

一类三阶积分边值问题的奇数个正解

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摘要:为了发展并完善常微分方程非局部问题的基本理论,本文利用 Guo-Krasnoselskill 不动点定理建立了一类三阶积分边值问题奇数个正解的存在性。首先,通过研究相应线性积分边值问题的格林函数得到解的形式,并讨论格林函数的性质和解的非负性、单调性等性质。其次,将三阶积分边值问题解的存在性转化为锥上算子的不动点问题,并验证该算子的全连续性。接下来,借助 Guo-Krasnoselskill 不动点定理证明非线性项满足特定增长性条件下该算子存在奇数个不动点,从而得到三阶积分边值问题奇数个正解的存在性。最后,给出具体的例子说明了研究结果的合理性。基于此,本文在方法上将常用于建立正解以及至少两个正解存在性的 Guo-Krasnoselskill 不动点定理运用到无穷多(奇数)个正解的存在性研究,推广并完善了三阶边值问题正解的研究结果,丰富了常微分方程边值问题的研究内容,为常微分方程非局部问题在应用数学、物理学领域的广泛应用提供了理论依据。

关键词:积分边值问题;不动点定理;正解

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Existence of Odd Number of Positive Solutions to a Class of Third-order Integral Boundary Value Problem

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Abstract: In order to develop and perfect the basic theory for nonlocal problems of ordinary differential equations, we establish the existence of odd number of positive solutions to a class of third-order integral boundary value problems by using Guo-Krasnoselskill fixed point theorem in this paper. Firstly, the form of solution is obtained by investigating the corresponding Green's function of linear integral boundary value problem, and the properties of Green's function and the nonnegativity, monotonicity and other properties of solutions are discussed at the same time. Secondly, the existence of solutions to the third-order integral boundary value problem is transformed into a fixed point problem to an operator defined on a cone, and the complete continuity of the operator is examined. Next, with the help of Guo-Krasnoselskill fixed point theorem, it is proven that the operator has an odd number of fixed points when the nonlinear term satisfies specific growth conditions, and thus the existence of odd number of positive solutions for the third-order integral boundary value problem is obtained. Finally, a concrete example is given to illustrate the rationality of our results. Based on the facts above, Guo-Krasnoselskill fixed point theorem, that is often used to establish the existence of one positive solution and two positive solutions at least, is applied to investigate the existence of infinite (odd) positive solutions in this paper, which extends and perfects the research results of positive solutions to third-order boundary value problems, enriches the research contents of boundary

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value problems for ordinary differential equations, and provides theoretical basis for wide applications of nonlocal problems of ordinary differential equations in applied mathematics and physics.

Key words: integral boundary value problem; fixed point theorem; positive solution

0 引言

三阶边值问题在应用数学、电磁波和重力驱动流等物理学领域的广泛应用,使相应常微分方程非局部问题的正解研究取得了相对成熟的结果^[1-9]。其中,热传导、等离子物理、热弹性等物理问题可以归结为带积分边界条件的常微分方程非局部问题,这类问题的正解引起了众多学者的关注,特别是三阶积分边值问题的正解及其多个正解的研究^[2-9]。例如,2022年,Smirnov Sergey^[4]利用 Schauder 不动点定理研究了含积分边界条件的三阶边值问题

$$\begin{cases} u''' + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0, u'(0) = 0, u(1) = \int_0^1 g(s)u(s)ds \end{cases} \quad (1)$$

正解的存在性。同年,潘凤等^[5]借助 Guo-Krasnoselskill 不动点定理建立了三阶积分边值问题

$$\begin{cases} u'''(t) + a(t)f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = \int_0^1 \alpha(s)u(s)ds, u'(0) = u'(1) = 0 \end{cases} \quad (2)$$

至少两个正解的存在性。此外,杨佳^[7]利用 Leggett-Williams 不动点定理考虑了三阶积分边值问题

$$\begin{cases} u'''(t) + f(t, u(t)) = 0, & t \in [0, 1], \\ u(0) = 0, u''(0) = 0, u'(1) = \int_0^1 g(s)u'(s)ds \end{cases} \quad (3)$$

的三个正解。基于 Leggett-Williams 不动点定理的三阶常微分方程非局部问题三个正解的研究还可参见文献[7-10]。众所周知,Leggett-Williams 不动点定理常用于常微分边值问题至少三个正解的存在性研究,而 Guo-Krasnoselskill 不动点定理多用于单个正解或至少两个正解的存在性^[5, 7-8]。但是,目前关于三阶常微分方程非局部问题,特别是三阶积分边值问题 $2m - 1 (m = 1, 2, \dots)$ 个正解的研究并不多见。2021年,张瑞燕^[11]借助 Guo-Krasnoselskill 不动点定理建立了三阶三点边值问题

$$\begin{cases} u'''(t) - \alpha u'(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, u'(1) = ku'(\xi) \end{cases} \quad (4)$$

$2m - 1 (m = 1, 2, \dots)$ (奇数)个正解的存在性。

受张瑞燕^[11]工作的启发,本文利用 Guo-Krasnoselskill 不动点定理研究三阶积分边值问题

$$\begin{cases} u'''(t) + \lambda a(t)f(t, u(t), u'(t)) = 0, & t \in (0, 1), \\ u(0) = \int_0^1 \alpha(s)u(s)ds, u'(0) = u'(1) = 0 \end{cases} \quad (5)$$

$2m - 1 (m = 1, 2, \dots)$ (奇数)个正解的存在性。因此,我们不仅将张瑞燕^[11]三阶三点边值问题推广到三阶积分边值问题,还将 Guo-Krasnoselskill 不动点定理运用到无穷多(奇数)个正解的存在性研究。注意到,当式(5)中的 $\lambda = 1, f(t, u(t), u'(t)) = f(t, u(t))$ 时,式(5)退化为式(2)。因此,本文将潘凤等^[5]至少两个正解的研究结果推广到 $2m - 1 (m = 1, 2, \dots)$ 个正解的同时,还完善了一些三阶边值问题正解的研究结果^[5, 7-8, 11-12]。

1 预备知识

首先给出本文用到的假设条件、相关定义和主要工具:

(A1) $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$;

(A2) $a \in C([0, 1], [0, +\infty))$, $a(t)$ 在 $(0, 1)$ 上不恒为零, 且 $0 < \int_0^1 a(s) ds < +\infty$, 参数 $\lambda > 0$;

(A3) $\alpha \in C([0, 1], [0, +\infty))$, 且满足 $\mu := \int_0^1 \alpha(s) ds < 1$ 。

引理 1^[11] (Guo-Krasnoselskii 不动点定理) 假设 X 是 Banach 空间, K 是 X 中的锥, $T: K \rightarrow K$ 是全连续算子且 $0 < r < R$ 。若下列两个条件之一成立:

1) $\|Tu\| \leq \|u\|, \forall u \in K, \|u\| = r$, 且 $\|Tu\| \geq \|u\|, \forall u \in K, \|u\| = R$;

2) $\|Tu\| \geq \|u\|, \forall u \in K, \|u\| = r$, 且 $\|Tu\| \leq \|u\|, \forall u \in K, \|u\| = R$,

则算子 T 有不动点 $u \in K$ 且 $r < \|u\| < R$ 。

引理 2 若条件(A1)–(A3)成立, 则边值问题式(5)有唯一解

$$u(t) = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds, \quad (6)$$

这里,

$$H(t, s) = G(t, s) + \frac{1}{1-\mu} \int_0^1 G(\tau, s) \alpha(\tau) d\tau, \quad (7)$$

其中,

$$G(t, s) = \frac{1}{2} \begin{cases} 2st - st^2 - s^2, & 0 \leq s \leq t \leq 1, \\ t^2(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

证明 设 $u(t)$ 为边值问题式(5)的解, 则当 $t \in [0, 1]$ 时, 有 $u'''(t) + \lambda a(t) f(t, u(t), u'(t)) = 0$, 对此式从 0 到 t 积分 3 次依次得

$$u''(t) = u''(0) - \int_0^t \lambda a(s) f(s, u(s), u'(s)) ds, \quad (8)$$

$$u'(t) = u'(0)t - \int_0^t \lambda(t-s) a(s) f(s, u(s), u'(s)) ds, \quad (9)$$

$$u(t) = u(0) + \frac{u''(0)}{2} t^2 - \int_0^t \frac{(t-s)^2}{2} \lambda a(s) f(s, u(s), u'(s)) ds, \quad (10)$$

由式(9)和边界条件 $u'(1) = 0$ 可知

$$u''(0) = \int_0^1 \lambda(1-s) a(s) f(s, u(s), u'(s)) ds, \quad (11)$$

再将式(11)和边界条件 $u(0) = \int_0^1 \alpha(s) u(s) ds$ 代入式(10)可得

$$u(t) = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds,$$

其中 $H(t, s), G(t, s)$ 的定义见式(7)。证毕。

引理 3 $G(t, s)$ 具有如下性质:

(i) 对 $\forall t, s \in [0, 1]$, 有 $\frac{1}{2} t^2 s(1-s) \leq G(t, s) \leq \frac{1}{2} s(1-s)$;

(ii) 对 $\forall t, s \in [0, 1]$, 有 $ts(1-t)(1-s) \leq G_1(t, s) = \frac{\partial}{\partial t} G(t, s) \leq s(1-s)$;

(iii) 当 $t \in [\frac{1}{4}, \frac{3}{4}]$, $s \in [0, 1]$ 时, 有 $G(t, s) \geq \frac{1}{32} s(1-s)$ 和 $G_1(t, s) \geq \frac{3}{16} s(1-s)$ 。

证明 (i) 注意到, 当 $0 \leq s \leq t \leq 1$ 时, 有

$$G(t, s) = \frac{1}{2} (2t - t^2 - s) s = \frac{1}{2} [1 - s - (1-t)^2] s \leq \frac{1}{2} (1-s) s,$$

$$G(t, s) = \frac{1}{2} (2t - t^2 - s) s = \frac{1}{2} t^2 (1-s) + \frac{1}{2} (1-t) [(t-s) + (1-s)t] s \geq \frac{1}{2} t^2 (1-s);$$

当 $0 \leq t \leq s \leq 1$ 时, 有

$$\frac{1}{2} t^2 (1-s) s \leq G(t, s) = \frac{1}{2} t^2 (1-s) \leq \frac{1}{2} s(1-s)。$$

(ii) 由 $G(t, s)$ 的表达式, 不难得到

$$G_1(t, s) = \frac{\partial}{\partial t} G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

当 $0 \leq s \leq t \leq 1$ 时, 有 $G_1(t, s) = s(1-t) \leq s(1-s)$, $G_1(t, s) = s(1-t) \geq ts(1-t)(1-s)$; 当 $0 \leq t \leq s \leq 1$ 时, 有 $G_1(t, s) = t(1-s) \leq s(1-s)$, $G_1(t, s) = t(1-s) \geq ts(1-t)(1-s)$, 从而 $ts(1-t)(1-s) \leq G_1(t, s) = \frac{\partial}{\partial t} G(t, s) \leq s(1-s)$ 。

(iii) 当 $t \in [\frac{1}{4}, \frac{3}{4}]$ 时, 根据 (i), (ii) 易知, $G(t, s) \geq \frac{1}{32}s(1-s)$, $G_1(t, s) \geq \frac{3}{16}s(1-s)$ 。证毕。

引理4 $H(t, s)$ 具有如下性质:

(i) 对 $\forall t, s \in [0, 1]$, 有 $0 \leq H(t, s) \leq \frac{1}{2(1-\mu)}s(1-s)$;

(ii) 对 $\forall t, s \in [0, 1]$, 有 $0 \leq H_1(t, s) = \frac{\partial}{\partial t} H(t, s) \leq s(1-s)$;

(iii) 当 $t \in [\frac{1}{4}, \frac{3}{4}]$, $s \in [0, 1]$ 时, 有 $H(t, s) \geq \frac{1}{32(1-\mu)}s(1-s)$ 和 $H_1(t, s) \geq \frac{3}{16}s(1-s)$ 。

证明 (i) 对 $\forall t, s \in [0, 1]$, 由式(7)可知 $H(t, s) \geq 0$ 显然成立。再由引理3, 可知

$$H(t, s) = G(t, s) + \frac{1}{1-\mu} \int_0^1 G(\tau, s) \alpha(\tau) d\tau \leq \frac{1}{2}s(1-s) + \frac{1}{1-\mu} \int_0^1 \frac{1}{2}s(1-s) \alpha(\tau) d\tau = \frac{1}{2(1-\mu)}s(1-s)。$$

(ii) 对 $\forall t, s \in [0, 1]$, 由式(7)可得

$$H_1(t, s) = \frac{\partial}{\partial t} H(t, s) = \frac{\partial}{\partial t} \left(G(t, s) + \frac{1}{1-\mu} \int_0^1 G(\tau, s) \alpha(\tau) d\tau \right) = \frac{\partial}{\partial t} G(t, s) = G_1(t, s), \tag{12}$$

显然有 $H_1(t, s) \geq 0$ 成立, 再由引理3不难证明 $H_1(t, s) = G_1(t, s) \leq s(1-s)$ 。

(iii) 对 $\forall t \in [\frac{1}{4}, \frac{3}{4}]$, 由式(7), 式(12)及引理3可知

$$H(t, s) = G(t, s) + \frac{1}{1-\mu} \int_0^1 G(\tau, s) \alpha(\tau) d\tau \geq \frac{1}{32}s(1-s) + \frac{1}{1-\mu} \int_0^1 \frac{1}{32}s(1-s) \alpha(\tau) d\tau \geq \frac{1}{32(1-\mu)}s(1-s)$$

和 $H_1(t, s) = G_1(t, s) \geq \frac{3}{16}s(1-s)$ 。证毕。

记 $E = C^1[0, 1]$, 对任意的 $u \in E$, 定义

$$\|u\|_1 = \|u\| + \|u'\| = \max_{t \in [0, 1]} |u(t)| + \max_{t \in [0, 1]} |u'(t)|,$$

其中 $\|u\| = \max_{t \in [0, 1]} |u(t)|$, $\|u'\| = \max_{t \in [0, 1]} |u'(t)|$, 不难验证 E 为 Banach 空间。

引理5 假设条件 (A1)–(A3) 成立, 则边值问题式(5)的唯一解 u 满足 $u(t) \geq 0$, $u'(t) \geq 0$ 且

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{16}\|u\|, \quad \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u'(t) \geq \frac{3}{16}\|u'\|。$$

证明 首先, 不难证明 $u(t) \geq 0$ 。其次, 根据式(6), 式(7)和引理4可得

$$u(t) = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds \leq \int_0^1 \lambda a(s) \frac{1}{2(1-\mu)} s(1-s) f(s, u(s), u'(s)) ds = \frac{1}{2(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds,$$

所以

$$\|u\| \leq \frac{1}{2(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds,$$

从而由引理4得

$$\begin{aligned} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds \geq \\ &\int_0^1 \lambda a(s) \frac{1}{32(1-\mu)} s(1-s) f(s, u(s), u'(s)) ds = \\ &\frac{1}{32(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds \geq \frac{1}{16} \|u\|. \end{aligned}$$

注意到, $u(t) = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds$, 则有

$$u'(t) = \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds, \quad (13)$$

故, $u'(t) \geq 0$ 显然成立。由引理4可知

$$u'(t) = \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds \leq \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds,$$

这表明 $\|u'\| \leq \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds$, 则由引理4得

$$\begin{aligned} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u'(t) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds \geq \int_0^1 \lambda a(s) \frac{3}{16} s(1-s) f(s, u(s), u'(s)) ds = \\ &\frac{3}{16} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds \geq \frac{3}{16} \|u'\|. \end{aligned}$$

证毕。

$$\text{令 } K = \left\{ u \in E: u(t) \geq 0, u'(t) \geq 0, t \in [0, 1], \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{16} \|u\|, \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u'(t) \geq \frac{3}{16} \|u'\| \right\}, \text{ 易知 } K$$

为 E 中的锥, 且对于任意的 $u \in K$, 有

$$u(t) + u'(t) \geq \frac{1}{16} \|u\| + \frac{3}{16} \|u'\| \geq \frac{1}{16} \|u\|_1, t \in [\frac{1}{4}, \frac{3}{4}]. \quad (14)$$

定义 K 上的算子 T 为

$$(Tu)(t) = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds, \quad (15)$$

则边值问题式(5)有解等价于算子 T 存在不动点。

引理6 若条件(A1)–(A3)成立, 则 $T: K \rightarrow K$ 是全连续算子。

证明 对任意的 $u \in K$, 有

$$(Tu)'(t) = \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds, \quad (16)$$

因此 $(Tu) \geq 0, (Tu)' \geq 0$ 。由式(15), 式(16)和引理4知

$$\begin{aligned} (Tu)(t) &= \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds \leq \int_0^1 \lambda a(s) \frac{1}{2(1-\mu)} s(1-s) f(s, u(s), u'(s)) ds = \\ &\frac{1}{2(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds, \end{aligned}$$

和 $(Tu)'(t) = \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds \leq \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds$, 所以

$$\|Tu\| \leq \frac{1}{2(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds. \quad (17)$$

$$\|(Tu)'\| \leq \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds. \tag{18}$$

因此,结合式(15)一式(18)及引理4可得

$$\begin{aligned} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} (Tu)(t) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds \geq \int_0^1 \lambda a(s) \frac{1}{32(1-\mu)} s(1-s) f(s, u(s), u'(s)) ds = \\ &= \frac{1}{32(1-\mu)} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds \geq \frac{1}{16} \|Tu\| \end{aligned}$$

和

$$\begin{aligned} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} (Tu)'(t) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds \geq \int_0^1 \lambda a(s) \frac{3}{16} s(1-s) f(s, u(s), u'(s)) ds = \\ &= \frac{3}{16} \int_0^1 \lambda a(s) s(1-s) f(s, u(s), u'(s)) ds \geq \frac{3}{16} \|(Tu)'\|, \end{aligned}$$

所以 $Tu \in K$ 。

若 $u_m, u \in K$ 且 $\|u_m - u\|_1 \rightarrow 0$ (当 $m \rightarrow \infty$), 则 $\exists M_1 > 0$, 使得对 $\forall m \in N$, 有 $\|u_m\|_1 \leq M_1$ 。令 $M_2 = \sup\{f(t, u, u') : (t, u, u') \in [0, 1] \times [0, M_1] \times [0, M_1]\}$, 那么, 对 $\forall t \in [0, 1]$, 由引理4可知

$$\begin{aligned} \lambda a(s) H(t, s) f(s, u_m(s), u'_m(s)) &\leq \lambda a(s) \frac{M_2 s(1-s)}{2(1-\mu)}, s \in [0, 1], \\ \lambda a(s) H_1(t, s) f(s, u_m(s), u'_m(s)) &\leq \lambda a(s) M_2 s(1-s), s \in [0, 1]. \end{aligned}$$

由勒贝格控制收敛定理可知, 当 $t \in [0, 1]$ 时, 有

$$\begin{aligned} \lim_{m \rightarrow \infty} (Tu_m)(t) &= \lim_{m \rightarrow \infty} \int_0^1 \lambda a(s) H(t, s) f(s, u_m(s), u'_m(s)) ds = \int_0^1 \lambda a(s) H(t, s) f(s, u(s), u'(s)) ds = (Tu)(t), \\ \lim_{m \rightarrow \infty} (Tu_m)'(t) &= \lim_{m \rightarrow \infty} \int_0^1 \lambda a(s) H_1(t, s) f(s, u_m(s), u'_m(s)) ds = \int_0^1 \lambda a(s) H_1(t, s) f(s, u(s), u'(s)) ds = (Tu)'(t). \end{aligned}$$

故 $\|(Tu_m) - (Tu)\|_1 = \|(Tu_m) - (Tu)\| + \|(Tu_m)' - (Tu)'\| \rightarrow 0$ (当 $m \rightarrow \infty$), 即 T 是连续算子。再借助 Arzela-Ascoli 定理可知, $T: K \rightarrow K$ 是全连续的。证毕。

$$\text{记 } N_1 = \max_{t \in [0, 1]} a(t), N_2 = \min_{t \in [0, 1]} a(t) \text{ 以及常数 } \Phi_1 = \frac{1-\mu}{\lambda N_2 \mu \min_{\tau, s \in [\frac{1}{4}, \frac{3}{4}]} G(\tau, s)}, \Phi_2 = \frac{8(1-\mu)}{\lambda N_1 (3-2\mu)}.$$

2 主要结论及其证明

定理 1 假设条件 (A1)–(A3) 成立, 对任意的 $m \in \mathbb{N} \cup \{+\infty\}$, 数列 $\{r_k\}_{k=1}^m$ 和 $\{R_k\}_{k=1}^m$ 满足 $r_{k+1} < R_{k+1}, R_{k+1} < r_k$ 和 $r_k < R_k, k = 1, 2, 3, \dots, m-1$ 。若 f 满足如下条件:

- (i) $f(t, u, v) \geq Br_k, \frac{1}{16} r_k \leq u + v \leq r_k, t \in [\frac{1}{4}, \frac{3}{4}], B \in (\Phi_1, +\infty)$;
- (ii) $f(t, u, v) \leq AR_k, 0 \leq u + v \leq R_k, t \in [0, 1], A \in (0, \Phi_2)$ 。

则边值问题式(5)存在 $2m-1$ ($m = 1, 2, \dots$) 个正解 $\{u_k\}_{k=1}^m$ 和 $\{u^*_k\}_{k=1}^{m-1}$, 其中 $\{u_k\}_{k=1}^m$ 满足 $r_k \leq \|u_k\| \leq R_k, (k = 1, 2, 3, \dots, m), \{u^*_k\}_{k=1}^{m-1}$ 满足 $R_{k+1} \leq \|u^*_k\| \leq r_k, (k = 1, 2, 3, \dots, m)$ 。

证明 记 $\{\Omega_{1,k}\}_{k=1}^m = \{u \in K : \|u\|_1 < r_k\}, k = 1, 2, 3, \dots, m; \{\Omega_{2,k}\}_{k=1}^m = \{u \in K : \|u\|_1 < R_k\}, k = 1, 2, 3, \dots, m$ 。显然 $\{\Omega_{1,k}\}_{k=1}^m$ 和 $\{\Omega_{2,k}\}_{k=1}^m$ 是 E 中的开子集。

为了建立边值问题式(5) $2m-1$ ($m = 1, 2, \dots$) 个正解的存在性, 分以下两步证明:

第一步 当 $u \in K \cap \partial\Omega_{1,k}$ ($k = 1, 2, \dots, m$) 时, 对 $\forall t \in [\frac{1}{4}, \frac{3}{4}]$, 有 $\frac{1}{16}\|u\|_1 \leq u + v \leq \|u\| + \|v\| = \|u\|_1 = r_k$, 由条件(i)可得

$$\begin{aligned} \|Tu\| &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H(t,s) f(s, u(s), v(s)) ds \right| = \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H(t,s) f(s, u(s), v(s)) ds \right| \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) \left[G(t,s) + \frac{1}{1-\mu} \int_0^1 G(\tau,s) \alpha(\tau) d\tau \right] f(s, u(s), v(s)) ds \geq \\ &= \int_0^1 \lambda a(s) \left[\frac{1}{1-\mu} \int_0^1 G(\tau,s) \alpha(\tau) d\tau \right] f(s, u(s), v(s)) ds \geq \frac{\lambda N_2 \mu}{1-\mu} \min_{\tau, s \in [\frac{1}{4}, \frac{3}{4}]} G(\tau,s) Br_k \end{aligned}$$

和 $\|(Tu)'\| = \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H_1(t,s) f(s, u(s), v(s)) ds \right| = \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) G_1(t,s) f(s, u(s), v(s)) ds \right| \geq 0$, 所以 $\|Tu\|_1 = \|Tu\| + \|(Tu)'\| \geq \frac{\lambda N_2 \mu}{1-\mu} \min_{\tau, s \in [\frac{1}{4}, \frac{3}{4}]} G(\tau,s) Br_k > r_k = \|u\|_1$.

令 $u \in K \cap \partial\Omega_{2,k}$, 对 $\forall t \in [0, 1]$, 有 $0 \leq u + v \leq \|u\| + \|v\| = \|u\|_1 = R_k$, 由条件(ii)可得

$$\begin{aligned} \|Tu\| &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H(t,s) f(s, u(s), v(s)) ds \right| = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) H(t,s) f(s, u(s), v(s)) ds = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) \left[G(t,s) + \frac{1}{1-\mu} \int_0^1 G(\tau,s) \alpha(\tau) d\tau \right] f(s, u(s), v(s)) ds \leq \\ &= \int_0^1 \lambda N_1 \left[\frac{1}{2} s(1-s) + \frac{1}{2} s(1-s) \frac{\mu}{1-\mu} \right] AR_k ds \leq \\ &= \lambda N_1 \left(\frac{1}{8} + \frac{\mu}{8(1-\mu)} \right) AR_k = \frac{\lambda N_1}{8(1-\mu)} AR_k, \end{aligned}$$

$$\begin{aligned} \|(Tu)'\| &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H_1(t,s) f(s, u(s), v(s)) ds \right| = \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) G_1(t,s) f(s, u(s), v(s)) ds \right| = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) G_1(t,s) f(s, u(s), v(s)) ds \leq \int_0^1 \lambda N_1 s(1-s) AR_k ds \leq \frac{\lambda N_1}{4} AR_k, \end{aligned}$$

所以 $\|Tu\|_1 = \|Tu\| + \|(Tu)'\| \leq \frac{\lambda N_1}{8(1-\mu)} AR_k + \frac{\lambda N_1}{4} AR_k = \frac{\lambda N_1(3-2\mu)}{8(1-\mu)} AR_k < R_k = \|u\|_1$. 根据引理1的结论2)易知, T 有不动点 $u \in K$ 且 $r_k < \|u\|_1 < R_k$. 又因为 $k = 1, 2, 3, \dots, m$, 所以 T 有 m 个不动点 $\{u_k\}_{k=1}^m$.

第二步 当 $u^* \in K \cap \partial\Omega_{2,k+1}$ ($k = 1, 2, 3, \dots, m-1$), 对 $\forall t \in [0, 1]$, 有 $u^* + v^* \leq \|u^*\| + \|v^*\| = \|u^*\|_1 = R_{k+1}$, 由条件(ii)可知

$$\begin{aligned} \|Tu^*\| &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H(t,s) f(s, u^*(s), v^*(s)) ds \right| = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) H(t,s) f(s, u^*(s), v^*(s)) ds = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) \left[G(t,s) + \frac{1}{1-\mu} \int_0^1 G(\tau,s) \alpha(\tau) d\tau \right] f(s, u^*(s), v^*(s)) ds \leq \\ &= \int_0^1 \lambda N_1 \left[\frac{1}{2} s(1-s) + \frac{1}{2} s(1-s) \frac{\mu}{1-\mu} \right] AR_{k+1} ds \leq \end{aligned}$$

$$\lambda N_1 \left(\frac{1}{8} + \frac{\mu}{8(1-\mu)} \right) AR_{k+1} = \frac{\lambda N_1}{8(1-\mu)} AR_{k+1}$$

和

$$\begin{aligned} \|(Tu^*)'\| &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) H_1(t,s) f(s, u^*(s), v^*(s)) ds \right| = \\ &= \max_{t \in [0,1]} \left| \int_0^1 \lambda a(s) G_1(t,s) f(s, u^*(s), v^*(s)) ds \right| = \\ &= \max_{t \in [0,1]} \int_0^1 \lambda a(s) G_1(t,s) f(s, u^*(s), v^*(s)) ds \leq \\ &= \int_0^1 \lambda N_1 s(1-s) AR_{k+1} ds \leq \frac{\lambda N_1}{4} AR_{k+1}, \end{aligned}$$

所以 $\|Tu^*\|_1 = \|Tu^*\| + \|(Tu^*)'\| \leq \frac{\lambda N_1}{8(1-\mu)} AR_{k+1} + \frac{\lambda N_1}{4} AR_{k+1} = \frac{\lambda N_1(3-2\mu)}{8(1-\mu)} AR_{k+1} < R_{k+1} = \|u^*\|_1$ 。

令 $u^* \in K \cap \partial\Omega_{1,k}$, 对 $\forall t \in [\frac{1}{4}, \frac{3}{4}]$, 由第一步可知 $\|Tu^*\|_1 \geq \|u^*\|_1$ 。因此由引理 1 的结论 1) 可知, T 存在不动点 $u^* \in K$ 且 $R_{k+1} < \|u^*\|_1 < r_k$ 。因为 $k=1, 2, 3, \dots, m-1$, 所以 T 有 $m-1$ 个不动点 $\{u_k^*\}_{k=1}^{m-1}$ 。综合第一步和第二步可知, T 共有 $2m-1$ ($m=1, 2, \dots$) 个不动点, 即边值问题式 (5) 存在 $2m-1$ ($m=1, 2, \dots$) 个正解。

注记 1 λ 的所属范围为 $\left(\frac{1-\mu}{N_2\mu B \min_{\tau,s \in [\frac{1}{4}, \frac{3}{4}]} G(\tau,s)}, \frac{8(1-\mu)}{N_1A(3-2\mu)} \right)$, 可以利用 Φ_1, Φ_2 以及 f 满足的条件 (i) 和 (ii)

推出。

3 应用举例

例 1 考虑如下三阶边值问题

$$\begin{cases} u'''(t) + (t + \frac{1}{5})f(t, u(t), u'(t)) = 0, & t \in (0, 1), \\ u(0) = \frac{1}{6} \int_0^1 su(s) ds, & u'(0) = u'(1) = 0, \end{cases} \tag{19}$$

这里, 边值问题式 (5) 中的参数 $\lambda = 1, a(t) = t + \frac{1}{5}, \alpha(t) = \frac{1}{6}t$ 。通过计算不难得到

$$\mu = \int_0^1 \alpha(t) dt = \frac{1}{12}, N_1 = \max_{t \in [0,1]} a(t) = \frac{6}{5}, N_2 = \min_{t \in [0,1]} a(t) = \frac{1}{5}, \min_{t,s \in [\frac{1}{4}, \frac{3}{4}]} G(t,s) = \frac{3}{512},$$

则

$$\Phi_1 = \frac{1-\mu}{\lambda N_2 \mu \min_{\tau,s \in [\frac{1}{4}, \frac{3}{4}]} G(\tau,s)} \approx 9387, \Phi_2 = \frac{8(1-\mu)}{\lambda N_1(3-2\mu)} \approx 2.1569.$$

令 $m \in \mathbb{N} \cup \{+\infty\}, \gamma \in \mathbb{R}, A \in (0, \Phi_2), B \in (\Phi_1, +\infty)$, 使得 $\gamma > \max\{9387, \sqrt{B/A}\}$, 记 $R_k = \gamma^{-4k}, r_k = \gamma^{-(4k+2)}, k \in \{1, 2, 3, \dots, m-1\}$, 则有 $r_{k+1} < R_{k+1} < r_k < R_k, R_{k+1} < r_k < Br_k, Br_k < AR_k$, 以及 $[9387^{-2}r_k, \frac{1}{16}r_k] \subset [R_{k+1}, r_k]$ 。定义函数 f 为

$$f(t, u, u') = \begin{cases} R_{k+1}, & u + u' \in [0, R_{k+1}], \\ \frac{Br_k - R_{k+1}}{9387^{-2}r_k - R_{k+1}}(u + u' - R_{k+1}) + R_{k+1}, & u + u' \in [R_{k+1}, 9387^{-2}r_k], \\ Br_k, & u + u' \in [9387^{-2}r_k, r_k], \\ \frac{Br_k - AR_k}{r_k - R_k}(u + u' - R_k) + AR_k, & u + u' \in [r_k, R_k], \\ AR_k, & u + u' \in [R_k, +\infty], \end{cases}$$

则函数 f 满足定理 1 的所有条件。那么由定理 1 不难得到, 边值问题式 (15) 存在 $2m - 1$ ($m = 1, 2, \dots$) 个正解 $\{u_k\}_{k=1}^m$ 和 $\{v_k\}_{k=1}^{m-1}$, 且满足 $\gamma^{-(4k+2)} \leq \|u_k\|_1 \leq \gamma^{-4k}$, $k = 1, 2, 3, \dots, m$; $\gamma^{-4(k+1)} \leq \|v_k\|_1 \leq \gamma^{-(4k+2)}$, $k = 1, 2, 3, \dots, m - 1$ 。

参考文献:

- [1] WANG C X, SUN H R. Positive Solutions for a Class of Singular Third-Order Three-Point Nonhomogeneous Boundary Value Problem[J]. *Dynam Syst Appl*, 2010, **19** (2): 225-234.
- [2] YANG C. Existence and Uniqueness of Solutions for a Third-order Three-point Boundary Value Problem via Measure of Noncompactness[J]. *J Math*, 2022, **2022**: 1-8. DOI: 10.1155/2022/1157154.
- [3] SMIRNOV S. Existence of a Unique Solution for a Third-order Boundary Value Problem with Nonlocal Conditions of Integral Type[J]. *Nonlinear Anal Model Control*, 2021, **26**(5): 914-927. DOI: 10.15388/namc.2021.26.23932.
- [4] SMIRNOV S. Green's Function and Existence of Solutions for a Third-order Boundary Value Problem Involving Integral Condition[J]. *Lith Math J*, 2022, **62**(4): 509-518. DOI: 10.1007/s10986-022-09576-7.
- [5] 潘凤, 武晨. 带有积分型边值条件的三阶边值问题两个正解的存在性[J]. *长春师范大学学报*, 2020, **39**(1): 1-4. DOI: 10.3969/j.issn.1008-178X.2020.01.001.
PAN F, WU C. The Existence of Two Positive Solutions for Third-order Boundary Value Problems with Integral Boundary Conditions[J]. *J Chang Norm Univ Nat Sci*, 2020, **39**(1): 1-4. DOI: 10.3969/j.issn.1008-178X.2020.01.001.
- [6] SUN B, LI J F. Positive Solutions to a Third-order Boundary Value Problem with Riemann Stieltjes Integral Boundary Conditions[J]. *J Phys: Conf Ser*, 2018, **1039**: 012004. DOI: 10.1088/1742-6596/1039/1/012004.
- [7] 杨佳. 几类带积分边界条件的三阶边值问题[D]. 兰州: 兰州理工大学, 2013.
YANG J. Several Classes of Third-Order Boundary Value Problems with Integral Boundary Conditions[D]. Lanzhou: Lanzhou University of Technology, 2013.
- [8] 李海宝. 两类带积分边界条件的三阶边值问题的正解[D]. 兰州: 兰州理工大学, 2011.
LI H B. Positive Solutions for Two Classes of Third-Order BVP with Integral Boundary Conditions[D]. Lanzhou: Lanzhou University of Technology, 2011.
- [9] 穆可旺, 杨赞瑞, 杨璐. 一类三阶 m 点边值问题的三个正解[J]. *滨州学院学报*, 2022, **38**(4): 49-55. DOI: 10.13486/j.cnki.1673-2618.2022.04.008.
MU K W, YANG Y R, YANG L. Three Positive Solutions to a Kind of M -point Third-order Boundary Value Problems[J]. *J Binzhou Univ*, 2022, **38**(4): 49-55. DOI: 10.13486/j.cnki.1673-2618.2022.04.008.
- [10] LI Y K, WANG L Y. Multiple Positive Solutions of Nonlinear Third-order Boundary Value Problems with Integral Boundary Conditions on Time Scales[J]. *Adv Differ Equ*, 2015, **2015**(1): 1-15. DOI: 10.1186/s13662-015-0396-8.
- [11] 张瑞燕. 非线性三阶三点边值问题多个正解的存在性[J]. *吉林大学学报(理学版)*, 2021, **59**(5): 1050-1056. DOI: 10.13413/j.cnki.jdxblxb.2021001.
ZHANG R Y. Existence of Multiple Positive Solutions for Nonlinear Third-order Three-point Boundary Value Problems[J]. *J Jilin Univ Sci Ed*, 2021, **59**(5): 1050-1056. DOI: 10.13413/j.cnki.jdxblxb.2021001.
- [12] 张海娥. 带参数的非线性奇异三阶非局部边值问题的单调正解[J]. *数学的实践与认识*, 2012, **42**(23): 262-266. DOI: 10.3969/j.issn.1000-0984.2012.23.035.
ZHANG H E. Monotone Positive Solutions for Nonlinear Singular Third-order Nonlocal BVPS with Parameter[J]. *Math Pract Theory*, 2012, **42**(23): 262-266. DOI: 10.3969/j.issn.1000-0984.2012.23.035.