和遗传特性，这些特性难以用整数阶模型精确描述．分数阶微分方程提供了一种新的工具，其概念最早可以追溯到17世纪，由数学家Leibniz和L'Hospital提出．随着分数阶微分方程理论的发展，研究者们提出了多种分数阶导数的定义[2-3]，如Riemann-Liouville分数阶导数[4-7]、Caputo分数阶导数[8]等．分数阶微分方程的求解比传统整数阶微分方程复杂得多，许多分数阶微分方程没有解析解，尤其是非线性分数阶微分方程．在这种情况下，积分形式解提供了一种解的表达．尤其是在求解分数阶微分方程边值问题[9-11]解析解和数值解[12]的过程中，积分形式解是分析解的稳定性的基础，使得通过数值方法得到解时能够确保其稳定性和精度．目前，分数阶微分方程解[13-20]的存在性和存在形式已经成为一个热门的研究方向．

对于Riemann-Liouville分数阶微分方程边值问题，许多数学工作者[17]进行了研究．2012年，Zhou[15]证明了下列非线性Riemann-Liouville分数阶三点边值问题正解的存在唯一性：

$$\begin{cases} D_{0^+}^{\alpha}u(x)+f(x,u(x))=0,\ 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ u^{(n-2)}(1)=\int_0^{\eta}u(t)\mathrm{d}t, \end{cases}$$

其中：$n-1<\alpha\leqslant n, n\geqslant 3$. 同年，Yang[16]借助于Banach不动点定理、Leray-Schauder型的非线性微分和范数型的锥展开和压缩不动点定理，给出了一类非线性Riemann-Liouville分数阶($1<\alpha,\beta\leqslant 2$)微分方程耦合系统积分边值问题

$$\begin{cases} D^{\alpha}u(x)+a(x)f(x,v(x))=0,\\ D^{\beta}v(x)+b(x)g(t,u(x))=0,\ 0<t<1,\\ u(0)=0,\ u(1)=\int_0^1\phi(t)u(t)\mathrm{d}t,\\ v(0)=0,\ v(1)=\int_0^1\psi(t)v(t)\mathrm{d}t \end{cases}$$

收稿日期：2024-12-23

基金项目：国家自然科学基金资助项目(42064004，12161067)；宁夏科技领军人才项目(2022GKLRLX04)

作者简介：史新杰(1997—)，女，博士研究生，主要从事分数阶微分方程边值问题研究，(电子信箱)15065890745@163.com.

* **通信联系人：**汪文帅(1980—)，男，教授，博士研究生导师，主要从事数学力学模型与分析研究，(电子信箱)wws@nxu.edu.cn.

引用格式：史新杰，汪文帅，龙品红.Riemann-Liouville分数阶微分方程边值问题的积分形式解[J].宁夏大学学报(自然科学版中英文)，2025，46(1)：7-15.

正解存在和不存在的充分条件，其中 $a,b\in C((0,1),[0,+\infty)),\phi,\psi\in L^1[0,1]$ 是非负函数，且 $f,g\in C([0,1]\times[0,+\infty),[0,+\infty))$．2014年，Jiang等[14]研究了一类具有非局部积分边界条件的Riemann-Liouville分数阶微分方程的边值问题：

$$\begin{cases} D_{0^+}^{\alpha}u(x)+f(x,u(x))=0,\ 0<t<1,\\ u(0)=u'(0)=u''(0)=0,\\ u(1)=\int_0^{\eta}u(t)\mathrm{d}t,\end{cases}$$

其中：$3<\alpha\leqslant 4,0<\eta<1,0<\frac{\eta^{\alpha}}{\alpha}<1$．然后构造了两个连续迭代序列，给出了微分方程非平凡变号解存在的条件．2024年，Zhu5]利用Schauder不动点定理和广义Ascoli-Arzela定理，证明了Riemann-Liouville分数阶微分方程

$$\begin{cases} D_{0^+}^{\beta}u(x)=f(x,u(x)),\ x\in(0,+\infty),\\ \lim\limits_{x\to 0^+}x^{1-\beta}u(x)=u_0\end{cases}$$

至少存在一个全局吸引解．

注意到，上述研究大多基于Dirichlet边值条件，对于Robin边值问题的研究不够深入．为此，本文主要讨论一类Riemann-Liouville分数阶微分广义Robin边值问题：

$$\begin{cases} D^{\alpha}u(x)=\rho(x,u(x)),x\in(0,1),\\ u(0)=\lambda_1u(1)+\mu_1,\\ u'(0)=\lambda_2u'(1)+\mu_2,\\ \lambda_1\neq 1,\lambda_2\neq 0,\end{cases}$$

其中 $1<\alpha\leqslant 2$，给出该边值问题的积分形式解，确定了该解存在且唯一的条件．最后给出几个数值算例证明结论的有效性．

1　基础知识

分数阶微积分是整数阶微积分的一种推广．首先，介绍一些与Riemann-Liouville分数阶微积分有关的基本概念和引理．

定义1[21-22]　对于实数 $\alpha(n-1<\alpha\leqslant n,n\in\mathbf{N})$，函数 $f(x)$ 的 α 阶Riemann-Liouville积分表示为

$$I^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}u(t)\mathrm{d}t,$$

其中 $\Gamma(\cdot)$ 是伽玛函数．当 $\alpha=n$ 时，有

$$I^{\alpha}f(x)=I^{n}f(x)=\frac{1}{\Gamma(n)}\int_a^x(x-t)^{n-1}f(t)\mathrm{d}t=\int_a^x\mathrm{d}t\int_a^t\mathrm{d}t_1\cdots\int_a^{t_{n-2}}f(t_{n-1})\mathrm{d}t_{n-1}.$$

定义2[21-22]　对于实数 $\alpha(n-1<\alpha\leqslant n,n\in\mathbf{N})$，函数 $f(x)$ 的 α 阶Riemann-Liouville微分表示为

$$D^{\alpha}u(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}(I^{n-\alpha}u(x))=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x(x-t)^{n-\alpha-1}u(t)\mathrm{d}t,$$

其中 n 是大于 α 的最小整数．

这些定义是分数阶微积分的基础，允许在分数阶微分和积分的情况下进行微积分运算．

引理1[23]　设 $n-1<\alpha\leqslant n,n\in\mathbf{N}$．若 $u(t)$ 在 $[a,b]$ 上 n 阶导数连续，则

$$D^{\alpha}I^{\alpha}u(x)=u(x).$$

引理2[23]　令 $n-1<\alpha\leqslant n,n\in\mathbf{N}$，若 $u(t)$ 在 $[a,b]$ 上 n 阶导数连续，则Riemann-Liouville型分数阶积分和Riemann-Liouville型分数阶导数运算的复合公式为

$$I^{\alpha}D^{\alpha}u(x)=u(x)-\sum_{j=1}^{n}\left[I_{a^+}^{j-\alpha}u(x)\right]_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}.$$

引理2的详细证明见附录．

注1　当$\alpha\in(1,2]$时，$n=2$，有

$$I^{\alpha}D^{\alpha}u(x)=u(x)-I^{2-\alpha}u(a)\frac{(x-a)^{\alpha-2}}{\Gamma(\alpha)}-I^{1-\alpha}u(a)\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}=u(x)-c_2(x-a)^{\alpha-2}-c_1(x-a)^{\alpha-1}.\tag{1}$$

2　主要结果及其证明

本节主要考虑一类Riemann-Liouville型分数阶微分方程边值问题

$$\begin{cases}D^{\alpha}u(x)=\rho(x,u(x)),x\in(0,1),\\u(0)=\lambda_1u(1)+\mu_1,\\u'(0)=\lambda_2u'(1)+\mu_2,\\\lambda_1\neq\lambda_2\neq0\end{cases}\tag{2}$$

解的存在条件及积分解的形式，其中:ρ是关于x的函数，u是$(0,1)$上关于x的函数，$\lambda_1,\lambda_2,\mu_1,\mu_2$是常数．然后通过几个算例说明结果的优越性,并给出数值解．

定理1　令$1<\alpha\leqslant2$. 对于Riemann-Liouville型分数阶微分方程边值问题(2),存在积分形式解:

$$u(x)=\int_0^1G(x,t)\rho(t,u(t))\mathrm{d}t+Mx^{\alpha-2}-\left(M+\frac{\mu_1}{\lambda_1}\right)x^{\alpha-1},\tag{3}$$

其中

$$G(x,t)=\begin{cases}\dfrac{(x-t)^{\alpha-1}+(1-\alpha)(1-t)^{\alpha-1}x^{\alpha-2}+(\alpha-2)(1-t)^{\alpha-1}x^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(x^{\alpha-2}-x^{\alpha-1})(1-t)^{\alpha-2}}{\Gamma(\alpha-1)},0\leqslant t\leqslant x\leqslant1,\\\dfrac{(1-\alpha)x^{\alpha-2}(1-t)^{\alpha-1}+(\alpha-2)x^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(x^{\alpha-2}-x^{\alpha-1})(1-t)^{\alpha-2}}{\Gamma(\alpha-1)},0\leqslant x\leqslant t\leqslant1,\end{cases}$$

$$M=(1-\alpha)\frac{\mu_1}{\lambda_1}-\frac{\mu_2}{\lambda_2}.$$

证明　在边值问题(2)中,对$D^{\alpha}u(x)$求α阶R-L积分:

$$I^{\alpha}D^{\alpha}u(x)=I^{\alpha}\rho(x,u(x)).$$

根据引理2,可得

$$u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\rho(t,u(t))\mathrm{d}t+c_2x^{\alpha-2}+c_1x^{\alpha-1}.\tag{4}$$

进一步对$u(x)$求一阶导数,得到方程组

$$\begin{cases}u(x)=\displaystyle\int_0^x\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+c_2x^{\alpha-2}+c_1x^{\alpha-1},\\u'(x)=\displaystyle\int_0^x\frac{(x-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t+c_2(\alpha-2)x^{\alpha-3}+c_1(\alpha-1)x^{\alpha-2}.\end{cases}$$

根据问题(2)中边值条件,可得

$$\begin{cases}u(0)=0=\lambda_1\left[\displaystyle\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+c_2+c_1\right]+\mu_1,\\u'(0)=0=\lambda_2\left[\displaystyle\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t+c_2(\alpha-2)+c_1(\alpha-1)\right]+\mu_2.\end{cases}\tag{5}$$

求解方程组(5),得到

$$\begin{cases}c_1=(\alpha-2)\dfrac{\mu_1}{\lambda_1}-\dfrac{\mu_2}{\lambda_2}+(\alpha-2)\displaystyle\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t-\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t;\\c_2=-(\alpha-1)\dfrac{\mu_1}{\lambda_1}+\dfrac{\mu_2}{\lambda_2}-(\alpha-1)\displaystyle\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t.\end{cases}$$

将c_1,c_2代入式(4)，得

$$
\begin{aligned}
u(x)=&\int_0^x\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+\\
&\left[-(\alpha-1)\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}-(\alpha-1)\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t\right]x^{\alpha-2}+\\
&\left[(\alpha-2)\frac{\mu_1}{\lambda_1}-\frac{\mu_2}{\lambda_2}+(\alpha-2)\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t-\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t\right]x^{\alpha-1}=\\
&\int_0^x\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+(1-\alpha)x^{\alpha-2}\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t+x^{\alpha-2}\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t+\\
&(\alpha-2)x^{\alpha-1}\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\rho(t,u(t))\mathrm{d}t-x^{\alpha-1}\int_0^1\frac{(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\rho(t,u(t))\mathrm{d}t+Mx^{\alpha-2}-\left(M+\frac{\mu_1}{\lambda_1}\right)x^{\alpha-1}=\\
&\int_0^1G(x,t)\rho(t)\mathrm{d}t+Mx^{\alpha-2}-\left(M+\frac{\mu_1}{\lambda_1}\right)x^{\alpha-1}.
\end{aligned}
$$

为方便，记

$$g(x)=Mx^{\alpha-2}-\left(M+\frac{\mu_1}{\lambda_1}\right)x^{\alpha-1},$$

则$u(x)=\int_0^1G(x,t)\rho(t,u(t))\mathrm{d}t+g(x)$.

定理2 假设$\rho(x,u(x))$是$[0,1]$上的实值连续函数且满足条件

$$|\rho(x,u)-\rho(x,v)|\leqslant\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}|u-v|,\tag{6}$$

则边值问题(2)存在唯一解.

证明 记$\Lambda=\dfrac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}$，给定一个算子Q：

$$Qu(x)=\int_0^1G(x,t)\rho(t,u(t))\mathrm{d}t+g(x),$$

则

$$
\begin{aligned}
\|Qu(x)-Qv(x)\|=&\left\|\int_0^1G(x,t)\rho(t,u(t))\mathrm{d}t-\int_0^1G(x,t)\rho(t,v(t))\mathrm{d}t\right\|=\\
&\left\|\int_0^tG(x,t)\left[\rho(t,u(t))-\rho(t,v(t))\right]\mathrm{d}t+\int_t^1G(x,t)\left[\rho(t,u(t))-\rho(t,v(t))\right]\mathrm{d}t\right\|\leqslant\\
&\left\|\int_0^t\left[\frac{(x-t)^{\alpha-1}+[(1-\alpha)x^{\alpha-2}+(\alpha-2)x^{\alpha-1}](1-t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(x^{\alpha-2}-x^{\alpha-1})(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left[\rho(t,u(t))-\rho(t,v(t))\right]\mathrm{d}t\right\|+\\
&\left\|\int_t^1\left[\frac{(1-\alpha)x^{\alpha-2}(1-t)^{\alpha-1}+(\alpha-2)x^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(x^{\alpha-2}-x^{\alpha-1})(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left[\rho(t,u(t))-\rho(t,v(t))\right]\mathrm{d}t\right\|\leqslant\\
&\left[\frac{x^{\alpha}}{\Gamma(\alpha)}+\frac{(1-\alpha)x^{\alpha-2}[1-(1-x)^{\alpha}]}{\Gamma(\alpha+1)}+\frac{(\alpha-2)x^{\alpha-1}[1-(1-x)^{\alpha}]}{\Gamma(\alpha+1)}+\frac{(x^{\alpha-2}-x^{\alpha-1})[1-(1-x)^{\alpha-1}]}{\Gamma(\alpha)}\right]\Lambda\|u(t)-v(t)\|+\\
&\left[\frac{(1-\alpha)x^{\alpha-2}(1-x)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\alpha-2)x^{\alpha-1}(1-x)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(x^{\alpha-2}-x^{\alpha-1})(1-x)^{\alpha-1}}{\Gamma(\alpha)}\right]\Lambda\|u(t)-v(t)\|\leqslant\\
&\left[\frac{x^{\alpha}}{\Gamma(\alpha)}+\frac{(x^{\alpha-2}-x^{\alpha-1})[1-(1-x)^{\alpha-1}]}{\Gamma(\alpha)}+\frac{(x^{\alpha-2}-x^{\alpha-1})(1-x)^{\alpha-1}}{\Gamma(\alpha)}\right]\Lambda\|u(t)-v(t)\|\leqslant\\
&\frac{2+x^{\alpha-2}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}\|u(t)-v(t)\|=\frac{1}{2}\|u(t)-v(t)\|.
\end{aligned}
$$

算子$Qu(t)$满足度量空间中Banach压缩定理条件，映射系数$k=\frac{1}{2}$，因此，可知边值问题(2)存在唯一解.

注2　当$\alpha=2$时，函数$\rho(x,u(x))$只需满足Lipschitz条件

$$|\rho(x,u(x))-\rho(x,v(x))|\leqslant\frac{1}{5}|u(x)-v(x)|,$$

即二阶边值问题

$$\begin{cases}D^2u(x)=\rho(x,u(x)),x\in(0,1),\\u(0)=\lambda_1u(1)+\mu_1,\\u'(0)=\lambda_2u'(1)+\mu_2,\\\lambda_1\neq\lambda_2\neq0\end{cases}$$

存在唯一解，其中$\frac{1}{5}$是Lipschitz常数.

例1　考虑Riemann-Liouville型分数阶微分方程的初值问题：

$$\begin{cases}D^{\frac{3}{2}}u(x)=\rho(x,u(x)),x\in(0,1),\\u(0)=2u(1)+1,\\u'(0)=2u'(1)+1,\end{cases}\tag{7}$$

其中$\rho(x,u(x))=\frac{\sqrt{\pi}}{4\left(2+x^{-\frac{1}{2}}\right)}u(x)$，满足

$$|\rho(x,u(x))-\rho(x,v(x))|=$$

$$\left|\frac{\sqrt{\pi}}{4\left(2+x^{-\frac{1}{2}}\right)}u(x)-\frac{\sqrt{\pi}}{4\left(2+x^{-\frac{1}{2}}\right)}v(x)\right|=\frac{\sqrt{\pi}}{4\left(2+x^{-\frac{1}{2}}\right)}|u(x)-v(x)|\leqslant\frac{\Gamma\left(\frac{3}{2}\right)}{2\left(2+x^{-\frac{1}{2}}\right)}|u(x)-v(x)|.$$

根据定理1知初值问题(7)存在唯一解，并且解的形式为

$$u(x)=\int_0^x\frac{2(x-t)^{\frac{1}{2}}-(1-t)^{\frac{1}{2}}\left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)+\left(x^{-\frac{1}{2}}-x^{\frac{1}{2}}\right)(1-t)^{-\frac{1}{2}}}{4\left(2+x^{-\frac{1}{2}}\right)}u(t)\mathrm{d}t+$$

$$\int_x^1\frac{\left(-x^{-\frac{1}{2}}-x^{\frac{1}{2}}\right)(1-t)^{\frac{1}{2}}+\left(x^{-\frac{1}{2}}-x^{\frac{1}{2}}\right)(1-t)^{-\frac{1}{2}}}{4\left(2+x^{-\frac{1}{2}}\right)}u(t)\mathrm{d}t+\frac{x^{-\frac{1}{2}}}{4}-\frac{3x^{\frac{1}{2}}}{4}=\frac{2}{4\left(2+x^{-\frac{1}{2}}\right)}\int_0^x(x-$$

$$t)^{\frac{1}{2}}u(t)\mathrm{d}t-\frac{x^{-\frac{1}{2}}+x^{\frac{1}{2}}}{4\left(2+x^{-\frac{1}{2}}\right)}\int_0^1(1-t)^{\frac{1}{2}}u(t)\mathrm{d}t+\frac{x^{-\frac{1}{2}}-x^{\frac{1}{2}}}{4\left(2+x^{-\frac{1}{2}}\right)}\int_0^1(1-t)^{-\frac{1}{2}}u(t)\mathrm{d}t+\frac{1}{4}x^{-\frac{1}{2}}-\frac{3}{4}x^{\frac{1}{2}}.$$

取$u_0(x)=\sqrt{x}$进行迭代，得到$u(x)$的数值解和迭代误差(图1、图2). 图2表明随着迭代次数的不断增加，迭代误差逐渐减小.

例2　考虑Riemann-Liouville型分数阶微分方程的初值问题：

$$\begin{cases}D^{\alpha}u(x)=\rho(x,u(x)),\\u(0)=\frac{1}{4}u(1)+\frac{3}{4},\\u'(0)=\frac{1}{3}u'(1)+\frac{2}{3},\end{cases}\tag{8}$$

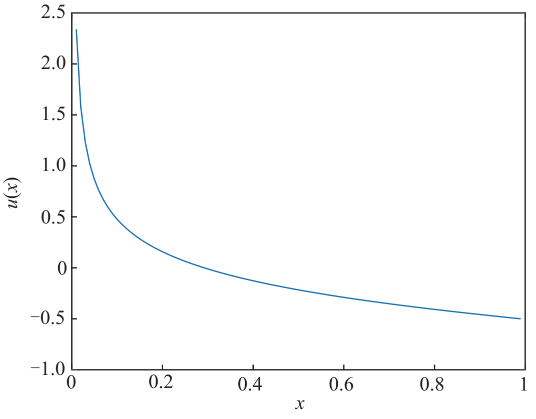

图 1 例 1 中 $u(x)$ 的数值解

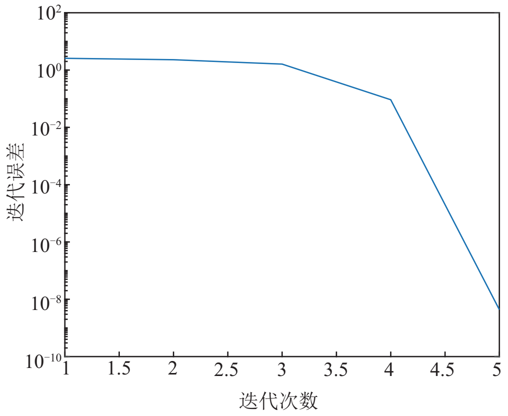

图 2 例 1 中 $u(x)$ 的迭代误差

其中 $\rho(x,u(x))=\dfrac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}\tanh(u(x))$,满足条件

$$|\rho(x,u(x))-\rho(x,v(x))|=\left|\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}\tanh(u(x))-\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}\tanh v(u(x))\right|=$$

$$\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}|\tanh(u(x))-\tanh(v(x))|\leqslant\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}|u(x)-v(x)|,$$

则边值问题(8)存在唯一解,并且其形式为

$$u(x)=\int_0^x\frac{(x-t)^{\alpha-1}}{2(2+x^{\alpha-2})}\tanh(u(t))\mathrm{d}t+$$

$$\int_0^1\left[\frac{(1-\alpha)x^{\alpha-2}(1-t)^{\alpha-1}+(\alpha-2)x^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(x^{\alpha-2}-x^{\alpha-1})(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\times$$

$$\frac{\Gamma(\alpha)}{2(2+x^{\alpha-2})}\tanh(u(t))\mathrm{d}t+(1-3\alpha)x^{\alpha-2}-(4-3\alpha)x^{\alpha-1}.$$

分别取 $\alpha=1.01,1.2,1.4,1.6,1.8,2$,得到 $u(x)$ 的数值解(图 3). 当 $\alpha=2$ 时,结果弱化为整数阶的结论,即求下列整数阶初值问题的数值解:

$$\begin{cases}u''(x)=\dfrac{1}{6}\tanh(u(x)),\\ u(0)=\dfrac{1}{4}u(1)+\dfrac{3}{4},\\ u'(0)=\dfrac{1}{3}u'(1)+\dfrac{2}{3}.\end{cases}$$

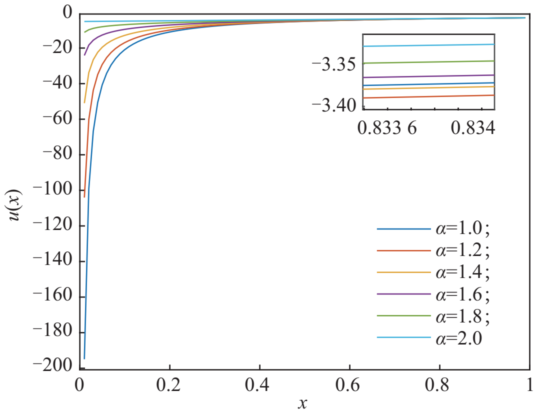

图 3 例 2 中 $u(x)$ 的数值解

例 3　对于例 2，考虑 $\alpha=\frac{4}{3}$ 时的初值问题：

$$\begin{cases} D^{\frac{4}{3}}u(x)=\rho(x,u(x)), \\ u(0)=\frac{1}{4}u(1)+\frac{3}{4}, \\ u'(0)=\frac{1}{3}u'(1)+\frac{2}{3}, \end{cases} \tag{9}$$

其中 $\rho(x,u(x))=\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u(x))$. 因为

$$|\rho(x,u(x))-\rho(x,v(x))|=\left|\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u(x))-\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(v(x))\right|=$$

$$\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}|\tanh(u(x))-\tanh(v(x))|\leqslant\frac{\Gamma\left(\frac{4}{3}\right)}{2\left(2+x^{-\frac{2}{3}}\right)}|u(x)-v(x)|.$$

所以，边值问题(9)存在唯一解，并且其形式为

$$u(x)=\int_0^x\frac{(x-t)^{\frac{4}{3}-1}+\left(1-\frac{4}{3}\right)(1-t)^{\frac{4}{3}-1}x^{\frac{4}{3}-2}+\left(\frac{4}{3}-2\right)(1-t)^{\frac{4}{3}-1}x^{\frac{4}{3}-1}}{\Gamma\left(\frac{4}{3}\right)}\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u)\mathrm{d}t+$$

$$\int_0^x\frac{\left(x^{\frac{4}{3}-2}-x^{\frac{4}{3}-1}\right)(1-t)^{\frac{4}{3}-2}}{\Gamma\left(\frac{4}{3}-1\right)}\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u)\mathrm{d}t+$$

$$\int_x^1\left[\frac{\left(1-\frac{4}{3}\right)x^{\frac{4}{3}-2}(1-t)^{\frac{4}{3}-1}+\left(\frac{4}{3}-2\right)x^{\frac{4}{3}-1}(1-t)^{\frac{4}{3}-1}}{\Gamma\left(\frac{4}{3}\right)}\right]\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u)\mathrm{d}t+$$

$$\int_x^1\frac{\left(x^{\frac{4}{3}-2}-x^{\frac{4}{3}-1}\right)(1-t)^{\frac{4}{3}-2}}{\Gamma\left(\frac{4}{3}-1\right)}\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\tanh(u)\mathrm{d}t-3x^{-\frac{2}{3}}=$$

$$\frac{1}{\Gamma\left(\frac{4}{3}\right)}\int_0^x(x-t)^{\frac{1}{3}}\tanh(u)\mathrm{d}t-\frac{\frac{1}{3}x^{-\frac{2}{3}}+\frac{2}{3}x^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\int_0^1(1-t)^{\frac{1}{3}}\tanh(u)\mathrm{d}t+$$

$$\frac{\left(x^{-\frac{2}{3}}-x^{\frac{1}{3}}\right)(1-t)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)}\frac{2}{5\left(2+x^{-\frac{2}{3}}\right)}\int_0^1(1-t)^{-\frac{2}{3}}\tanh(u)\mathrm{d}t-3x^{-\frac{2}{3}}.$$

对于 $u(x)$ 积分形式解，得到其数值解及迭代误差(图 4、图 5).

图 5 表明，当迭代次数大于 3 时，随着迭代次数的增加，误差快速趋于 0，可见收敛速度越快，求解效率越高.

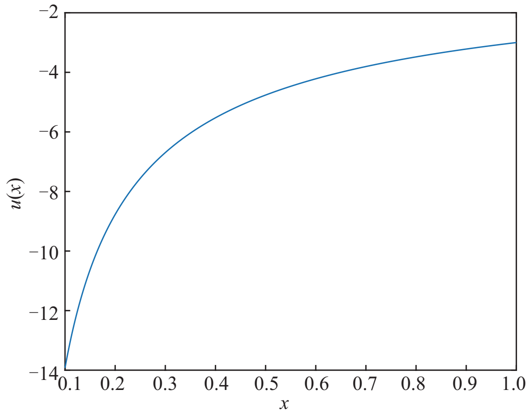

图4　例3中$u(x)$的数值解

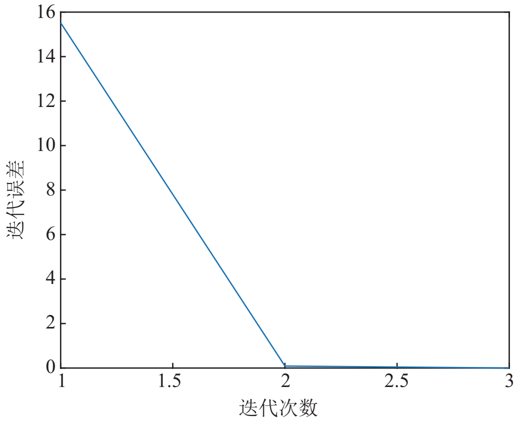

图5　例3中$u(x)$的迭代误差

3　总结

本文主要以积分形式给出了一类Riemann-Liouville分数阶微分方程边值问题的解，并给出了该边值问题解存在且唯一的条件．最后，给出两个求Riemann-Liouville分数阶微分方程边值问题数值解的例子，证明了结论的有效性．

参考文献：

[1]　冯雪．微分方程周期积分边值问题[D]．长春：吉林大学，2018.

[2]　MILLER K S，ROSS B．An introduction to the fractional calculus and fractional differential equations[M]．New York：Wiley，1993.

[3]　SPANIER J，OLDHAM K B．The fractional calculus：Theory and applications of differentiation and integration to arbitrary order[M]．Amsterdam：Elsevier，1974.

[4]　ALDAWISH I，IBRAHIM R W．Studies on a new K-symbol analytic functions generated by a modified K-symbol Riemann-Liouville fractional calculus[J]．MethodsX，2023，11：102398.

[5]　ZHU Tao．Global attractivity for fractional differential equations of Riemann-Liouville type[J]．Fractional Calculus and Applied Analysis，2023，26(5)：2264-2280.

[6]　TURKI S．On existence and asymptotic behavior of positive solutions for a fractional order differential system involving Riemann-Liouville derivatives[J]．Differential Equations and Dynamical Systems，2023．https://doi.org/10.1007/s12591-023-00634-1.

[7]　LAN Kunquan．Linear first order Riemann-Liouville fractional differential and perturbed Abel's integral equations[J]．Journal of Differential Equations，2022，306：28-59.

[8]　BAIHI A，KAJOUNI A，HILAL K，et al．Laplace transform method for a coupled system of (p,q)-Caputo fractional differential equations[J]．Journal of Applied Mathematics and Computing，2024：1-20．https://doi.org/10.1007/s12190-024-02254-6.

[9]　LIANG Sihua，ZHANG Jihui．Positive solutions for boundary value problems of nonlinear fractional differential equation[J]．Nonlinear Analysis：Theory，Methods and Applications，2009，71(11)：5545-5550.

[10]　NYAMORADI N．Positive solutions for multi-point boundary value problems for nonlinear fractional differential equations[J]．Journal of Contemporary Mathematical Analysis，2013，48(4)：145-157.

[11]　JLELI M，SAMET B．Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method[J]．Nonlinear Analysis：Modelling and Control，2015，20(3)：367-376.

[12]　肖静宇．几类分数阶微分方程的数值方法研究[D]．哈尔滨：哈尔滨工业大学，2013.

[13]　CHEN Taiyong，LIU Wenbin，HU Zhigang．A boundary value problem for fractional differential equation with p-Laplacian operator at resonance[J]．Nonlinear Analysis：Theory，Methods and Applications，2012，75(6)：3210-3217.

[14]　JIANG Min，ZHONG Shouming．Successively iterative method for fractional differential equations with integral boundary conditions[J]．Applied Mathematics Letters，2014，38：94-99.

[15]　ZHOU Chenxing．Existence and uniqueness of positive solutions to higher-order nonlinear fractional differential equation with integral boundary conditions[J]．Electronic Journal of Differential Equations，2012，234：1-11．https://ejde.math.txstate.edu.

[16]　YANG Wengui．Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions[J]．Computers and Mathematics with Applications，2012，63(1)：288-297.

[17] 丰文泉. 几类分数阶微分方程奇异边值问题及其应用[D]. 济南:济南大学,2015.

[18] 罗华. 几类非线性分数阶微分方程多点边值问题[D]. 伊犁:伊犁师范学院,2013.

[19] 赵以阁. 分数阶微分方程边值问题解的存在性[D]. 济南:济南大学,2011.

[20] HENRY B I, WEARNE S L. Existence of turing instabilities in a two-species fractional reaction-diffusion system[J]. SIAM Journal on Applied Mathematics, 2002, 62(3): 870-887.

[21] 叶俊杰,钱德亮. Riemann-Liouville型分数阶微分方程的微分变换方法[J]. 应用数学与计算数学学报,2009,23(2):111-120.

[22] GOODRICH C S. Existence of a positive solution to a class of fractional differential equations[J]. Applied Mathematics Letters, 2010, 23(9): 1050-1055.

[23] BAI Zhanbing, LÜ Haishen. Positive solutions for boundary value problem of nonlinear fractional differential equation[J]. Journal of Mathematical Analysis and Applications, 2005, 311(2): 495-505.

Integral Solutions to Boundary Value Problems of Riemann-Liouville Fractional Differential Equations

SHI Xinjie[1,2], *WANG Wenshuai*[1,2*], *LONG Pinhong*[1,2]
(1. School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China;
2. Ningxia Basic Science Research Center of Mathematics, Yinchuan 750021, China)

Abstract: In this paper, the existence and uniqueness of solutions to a class of Robin boundary value problems for Riemann-Liouville fractional differential equations are studied, which are presented in integral form. For these problems, the existence and uniqueness of solutions can be proved under certain conditions. Furthermore, some examples of numerical solutions to boundary value problems for Riemann-Liouville fractional differential equations are provided to illustrate the validity of our conclusions.

Key words: integral solution; fractional differential equation; boundary value problem

附录

引理2的证明:因为

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha D^\alpha u(t)\mathrm{d}t=\frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha \mathrm{d}D^{n-1}(I^n D^\alpha u(t))=\\
&\quad\frac{1}{\Gamma(\alpha+1)}\left[-(x-a)^\alpha D^{n-1}I^n D^\alpha u(t)|_{t=a}+\alpha\int_a^x (x-t)^{\alpha-1}D^{n-1}(I^n D^\alpha u(t))\mathrm{d}t\right]=\\
&\quad\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}D^{n-1}(I^n D^\alpha u(t))\mathrm{d}t-\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}D^{n-1}I^n D^\alpha u(t)|_{t=a}=\\
&\quad\frac{1}{\Gamma(\alpha-1)}\int_a^x (x-t)^{\alpha-2}D^{n-2}(I^n D^\alpha u(t))\mathrm{d}t-\\
&\quad\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}D^{n-2}I^n D^\alpha u(t)|_{t=a}-\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}D^{n-1}I^n D^\alpha u(t)|_{t=a}=\cdots=\\
&\quad\frac{1}{\Gamma(\alpha-n+1)}\int_a^x (x-t)^{\alpha-n}(I^n D^\alpha u(t))\mathrm{d}t-\sum_{j=1}^{n}\frac{(x-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)}I^j D^\alpha u(t)|_{t=a}=\\
&\quad Iu(t)-\sum_{j=1}^{n}\frac{(x-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)}I^j D^\alpha u(t)|_{t=a},
\end{aligned}$$

所以,对$D^\alpha u(x)$求α阶R-L积分,得到

$$\begin{aligned}
&I^\alpha D^\alpha u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}D^\alpha u(t)\mathrm{d}t=D\left[\frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha D^\alpha u(t)\mathrm{d}t\right]=\\
&\quad D\left[Iu(t)-\sum_{j=1}^{n}\frac{(x-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)}I^j D^\alpha u(x)|_{x=a}\right]=u(x)-\sum_{j=1}^{n}\left[I^{j-\alpha}u(x)\right]_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}.
\end{aligned}$$

(责任编辑　张　娣)