

广义非线性薛定谔方程解的构建

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摘要:对广义非线性薛定谔方程的解进行了构建和研究。首先通过行波变换,将非线性薛定谔方程转化为常微分方程,并得到系统色散关系,再利用 (G'/G^2) 展开法进行解的构造,构造出了广义非线性薛定谔方程的一系列精确解。同时表明所用方法和过程对构造非线性薛定谔方程的精确解具有有效性和适用性。

关键词:广义非线性薛定谔方程; (G'/G^2) 展开法; 精确解

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Construction of Solutions of Generalized Nonlinear Schrödinger Equation

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Abstract: The construction of the solution of the generalized nonlinear Schrödinger equation is analyzed. Firstly, the nonlinear Schrödinger equation is transformed into an ordinary differential equation by traveling wave transformation, and the system dispersion relation is obtained. Then the solution is constructed by (G'/G^2) expansion method, and a series of exact solutions of generalized nonlinear Schrödinger equation are constructed. At the same time, it is proved that the method and procedure are effective and applicable to construct the exact solution of nonlinear Schrödinger equation.

Key words: generalized nonlinear Schrödinger equation; (G'/G^2) - expansion method; exact solution

0 引言

众所周知,诸多科学及工程领域中的许多复杂现象的变化规律都可以用非线性偏微分方程来描述,例如流体力学、量子场论、控制理论、光纤通信等。为了建立和求解非线性偏微分方程的精确解,近年来学者们提出了行之有效的求解方法,例如齐次平衡法^[1]、tanh函数法^[2]、雅可比椭圆函数展开法^[3]、动力系统分支理论^[4]、符号运算法^[5]、首次积分法^[6]、 (G'/G^2) 展开法^[7-11]等。

本文考虑如下的广义非线性薛定谔方程^[12-13]。

$$iq_t + q_{xx} + \alpha|q|^2q - i\gamma q_{xxx} + im|q|^2q_x + in(|q|^2)_x q = 0, \quad (1)$$

式(1)描述了在非线性光纤中光孤子的传播规律,并考虑了色散、散射等多种因素。其中 i 是虚数单位, $i^2 = -1$,复函数 $q = q(x, t)$, x 表示沿传播方向的距离, t 表示延迟时间。 γ 是高阶色散系数, m 是拉曼散射系数, n 是非线性色散项的系数。近几年已有一些学者对方程进行了研究,Du等^[14]运用动力系统分支方法获得了系统的哈密顿量,然后通过沿差分轨道积分构造了该方程的多个精

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确解;Mahak等^[15]用改进的辅助方程法得到了奇异复波和非奇异复波的精确行波解,讨论了非奇异复波解几何结构的物理意义;Miao等^[16]通过修正的 (G'/G) 展开法得到了该方程的有理函数解、三角函数解、双曲函数解;石兰芳等^[17]用Riccati-Bernoulli辅助方程法得到了该方程的行波解;何黎霞等^[18]应用完全判别系统法得到了一系列显示精确解,包括扭结波解、暗孤立波解、Jacobi椭圆函数解、周期波解等。张艳妮等^[19]运用含非负幂项的 (G'/G^2) 展开法得到了三个解,为探索更多更新的解结构,本文拟采用 (G'/G^2) 为基函数的正负级展开法进行研究,具体过程结果如下所述。

1 (G'/G^2) 展开法

首先考虑含有独立变量 x, t 的非线性偏微分方程

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (2)$$

其中 $u = u(x, t)$ 是未知函数, P 是一个关于 $u = u(x, t)$ 及其各阶偏导数的多项式。

步骤1:通过行波变换 $u(x, t) = u(\xi), \xi = x - Vt$,将式(2)转化为关于行波变量 ξ 的常微分方程

$$P(u, -Vu', u', Vu'', u'', \dots) = 0, \quad (3)$$

步骤2:假定式(3)的解可以表示成如下形式

$$u(\xi) = a_0 + \sum_{i=1}^m \left[a_i \left(\frac{G'}{G^2} \right)^i + b_i \left(\frac{G'}{G^2} \right)^{-i} \right], \quad (4)$$

其中 $G = G(\xi)$ 满足如下的二阶线性常微分方程

$$\left(\frac{G'}{G^2} \right)' = \mu + \lambda \left(\frac{G'}{G^2} \right)^2, \quad (5)$$

式(4)与式(5)中的 $a_0, a_i, b_i (i = 1, 2, \dots, m)$ 以及 λ, μ 均为待定常数,且 $\mu \neq 1, \lambda \neq 0$,未知常数 $a_i, b_i (i = 1, 2, \dots, m)$ 可能为0。正整数 m 由齐次平衡法确定,即通过平衡式(3)中出现的最高阶导数项和非线性项。

式(5)的通解通常可表示为以下三种情况:

当 $\mu\lambda > 0$ 时,

$$\frac{G'}{G^2} = \sqrt{\frac{\mu}{\lambda}} \left(\frac{E \cos(\sqrt{\mu\lambda} \xi) + F \sin(\sqrt{\mu\lambda} \xi)}{F \cos(\sqrt{\mu\lambda} \xi) - E \sin(\sqrt{\mu\lambda} \xi)} \right). \quad (6)$$

当 $\mu\lambda < 0$ 时,

$$\frac{G'}{G^2} = \frac{1}{2\lambda} \left(2\sqrt{|\mu\lambda|} - \frac{4E\sqrt{|\mu\lambda|}e^{2\xi\sqrt{|\mu\lambda|}}}{Ee^{2\xi\sqrt{|\mu\lambda|}} - F} \right), \quad (7)$$

也可以写成

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left(\frac{E \sinh(2\sqrt{|\mu\lambda|} \xi) + F \cosh(2\sqrt{|\mu\lambda|} \xi) + F}{E \sinh(2\sqrt{|\mu\lambda|} \xi) + F \cosh(2\sqrt{|\mu\lambda|} \xi) - F} \right). \quad (8)$$

当 $\mu = 0, \lambda \neq 0$ 时,

$$\frac{G'}{G^2} = -\frac{E}{\lambda(E\xi + F)}, \quad (9)$$

其中 E 和 F 是任意常数。

步骤3:将式(4)代入式(3),并利用常微分方程(5)整理可得一个关于 (G'/G^2) 的多项式,合并 (G'/G^2) 的相同幂次项,并令该多项式的 (G'/G^2) 各次幂系数为零,导出一个关于 $a_i, b_i (i = 1, 2, \dots, m), \lambda, \mu$ 的非线性代数方程组,求解此方程组,从而得到非线性偏微分方程(2)的精确解。

2 广义非线性薛定谔方程的精确解

假设式(1)具有如下形式的行波解

$$q(x, t) = u(\xi) e^{i\phi(x, t)}, \quad \xi = x - ct, \quad \phi = kx + \omega t, \quad (10)$$

其中 u 和 ϕ 分别代表 q 的振幅分量和相位分量, c 是波的传播速度。将式(10)代入式(1), 可得常微分方程

$$i(-cu' + 2ku' - \gamma u''' + 3\gamma k^2 u' + mu'u^2 + 2nu'u^2) + (-\omega u + u'' - k^2 u + \alpha u^3 + 3\gamma k u'' - \gamma k^3 u - mku^3) = 0, \quad (11)$$

其中 γ, m, n, α 是正常数, (\prime) 表示关于 ξ 的导数。

令式(11)实部和虚部分别为0, 由虚部得

$$-\gamma u''' + (2k - c + 3\gamma k^2)u' + (m + 2n)u'u^2 = 0, \quad (12)$$

由实部得

$$(1 + 3\gamma k)u'' - (\omega + k^2 + \gamma k^3)u + (\alpha - mk)u^3 = 0, \quad (13)$$

对式(12)积分一次可得

$$-\gamma u'' + (2k - c + 3\gamma k^2)u + \frac{1}{3}(m + 2n)u^3 = 0, \quad (14)$$

由式(13)(14)得

$$c = \frac{2k + 8\gamma k^2 + 8\gamma^2 k^3 - \omega\gamma}{1 + 3\gamma k}, \quad (15)$$

通过式(14), 可得到如下形式

$$Au'' + Bu + Cu^3 = 0, \quad (16)$$

其中 $A = -\gamma, B = 2k - c + 3\gamma k^2, C = \frac{m + 2n}{3}$ 。

平衡式(16)的最高阶导数项和非线性项, 然后利用 (G'/G^2) 展开法, 式(16)的精确解形式如下

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G^2} \right) + b_1 \left(\frac{G'}{G^2} \right)^{-1}, \quad (17)$$

其中 a_0, a_1, b_1 为未知常数, 现将式(17)和式(5)代入式(16)中, 合并 (G'/G^2) 的同次幂并使各幂次系数为零, 得到的代数方程组如下

$$\begin{aligned} \left(\frac{G'}{G^2} \right)^{-3} : 2\mu^2 b_1 A + b_1^3 C &= 0, \\ \left(\frac{G'}{G^2} \right)^{-2} : 3a_0 b_1^2 C &= 0, \\ \left(\frac{G'}{G^2} \right)^{-1} : 2\lambda \mu b_1 A + b_1 B + (3a_0^2 b_1 + 3b_1^2 a_1) C &= 0, \\ \left(\frac{G'}{G^2} \right)^0 : a_0 B + (a_0^3 + 6a_0 a_1 b_1) C &= 0, \\ \left(\frac{G'}{G^2} \right)^1 : 2\lambda \mu a_1 A + a_1 B + (3a_0^2 a_1 + 3a_1^2 b_1) C &= 0, \\ \left(\frac{G'}{G^2} \right)^2 : 3a_0 a_1^2 C &= 0, \\ \left(\frac{G'}{G^2} \right)^3 : 2\lambda^2 a_1 A + a_1^3 C &= 0. \end{aligned} \quad (18)$$

求解代数系统(18), 可以得到以下情况

$$a_0 = \pm \sqrt{-\frac{B}{C}}, a_1 = 0, b_1 = 0, \lambda = \lambda; \quad (19)$$

$$a_0 = 0, a_1 = 0, b_1 = \pm \sqrt{-\frac{2A}{C}} \mu, \lambda = -\frac{B}{2A\mu}; \quad (20)$$

$$a_0 = 0, a_1 = \pm \frac{B}{\mu\sqrt{-2AC}}, b_1 = 0, \lambda = -\frac{B}{2A\mu}; \quad (21)$$

$$a_0 = 0, a_1 = \mp \frac{B}{2\mu\sqrt{-2AC}}, b_1 = \pm \sqrt{-\frac{2A}{C}} \mu, \lambda = \frac{B}{4A\mu}; \quad (22)$$

$$a_0 = 0, a_1 = \mp \frac{B}{4\mu\sqrt{-2AC}}, b_1 = \pm \sqrt{-\frac{2A}{C}} \mu, \lambda = -\frac{B}{8A\mu}. \quad (23)$$

现将上述结果及式(6)、式(8)和式(9)代入式(17)。根据解集(19),假设 $\frac{B}{C} < 0$ 可得

$$q_1 = \pm \sqrt{-\frac{B}{C}} e^{i(kx + \omega t)}. \quad (24)$$

根据解集(20)–(23),有以下几种情况:

(i) 假设 $\frac{A}{C} < 0, \frac{B}{A} < 0$ 可得

$$q_2 = \left[\pm \sqrt{-\frac{2A}{C}} \mu \left(\sqrt{-\frac{2A\mu^2}{B}} \frac{E \cos\left(\sqrt{-\frac{B}{2A}} \xi\right) + F \sin\left(\sqrt{-\frac{B}{2A}} \xi\right)}{F \cos\left(\sqrt{-\frac{B}{2A}} \xi\right) - E \sin\left(\sqrt{-\frac{B}{2A}} \xi\right)} \right) \right]^{-1} e^{i(kx + \omega t)}, \quad (25)$$

$$q_3 = \left[\pm \frac{B}{\mu\sqrt{-2AC}} \left(\sqrt{-\frac{2A\mu^2}{B}} \frac{E \cos\left(\sqrt{-\frac{B}{2A}} \xi\right) + F \sin\left(\sqrt{-\frac{B}{2A}} \xi\right)}{F \cos\left(\sqrt{-\frac{B}{2A}} \xi\right) - E \sin\left(\sqrt{-\frac{B}{2A}} \xi\right)} \right) \right] e^{i(kx + \omega t)}, \quad (26)$$

$$q_4 = \left[\pm \frac{B}{2\mu\sqrt{-2AC}} \left(\sqrt{-\frac{4A}{B}} \mu \frac{E \sinh\left(\sqrt{-\frac{B}{A}} \xi\right) + E \cosh\left(\sqrt{-\frac{B}{A}} \xi\right) + F}{E \sinh\left(\sqrt{-\frac{B}{A}} \xi\right) + E \cosh\left(\sqrt{-\frac{B}{A}} \xi\right) - F} \right) \right] e^{i(kx + \omega t)} \mp$$

$$\left[\sqrt{-\frac{2A}{C}} \mu \left(\sqrt{-\frac{4A}{B}} \mu \frac{E \sinh\left(\sqrt{-\frac{B}{A}} \xi\right) + E \cosh\left(\sqrt{-\frac{B}{A}} \xi\right) + F}{E \sinh\left(\sqrt{-\frac{B}{A}} \xi\right) + E \cosh\left(\sqrt{-\frac{B}{A}} \xi\right) - F} \right) \right]^{-1} e^{i(kx + \omega t)}, \quad (27)$$

$$q_5 = \left[\mp \frac{B}{4\mu\sqrt{-2AC}} \left(\sqrt{-\frac{8A\mu^2}{B}} \frac{E \cos\left(\sqrt{-\frac{B}{8A}} \xi\right) + F \sin\left(\sqrt{-\frac{B}{8A}} \xi\right)}{F \cos\left(\sqrt{-\frac{B}{8A}} \xi\right) - E \sin\left(\sqrt{-\frac{B}{8A}} \xi\right)} \right) \right] e^{i(kx + \omega t)} \pm$$

$$\left[\sqrt{-\frac{2A}{C}} \mu \left(\sqrt{-\frac{8A\mu^2}{B}} \frac{\left(E \cos\left(\sqrt{-\frac{B}{8A}} \xi\right) + F \sin\left(\sqrt{-\frac{B}{8A}} \xi\right) \right)}{\left(F \cos\left(\sqrt{-\frac{B}{8A}} \xi\right) - E \sin\left(\sqrt{-\frac{B}{8A}} \xi\right) \right)} \right) \right]^{-1} e^{i(kx+\omega t)}. \quad (28)$$

特别地, 取 $\tan \eta = \frac{E}{F}$, 整理后得到 q_2, q_3, q_5 的另一种形式

$$q_{2,1} = \pm \sqrt{\frac{B}{C}} \cot\left(\sqrt{-\frac{B}{2A}} \xi + \eta\right) e^{i(kx+\omega t)}, \quad (29)$$

$$q_{3,1} = \pm \sqrt{\frac{B}{C}} \tan\left(\sqrt{-\frac{B}{2A}} \xi + \eta\right) e^{i(kx+\omega t)}, \quad (30)$$

$$q_{5,1} = \pm \sqrt{\frac{B}{4C}} \left[\tan\left(\sqrt{-\frac{B}{8A}} \xi + \eta\right) - \cot\left(\sqrt{-\frac{B}{8A}} \xi + \eta\right) \right] e^{i(kx+\omega t)}, \quad (31)$$

取 $E = F$, q_4 的另一种形式为

$$q_{4,1} = \pm \sqrt{\frac{B}{2C}} \left[\coth\left(\sqrt{-\frac{B}{4A}} \xi\right) - \tanh\left(\sqrt{-\frac{B}{4A}} \xi\right) \right] e^{i(kx+\omega t)}. \quad (32)$$

(ii) 假设 $\frac{A}{C} < 0, \frac{B}{A} > 0$ 可得

$$q_6 = \left[\pm \sqrt{-\frac{2A}{C}} \mu \left(\sqrt{\frac{2A}{B}} \mu \frac{\left(E \sinh\left(\sqrt{\frac{2B}{A}} \xi\right) + E \cosh\left(\sqrt{\frac{2B}{A}} \xi\right) + F \right)}{\left(E \sinh\left(\sqrt{\frac{2B}{A}} \xi\right) + E \cosh\left(\sqrt{\frac{2B}{A}} \xi\right) - F \right)} \right) \right]^{-1} e^{i(kx+\omega t)}, \quad (33)$$

$$q_7 = \left[\pm \frac{B}{\mu \sqrt{-2AC}} \left(\sqrt{\frac{2A}{B}} \mu \frac{\left(E \sinh\left(\sqrt{\frac{2B}{A}} \xi\right) + E \cosh\left(\sqrt{\frac{2B}{A}} \xi\right) + F \right)}{\left(E \sinh\left(\sqrt{\frac{2B}{A}} \xi\right) + E \cosh\left(\sqrt{\frac{2B}{A}} \xi\right) - F \right)} \right) \right] e^{i(kx+\omega t)}, \quad (34)$$

$$q_8 = \left[\mp \frac{B}{2\mu \sqrt{-2AC}} \left(\sqrt{\frac{4A\mu^2}{B}} \frac{\left(E \cos\left(\sqrt{\frac{B}{4A}} \xi\right) + F \sin\left(\sqrt{\frac{B}{4A}} \xi\right) \right)}{\left(F \cos\left(\sqrt{\frac{B}{4A}} \xi\right) - E \sin\left(\sqrt{\frac{B}{4A}} \xi\right) \right)} \right) \right] e^{i(kx+\omega t)} \pm \quad (35)$$

$$\left[\sqrt{-\frac{2A}{C}} \mu \left(\sqrt{\frac{4A\mu^2}{B}} \frac{\left(E \cos\left(\sqrt{\frac{B}{4A}} \xi\right) + F \sin\left(\sqrt{\frac{B}{4A}} \xi\right) \right)}{\left(F \cos\left(\sqrt{\frac{B}{4A}} \xi\right) - E \sin\left(\sqrt{\frac{B}{4A}} \xi\right) \right)} \right) \right]^{-1} e^{i(kx+\omega t)},$$

$$q_9 = \left[\mp \frac{B}{4\mu\sqrt{-2AC}} \left(\sqrt{\frac{8A}{B}} \mu \frac{E \sinh\left(\sqrt{\frac{B}{2A}} \xi\right) + E \cosh\left(\sqrt{\frac{B}{2A}} \xi\right) + F}{E \sinh\left(\sqrt{\frac{B}{2A}} \xi\right) + E \cosh\left(\sqrt{\frac{B}{2A}} \xi\right) - F} \right) \right] e^{i(kx+\omega t)} \pm \left[\sqrt{-\frac{2A}{C}} \mu \left(\sqrt{\frac{8A}{B}} \mu \frac{E \sinh\left(\sqrt{\frac{B}{2A}} \xi\right) + E \cosh\left(\sqrt{\frac{B}{2A}} \xi\right) + F}{E \sinh\left(\sqrt{\frac{B}{2A}} \xi\right) + E \cosh\left(\sqrt{\frac{B}{2A}} \xi\right) - F} \right) \right]^{-1} e^{i(kx+\omega t)}. \tag{36}$$

特别地,取 $\tan \eta = \frac{E}{F}$, 得到 q_8 的另一种形式

$$q_{8,1} = \pm \sqrt{-\frac{B}{2C}} \left[\tan\left(\sqrt{-\frac{B}{4A}} \xi + \eta\right) - \cot\left(\sqrt{-\frac{B}{4A}} \xi + \eta\right) \right] e^{i(kx+\omega t)}, \tag{37}$$

取 $E = F$, 得到 q_6, q_7, q_9 的另一种形式

$$q_{6,1} = \pm \sqrt{-\frac{B}{C}} \tanh\left(\sqrt{\frac{B}{2A}} \xi\right) e^{i(kx+\omega t)}, \tag{38}$$

$$q_{7,1} = \pm \sqrt{-\frac{B}{C}} \coth\left(\sqrt{\frac{B}{2A}} \xi\right) e^{i(kx+\omega t)}, \tag{39}$$

$$q_{9,1} = \pm \sqrt{-\frac{B}{4C}} \left[\coth\left(\sqrt{\frac{B}{8A}} \xi\right) - \tanh\left(\sqrt{\frac{B}{8A}} \xi\right) \right] e^{i(kx+\omega t)}. \tag{40}$$

为了更直观地理解这些显示解,将所求得解中的参数取特殊值,绘制成函数图像,一些典型的结果如图1—图3所示。

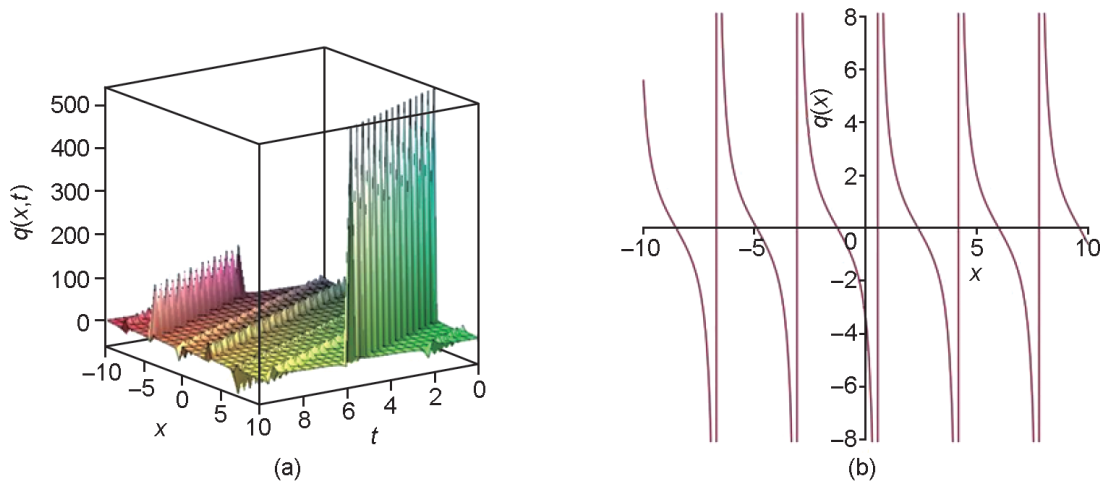


图1 当 $A = 2, B = -3, C = -1, c = 1, \mu = 1, E = 1$ 和 $F = -2$ 时,周期波解 q_2 的图像
(a) 3D 图像; (b) 当 $t = 0$ 时的 2D 图像

Fig. 1 The periodic wave solution for q_2 when $A = 2, B = -3, C = -1, c = 1, \mu = 1, E = 1$, and $F = -2$
(a) 3D graph; (b) 2D graph when $t = 0$

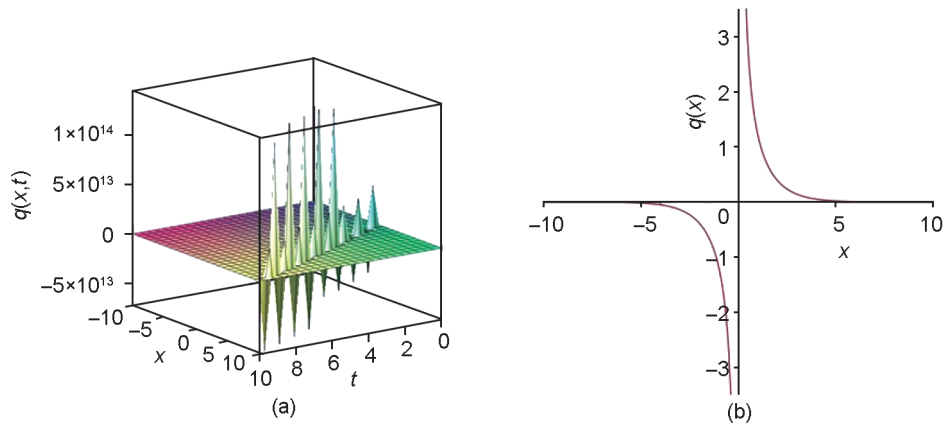


图2 当 $A = 2, B = C = -2, c = 1, \mu = 1$ 和 $E = F = 1$ 时,奇异波解 q_4 的图像
(a) 3D图像; (b) 当 $t=0$ 时的2D图像

Fig. 2 The singular wave solution for q_4 when $A = 2, B = C = -2, c = 1, \mu = 1$, and $E = F = 1$
(a) 3D graph; (b) 2D graph when $t = 0$

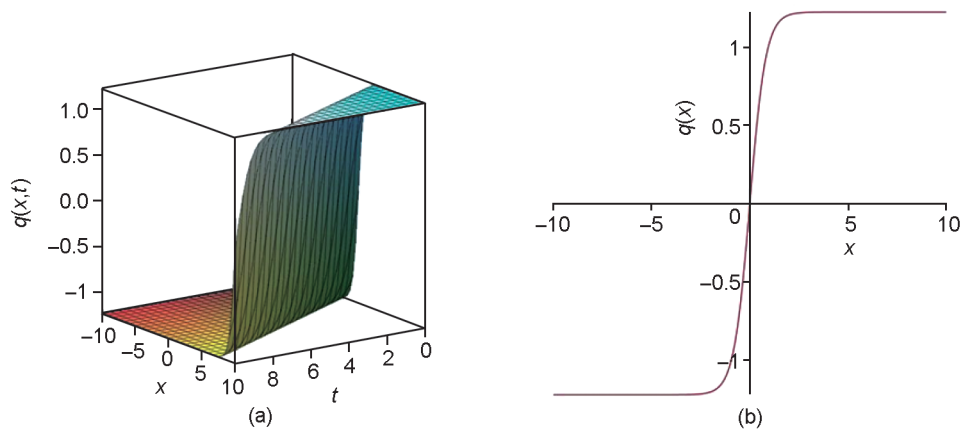


图3 当 $A = 2, B = 3, C = -2, c = 1, \mu = 0.5$ 和 $E = F = 2$ 时,扭结波解 q_6 的图像
(a) 3D图像; (b) 当 $t=0$ 时的2D图像

Fig. 3 The kink wave solution for q_6 when $A = 2, B = 3, C = -2, c = 1, \mu = 0.5$, and $E = F = 2$
(a) 3D graph; (b) 2D graph when $t = 0$

3 结论

本文以考虑了色散、散射等多种因素的广义非线性薛定谔方程为研究对象,应用 (G'/G^2) 展开法对其精确解进行了构建,得到一系列带参数的精确解,在参数取特殊值时得到周期波解、奇异波解、扭结波解等形式的解,体现了该模型的物理特性。与文献[14-19]中的结果进行对比, $q_1, q_2, q_3, q_4, q_5, q_{4,1}, q_{5,1}, q_{8,1}, q_{9,1}$ 所求解的表达形式是新的,特别地,与文献[19]相比,本文运用了含负幂项的 (G'/G^2) 展开法所求解的形式更多,丰富了解的体系。文中所求精确解有助于理解非线性光纤中光孤子的传播规律,对进一步了解和应用非线性薛定谔方程提供了有益帮助,同时表明 (G'/G^2) 展开法可提供多种解决方案,体现了对构建非线性薛定谔方程的精确解的重要意义。

参考文献:

- [1] WANG M L. Solitary Wave Solutions for Variant Boussinesq Equations[J]. *Phys Lett A*, 1995, **199**(3/4): 169-172. DOI: 10.1016/0375-9601(95)00092-h.
- [2] RABIE W B, AHMED H M. Dynamical Solitons and Other Solutions for Nonlinear Biswas-Milovic Equation with Kudryashov's Law by Improved Modified Extended Tanh-Function Method[J]. *Optik*, 2021, **245**: 167665.

- DOI: 10.1016/j.ijleo.2021.167665.
- [3] AHMED H M, RABIE W B, ALESSANDRA RAGUSA M. Optical Solitons and Other Solutions to Kaup-Newell Equation with Jacobi Elliptic Function Expansion Method[J]. *Anal Math Phys*, 2021, **11**(1): 1–16. DOI: 10.1007/s13324-020-00464-2.
- [4] 陈小燕, 刘妍丽. 广义 Gilson-Pickering 方程的分支解[J]. *应用数学*, 2023, **36**(2): 343–352. DOI: 10.13642/j.cnki.42-1184/o1.2023.02.015.
- CHEN X Y, LIU Y L. Bifurcation Solutions of a Generalized Gilson-Pickering Equation[J]. *Math Appl*, 2023, **36**(2): 343–352. DOI: 10.13642/j.cnki.42-1184/o1.2023.02.015.
- [5] 汪春江, 舒级, 李倩, 等. 一类(3+1)维 KdV 方程的有理解及其怪波[J]. *四川师范大学学报(自然科学版)*, 2017, **40**(2): 157–162. DOI: 10.3969/j.issn.1001-8395.2017.02.003.
- WANG C J, SHU J, LI Q, *et al.* Rogue Waves and Rational Solutions for a Class of (3 + 1)-dimensional KdV Equation[J]. *J Sichuan Norm Univ Nat Sci*, 2017, **40**(2): 157–162. DOI: 10.3969/j.issn.1001-8395.2017.02.003.
- [6] ADERYANI S R, SAADATI R, VAHIDI J, *et al.* The Exact Solutions of Conformable Time-fractional Modified Nonlinear Schrödinger Equation by First Integral Method and Functional Variable Method[J]. *Opt Quantum Electron*, 2022, **54**(4): 1–17. DOI: 10.1007/s11082-022-03605-y.
- [7] HASSABALLA A A. The G'/G^2 -Expansion Method for Solving Fractional Burgers-Fisher and Burgers Equations[J]. *Appl Comput Math*, 2020, **9**(3): 56. DOI: 10.11648/j.acm.20200903.12.
- [8] 陈继培, 陈浩. (g'/g^2) 展开法及其在耦合非线性 Klein-Gordon 方程中的应用[J]. *华南师范大学学报(自然科学版)*, 2012, **44**(2): 63–66. DOI: 10.6054/j.jscnun.2012.05.014.
- CHEN J P, CHEN H. The (g'/g^2) -expansion Method and Its Application to Coupled Nonlinear Klein-Gordon Equation[J]. *J South China Norm Univ Nat Sci Ed*, 2012, **44**(2): 63–66. DOI: 10.6054/j.jscnun.2012.05.014.
- [9] BILAL M, UR-REHMAN S, AHMAD J. Dynamics of Diverse Optical Solitary Wave Solutions to the Biswas-arsheed Equation in Nonlinear Optics[J]. *Int J Appl Comput Math*, 2022, **8**(3): 1–16. DOI: 10.1007/s40819-022-01309-1.
- [10] ADERYANI S R, SAADATI R, O'REGAN D, *et al.* Describing Water Wave Propagation Using the G'/G^2 -expansion Method[J]. *Mathematics*, 2023, **11**(1): 191. DOI: 10.3390/math11010191.
- [11] ARSHED S, RAZA N. Optical Solitons Perturbation of Fokas-Lenells Equation with Full Nonlinearity and Dual Dispersion[J]. *Chin J Phys*, 2020, **63**: 314–324. DOI: 10.1016/j.cjph.2019.12.004.
- [12] KODAMA Y. Optical Solitons in a Monomode Fiber[J]. *J Stat Phys*, 1985, **39**(5): 597–614. DOI: 10.1007/BF01008354.
- [13] POTASEK M J. Novel Femtosecond Solitons in Optical Fibers, Photonic Switching, and Computing[J]. *J Appl Phys*, 1989, **65**(3): 941–953. DOI: 10.1063/1.342996.
- [14] DU L X, SUN Y H, WU D S. Bifurcations and Solutions for the Generalized Nonlinear Schrödinger Equation[J]. *Phys Lett A*, 2019, **383**(36): 126028. DOI: 10.1016/j.physleta.2019.126028.
- [15] MAHAK N, AKRAM GHAZALA A. The Modified Auxiliary Equation Method to Investigate Solutions of the Perturbed Nonlinear Schrödinger Equation with Kerr Law Nonlinearity[J]. *Optik*, 2020, **207**: 164467. DOI: 10.1016/j.ijleo.2020.164467.
- [16] MIAO X J, ZHANG Z Y. The Modified G'/G -expansion Method and Traveling Wave Solutions of Nonlinear the Perturbed Nonlinear Schrödinger's Equation with Kerr Law Nonlinearity[J]. *Commun Nonlinear Sci Numer Simul*, 2011, **16**(11): 4259–4267. DOI: 10.1016/j.cnsns.2011.03.032.
- [17] 石兰芳, 王明灿, 钱正雅. 应用 Riccati-Bernoulli 辅助方程求解广义非线性 Schrödinger 方程和(2+1)维非线性 Ginzburg-Landau 方程[J]. *应用数学和力学*, 2020, **41**(7): 786–795. DOI: 10.21656/1000-0887.400271.
- SHI L F, WANG M C, QIAN Z Y. Solution of Generalized Nonlinear Schrödinger Equations and (2+1)-dimensional Nonlinear Ginzburg-landau Equations with a Riccati-bernoulli Auxiliary Equation Method[J]. *Appl Math Mech*, 2020, **41**(7): 786–795. DOI: 10.21656/1000-0887.400271.
- [18] 何黎霞, 孙峪怀, 胡艳. 广义非线性 Schrödinger 方程的显示解[J]. *内江师范学院学报*, 2021, **36**(6): 24–28. DOI: 10.13603/j.cnki.51-1621/z.2021.06.005.
- HE L X, SUN Y H, HU Y. The Explicit Solutions of the Generalized Nonlinear Schrödinger Equation[J]. *J Neijiang Normal Univ*, 2021, **36**(6): 24–28. DOI: 10.13603/j.cnki.51-1621/z.2021.06.005.
- [19] 张艳妮, 张丽萍, 庞晶. 应用 (G'/G^2) 展开法求解含三阶色散项的薛定谔方程[J]. *应用数学进展*, 2017, **6**(2): 212–217. DOI: 10.12677/aam.2017.62024.
- ZHANG Y N, ZHANG L P, PANG J. Application of (G'/G^2) Expansion Method for Solving Schrödinger's Equation with Three-Order Dispersion[J]. *Adv Appl Math*, 2017, **6**(2): 212–217. DOI: 10.12677/aam.2017.62024.