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一类分数阶脉冲微分方程边值问题解的存在性

黎文博,周文学*,张敏
(兰州交通大学 数理学院,甘肃 兰州 730070)

摘 要:本文基于对一类带p-Laplacian算子的分数阶微分方程的研究,讨论了同时具有瞬时脉冲和非瞬时脉冲时该类分数阶微分方程在Dirichlet边值条件下解的存在性问题。利用Ricceri提出的临界点理论,证明了该类边值问题至少存在三个解,且得到了该类边值问题解的多重性依赖于两个参数的结论。最后举例说明了结果的适用性。

关键词:变分法;瞬时和非瞬时脉冲;p-Laplacian算子;临界点理论
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Existence of Solutions to Boundary Value Problems of a Class of Fractional Order Impulsive Differential Equations

LI Wenbo, ZHOU Wenxue*, ZHANG Min
(*College of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou* 730070, *China*)

Abstract: In this paper, based on the study of a class of fractional-order differential equations with p-Laplacian operator, we discuss the existence of solutions for the class of fractional-order differential equations with both instantaneous and non-instantaneous pulses under Dirichlet boundary value conditions. By using the critical point theory proposed by Ricceri, it is proved that there are at least three solutions of this kind of boundary value problem, and the multiplicity of solutions of this kind of boundary value problem depends on two parameters. Finally, an example is given to illustrate the applicability of the results.

Key words: variational method; instantaneous and non-instantaneous pulses; p-Laplacian operator; critical point theory

0 引言

分数阶微分方程是近年来备受关注的一个领域。这既是由于分数阶微积分理论本身的深入发展,也是由于分数阶微积分在物理、力学、化学、工程等各个科学领域的应用[1-4]。

脉冲微分方程是一类用来描述具有跳跃点(脉冲)的不连续发展过程的微分方程,它有着十分广泛的应用背景,如控制论、医学和生物学、火箭与宇宙飞船运动以及力学等领域中都会涉及此类方程[5-8]。关于含有瞬时脉冲的分数阶微分方程解的存在性问题的研究已经取得了很大的进展。然而,许多现象不能用分数阶瞬时脉冲微分方程解的存在性问题来模拟,例如在某些演化过程的动力学中。2013年,Hernández和O'Regan首次提出了非瞬时脉冲的概念[9],它可以刻画在任意固定点处突然开始并在有限时间区间内持续的脉冲跳跃。这种类型的脉冲更多是在模拟演化过程的动力学方面具有优势,因此,涉及这种脉冲的微分方程受到了广泛的关注[10-11]。Khaliq和Rehm-

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作者简介:黎文博(1998—),男,甘肃天水人,硕士研究生,研究方向为分数阶微分方程。E-mail:wbli2022@126.com
* **通信作者**:周文学(ZHOU Wenxue),E-mail:wxzhou2006@126.com
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an 在文献[12]中利用 Lax-Milgram 定理研究了下面一类分数阶非瞬时脉冲微分方程边值问题的弱解存在性,

$$\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)=f_i(t), & t\in(s_i,t_{i+1}],\ i=0,1,\cdots,n, \\ {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(t)\right)=c_i, & t\in(t_i,s_i],\ i=1,2,\cdots,n, \\ {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(s_i^-)\right)={}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(s_i^+)\right), & i=1,2,\cdots,n, \\ {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(s_i^-)\right)=c_0, & u(0)=u(T)。 \end{cases} \tag{1}$$

文献[13]中利用变分法得到了同时具有瞬时脉冲和非瞬时脉冲的分数阶脉冲微分方程边值问题的一个经典解,

$$\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)=f_i(t,u(t)), & t\in(s_i,t_{i+1}],\ i=0,1,\cdots,n, \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_i)=I_i(u(t_i)), & i=1,2,\cdots,n, \\ {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(t)\right)={}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(t_i^+)\right), & t\in(t_i,s_i],\ i=1,2,\cdots,n, \\ {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(s_i^-)\right)={}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(s_i^+)\right), & i=1,2,\cdots,n;\ u(0)=u(T)。 \end{cases} \tag{2}$$

受上述工作的启发,本文研究了如下一类同时具有瞬时和非瞬时脉冲的带 p-Laplacian 算子的分数阶微分方程边值问题

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{1}{2}\left({}_0^cD_t^{\alpha}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)\right]+\lambda\nabla F(t,u(t))=0, & t\in(s_j,t_{j+1}],\ j=0,1,\cdots,m, \\ \Delta\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)(t_j)=\mu I_ju(t_j), & j=1,2,\cdots,m, \\ \left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)(t)= & \\ \quad\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)(t_j^+), & t\in(t_j,s_j],\ j=1,2,\cdots,m, \\ \left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)(t_j^-)= & \\ \quad\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u(t)\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u(t)\right)\right)(t_j^+), & j=1,2,\cdots,m;\ u(0)=u(T)=0 \end{cases} \tag{3}$$

解的存在性。与问题(2)相比,我们在更一般的 $p>1$ 的情况下寻找问题(3)的解,并将线性微分算子 ${}_0^cD_t^{\alpha}{}_0D_t^{\alpha}$ 和 ${}_t^cD_T^{\alpha}{}_tD_T^{\alpha}$ 推广到非线性微分算子 ${}_0^cD_t^{\alpha}\Phi_p\left({}_0D_t^{\alpha}\right)$ 和 ${}_t^cD_T^{\alpha}\Phi_p\left({}_tD_T^{\alpha}\right)$,这意味着我们的问题(3)包含了问题(2)在 $p=2$ 和 $\mu=\lambda=1$ 时的特例。其中,$1<p<\infty$,λ 和 μ 是两个正实参数,$\Phi_p(s)=|s|^{p-2}s(s\neq 0)$, $\Phi_p(0)=0$,${}_0D_t^{\alpha}$ 和 ${}_tD_T^{\alpha}$ 分别是左、右 Riemann-Liouville 分数阶导数,${}_0^cD_t^{\alpha}$ 和 ${}_t^cD_T^{\alpha}$ 分别是左、右 Caputo 分数阶导数,$\alpha\in\left(\dfrac{1}{p},1\right]$。$F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ 是一个连续函数,$\nabla F(t,u(t))$ 是 F 在 $u(t)$ 处的梯度。对于任意的 $j=1,2,\cdots,m$,都有 $0=s_0<t_1<s_1<\cdots<s_m<t_{m+1}=T$,瞬时脉冲 $I_j\in C(\mathbb{R},\mathbb{R})$ 在点 t_j 处开始突变,非瞬时脉冲在有限区间 $(t_j,s_j]$ 内持续,且有

$$\begin{aligned} {}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)(s_j^{\pm})&=\lim_{t\to s_j^{\pm}}\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)(t)\right), \\ {}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)(t_j^{\pm})&=\lim_{t\to t_j^{\pm}}\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)(t)\right), \\ \Delta\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)\right)(t_j)&=\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)\right)(t_j^+)- \\ &\quad\left({}_0^cD_t^{\alpha-1}\Phi_p\left({}_0D_t^{\alpha}u\right)-{}_t^cD_T^{\alpha-1}\Phi_p\left({}_tD_T^{\alpha}u\right)\right)(t_j^-)。 \end{aligned}$$

1 预备知识

设 X 是一个实 Banach 空间。用 Γ_X 表示所有满足如下性质的泛函 $\varphi:X\to\mathbb{R}$ 的类:如果序列

$\{x_n\}_{n\in N}$弱收敛于X且$\liminf\limits_{n\to\infty}\varphi(x_n)\leqslant\varphi(x)$,那么,存在一个$\{x_n\}_{n\in N}$的子序列强收敛于$x$。

定理 1[14] 设X是一个可分自反的实Banach空间,$\varphi: X\to\mathbb{R}$是一个强制的、弱下半连续的C^1函数,在X的每个有界子集上有界,与Γ_X有关,其导数在X^*上有一个连续的逆。$\omega: X\to\mathbb{R}$是一个具有紧导数的C^1函数。假设φ存在一个严格的局部最小值u_0,使得$\varphi(u_0)=\omega(u_0)=0$。最后,令

$$\rho_1=\max\left\{0,\ \lim_{\|u\|\to+\infty}\sup\frac{\omega(u)}{\varphi(u)},\ \lim_{u\to u_0}\sup\frac{\omega(u)}{\varphi(u)}\right\},\ \rho_2=\sup_{u\in\varphi^{-1}(0,+\infty)}\frac{\omega(u)}{\varphi(u)},\tag{4}$$

假设$\rho_1<\rho_2$,则对于每个紧区间$[\theta_1,\theta_2]\in\left(\frac{1}{\rho_2},\ \frac{1}{\rho_1}\right)$(规定$\frac{1}{0}=+\infty,\ \frac{1}{+\infty}=0$),存在$R>0$满足以下性质:对于每个$\lambda\in[\theta_1,\theta_2]$以及任意具有紧导数的$C^1$函数$\varphi: X\to\mathbb{R}$,存在$\xi>0$,使得对于每一个$\mu\in[0,\xi]$,方程$\varphi'(u)-\mu\varphi'(u)-\lambda\omega'(u)=0$在$X$中至少有三个范数小于$R$的解。

定义 1[15-16] 设$n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $t\in[0,T]$。分别用${}_0D_t^\alpha\big(u(t)\big)$和${}_tD_T^\alpha\big(u(t)\big)$表示$u(t)$的左右Riemann-Liouville分数阶导数,且定义如下

$${}_0D_t^\alpha u(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,{}_0D_t^{\alpha-n}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_0^t(t-\tau)^{n-\alpha-1}u(\tau)\mathrm{d}\tau\right),$$

$${}_tD_T^\alpha u(t)=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,{}_tD_T^{\alpha-n}u(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_t^T(\tau-t)^{n-\alpha-1}u(\tau)\mathrm{d}\tau\right)。$$

定义 2[15-16] 设$n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $u(t)\in AC^n\big([0,T],\mathbb{R}\big)$。分别用${}_0^cD_t^\alpha\big(u(t)\big)$和${}_t^cD_T^\alpha\big(u(t)\big)$表示$u(t)$的左右Caputo分数阶导数,且定义如下

$${}_0^cD_t^\alpha u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\int_0^t(t-\tau)^{n-\alpha-1}u^n(\tau)\mathrm{d}\tau\right),$$

$${}_t^cD_T^\alpha u(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\left(\int_t^T(\tau-t)^{n-\alpha-1}u^n(\tau)\mathrm{d}\tau\right),$$

且有

$${}_0^cD_t^\alpha u(t)={}_0D_t^\alpha u(t)-\sum_{i=0}^{n-1}\frac{u^i(0)}{\Gamma(i-\alpha+1)}(t)^{i-\alpha},\ {}_t^cD_T^\alpha u(t)={}_tD_T^\alpha u(t)-\sum_{i=0}^{n-1}\frac{u^i(T)}{\Gamma(i-\alpha+1)}(T-t)^{i-\alpha}。\tag{5}$$

现在,我们定义一个赋范空间$\left(X_\alpha^p,\|\cdot\|_\alpha^p\right)$

$$X_\alpha^p=\left\{u(t)\in L^p\big([0,T],\mathbb{R}\big)\,|\,{}_0D_t^\alpha u(t)\in L^p\big([0,T],\mathbb{R}\big),\ u(0)=u(T)=0\right\},\tag{6}$$

且对于任意的$u(t)\in X_\alpha^p$有范数

$$\|u\|_\alpha^p=\left(\int_0^T|u(t)|^p\mathrm{d}t+\int_0^T|{}_0D_t^\alpha u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}。\tag{7}$$

如文献[16]所示,X_α^p是一个自反的Banach空间。特别是,鉴于式(5),对于任意的$u(t)\in X_\alpha^p$我们可以推断出

$${}_0D_t^\alpha u(t)={}_0^cD_t^\alpha u(t),\ {}_tD_T^\alpha u(t)={}_t^cD_T^\alpha u(t)。\tag{8}$$

引理 1(文献[17]命题3.3) 设$0<\alpha\leqslant1$, $1\leqslant p<\infty$。对于所有的$u\in X_\alpha^p$,有

$$\|u\|_{L^p}\leqslant\frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_{L^p}。$$

此外,如果$\alpha>\frac{1}{\mathrm{p}}$和$\frac{1}{p}+\frac{1}{q}=1$,那么

$$\|u\|_\infty\leqslant\frac{T^{\alpha-1/P}}{\Gamma(\alpha)\big((\alpha-1)q+1\big)^{1/q}}\|{}_0D_t^\alpha u\|_{L^p}。$$

其中$\|u\|_{L^p}=\left(\int_0^T\left|u(t)\right|^p\mathrm{d}t\right)^{1/p}$和$\|u\|_\infty=\max\limits_{t\in[0,T]}\left|u(t)\right|$。令$U=\dfrac{T^\alpha}{\Gamma(\alpha+1)}$,$B=\dfrac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{\frac{1}{q}}}$。

注1[17] 根据引理1,范数(7)式与$\|u\|_\alpha^p=\|{}_0D_t^\alpha u\|_{L^p}$相同。

为了在进一步的讨论中简化符号,在本文的其余部分中,用$(X,\|\cdot\|)$代替$\left(X_\alpha^p,\|\cdot\|_\alpha^p\right)$来表示赋范空间。

引理2(文献[17]命题3.4) 设$\alpha\in\left(\dfrac{1}{p},\ 1\right]$,$1<p<\infty$。假设任意序列$\{u_n\}_{n\in N}$在$X$中弱收敛于$u$。那么在$C([0,T],\mathbb{R})$中当$n\to\infty$时有$u_k\to u$,即$n\to\infty$时,$\|u_n-u\|_\infty\to 0$。

引理3[16] 设$\alpha>0$,$p\geqslant 1$,$q\geqslant 1$,$\dfrac{1}{p}+\dfrac{1}{q}\leqslant 1+\alpha$(当$p$,$q\neq 1$时$\dfrac{1}{p}+\dfrac{1}{q}=1+\alpha$)。如果$u\in L^p(a,b)$,$v\in L^q(a,b)$,则有$\int_a^b\left({}_aD_t^\alpha u(t)\right)v(t)\mathrm{d}t=\int_a^b u(t)\left({}_tD_b^\alpha v(t)\right)\mathrm{d}t$。

引理4 假设$u(t)\in X_\alpha^p$是边值问题(Boundary Value Problems, BVP)(3)式的一个弱解,如果满足下面定义的关系,即对于任意的$v(t)\in X$有

$$\frac{1}{2}\left[\Phi_p\left({}_0D_t^\alpha u(t)\right){}_tD_T^\alpha v(t)+\Phi_p\left({}_tD_T^\alpha u(t)\right){}_0D_t^\alpha v(t)\right]\mathrm{d}t-\mu\sum_{j=1}^m I_j\left(u(t_j)\right)v(t_j)=$$
$$\lambda\sum_{j=0}^m\int_{s_j}^{t_j+1}\nabla F_j\left(t,\ u(t)\right)v(t)\mathrm{d}t。$$

证明 由引理3,${}_tD_T^\alpha u(t)=-\dfrac{\mathrm{d}}{\mathrm{d}t}{}_tD_T^{\alpha-1}u(t)$和(8)式,我们得到对于任意的$u,v\in X$,有

$$\begin{aligned}
&\frac{1}{2}\int_0^T\left[\left(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0D_t^\alpha u(t)\right),\ {}_tD_T^\alpha v(t)\right)+\left(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_tD_T^\alpha u(t)\right),\ {}_0D_t^\alpha v(t)\right)\right]\mathrm{d}t=\\
&\frac{1}{2}\int_0^T\left[\left(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right),\ {}_t^cD_T^\alpha v(t)\right)+\left(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right),\ {}_0^cD_t^\alpha v(t)\right)\right]\mathrm{d}t=\\
&\frac{1}{2}\int_0^T\left[{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v'(t)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v'(t)\right]\mathrm{d}t=\\
&\frac{1}{2}\int_0^T\left[\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_t^cD_T^\alpha v(t)+\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_t^cD_T^\alpha u(t){}_0^cD_t^\alpha v(t)\right]\mathrm{d}t=\\
&\frac{1}{2}\sum_{j=0}^m\int_{s_j}^{t_{j+1}}\left[{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v'(t)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v'(t)\right]\mathrm{d}t+\\
&\frac{1}{2}\sum_{j=1}^m\int_{t_j}^{s_j}\left[{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v'(t)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v'(t)\right]\mathrm{d}t=\\
&\frac{1}{2}\sum_{j=0}^m{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v(t)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v(t)\Big|_{s_j^+}^{t_{j+1}^-}+\\
&\frac{1}{2}\sum_{j=0}^m\int_{s_j}^{t_{j+1}}\left[{}_0D_t^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v(t)-{}_tD_T^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v(t)\right]\mathrm{d}t+\\
&\frac{1}{2}\sum_{j=1}^m{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v(t)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v(t)\Big|_{t_j^+}^{s_j^-}+\\
&\frac{1}{2}\sum_{j=1}^m\int_{t_j}^{s_j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left({}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)\right)v(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left({}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)\right)v(t)\right]\mathrm{d}t,
\end{aligned}\tag{9}$$

然后,问题(3)上的给定条件,将(9)式变化

$$\frac{1}{2}\int_0^T\left[\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\,{}_t^cD_T^\alpha v(t)+\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_t^cD_T^\alpha u(t)\,{}_0^cD_t^\alpha v(t)\right]\mathrm{d}t=$$

$$\frac{1}{2}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\left[{}_0D_t^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v(t)-{}_tD_T^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v(t)\right]\mathrm{d}t-$$

$$\sum_{j=1}^{m}\Delta\left[\left({}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(t_j)\right|^{p-2}\left({}_0^cD_t^\alpha u(t_j)\right)\right)v(t_j)-\left({}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(t_j)\right|^{p-2}\left({}_t^cD_T^\alpha u(t_j)\right)\right)v(t_j)\right]+$$

$$\frac{1}{2}\sum_{j=1}^{m}{}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(s_j^-)\right|^{p-2}\left({}_0^cD_t^\alpha u(s_j^-)\right)v(s_j)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(s_j^-)\right|^{p-2}\left({}_t^cD_T^\alpha u(s_j^-)\right)v(s_j)-$$

$$\left({}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(s_j^+)\right|^{p-2}\left({}_0^cD_t^\alpha u(s_j^+)\right)v(s_j)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(s_j^+)\right|^{p-2}\left({}_t^cD_T^\alpha u(s_j^+)\right)v(s_j)\right)+$$

$${}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(T)\right|^{p-2}\left({}_0^cD_t^\alpha u(T)\right)v(T)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(T)\right|^{p-2}\left({}_t^cD_T^\alpha u(T)\right)v(T)-$$

$$\left({}_0D_t^{\alpha-1}\left|{}_0^cD_t^\alpha u(0)\right|^{p-2}\left({}_0^cD_t^\alpha u(0)\right)v(0)-{}_tD_T^{\alpha-1}\left|{}_0^cD_t^\alpha u(0)\right|^{p-2}\left({}_t^cD_T^\alpha u(0)\right)v(0)\right)=$$

$$\frac{1}{2}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\left[{}_0D_t^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_0^cD_t^\alpha u(t)\right)v(t)-{}_tD_T^\alpha\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}\left({}_t^cD_T^\alpha u(t)\right)v(t)\right]\mathrm{d}t-\mu\sum_{j=1}^{m}I_j\left(u(t_j)\right)v(t_j)。\tag{10}$$

将 $v(t)$ 代入(3)式的方程,先从 s_j 到 t_{j+1} 两边积分,再从 $j=0$ 到 $j=m$ 相加,然后从(10)式中推导得

$$\frac{1}{2}\int_0^T\Phi_p\left({}_0D_t^\alpha u(t)\right){}_tD_T^\alpha v(t)+\Phi_p\left({}_tD_T^\alpha u(t)\right){}_0D_t^\alpha v(t)\mathrm{d}t-\mu\sum_{j=1}^{m}I_j\left(u(t_j)\right)v(t_j)=$$

$$\lambda\sum_{j=0}^{m}\int_{s_j}^{t_j+1}\nabla F_j\left(t,\ u(t)\right)v(t)\mathrm{d}t。$$

证毕。

接下来,我们定义一些函数 $\varphi,\ \phi,\ \omega: X\to\mathbb{R}$ 如下

$$\varphi(u)=-\frac{1}{p}\int_0^T\left({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\right)\mathrm{d}t,\tag{11}$$

$$\phi(u)=-\sum_{j=1}^{m}\int_0^{(u(t_j))}I_j(s)\mathrm{d}s,\ \omega(u)=\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j\left(t,\ u(t)\right)\mathrm{d}t。\tag{12}$$

通过标准论证表明,$\varphi\in C^1(X,\mathbb{R})$ 和 ϕ 和 ω 具有连续的 Gâteaux 导数。对于任何 $v(t)\in X$,它们的 Gâteaux 导数如下

$$\varphi'(u)(v)=-\frac{1}{p}\left[\Phi_p\left({}_0D_t^\alpha u(t)\right){}_tD_T^\alpha v(t)+\Phi_p\left({}_tD_T^\alpha u(t)\right){}_0D_t^\alpha v(t)\right]\mathrm{d}t,\tag{13}$$

$$\phi'(u)(v)=-\sum_{j=1}^{m}I_j\left(u(t_j)\right)v(t_j),\ \omega'(u)(v)=\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\nabla F_j\left(t,\ u(t)\right)v(t)\mathrm{d}t。\tag{14}$$

显然,我们寻找问题(3)的弱解的一种方法是求函数 $\varphi-\mu\phi-\lambda\omega$ 的临界点。此外,与文献[13]中的引理3类似,BVP(3)的弱解也是一个经典解。

2 主要结果

在本节中,我们利用定理1,建立了BVP(3)解的多重性。

定理2 假设存在非负常数 $k_j,j=0,1,\cdots,m$ 和一个函数 $\eta(t)\in X\backslash\{0\}$,使得

$$\max\left\{\limsup_{|x|\to0}\frac{F_j(t,x)}{|x|^p},\ \limsup_{|x|\to\infty}\frac{F_j(t,x)}{|x|^p}\right\}\leqslant k_j,\ j=0,1,\cdots,m,\tag{15}$$

和

$$\frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j\big(t,\eta(t)\big)\mathrm{d}t}{\|\eta\|^p}>TB^p\max_{0\leqslant j\leqslant m}\{k_j\}。\tag{16}$$

则对于每个紧区间$[\theta_1,\theta_2]\subset\left(\frac{1}{\rho_2},\frac{1}{\rho_1}\right)$($\rho_1,\rho_2$的定义见定理1),存在$R>0$,使得对于每一个$\lambda\in[\theta_1,\theta_2]$,存在$\xi>0$,对于每一个$\mu\in[0,\xi]$,BVP(3)在$X$中至少有三个解$x_i$,其中$\|x_i\|<R$,$i=1,2,3$。

证明 首先,我们考虑泛函φ。假设M是X的一个子集的界,即$\|x\|\leqslant M$在X的一个子集上。那么,由(11)式,我们有$|\varphi(x)|\leqslant\frac{1}{p}\|x\|^p\leqslant\frac{M^p}{p}$,这表明$\varphi$是$X$的每个有界子集上的有界泛函。进一步,由(13)式,我们得到

$$\begin{aligned}&\big(\varphi'(x)-\varphi'(y)\big)(x-y)=\\&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_0D_t^\alpha x(t)\big)\,{}_tD_T^\alpha\big(x(t)-y(t)\big)+\Phi_p\big({}_tD_T^\alpha x(t)\big)\,{}_0D_t^\alpha\big(x(t)-y(t)\big)\Big]\mathrm{d}t=\\&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_0D_t^\alpha y(t)\big)\,{}_tD_T^\alpha\big(x(t)-y(t)\big)+\Phi_p\big({}_tD_T^\alpha y(t)\big)\,{}_0D_t^\alpha\big(x(t)-y(t)\big)\Big]\mathrm{d}t=\\&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_0D_t^\alpha x(t)\big)-\Phi_p\big({}_0D_t^\alpha y(t)\big)\Big]\,{}_tD_T^\alpha\big(x(t)-y(t)\big)\mathrm{d}t=\\&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_tD_T^\alpha x(t)\big)-\Phi_p\big({}_tD_T^\alpha y(t)\big)\Big]\,{}_0D_t^\alpha\big(x(t)-y(t)\big)\mathrm{d}t。\end{aligned}\tag{17}$$

考虑文献[18]中引入的不等式,即

$$\big(|r_1|^{p-2}r_1-|r_2|^{p-2}r_2\big)(r_1-r_2)\geqslant\begin{cases}|r_1-r_2|^p, & p\geqslant2,\\ \dfrac{|r_1-r_2|^2}{\big(|r_1|+|r_2|\big)^{2-p}}, & 1<p<2,\end{cases}\tag{18}$$

并且利用Hölder不等式,当$1<p<2$时,我们得到

$$\begin{aligned}&\frac{1}{2}\int_0^T\big|{}_0D_t^\alpha x(t)-{}_0D_t^\alpha y(t)\big|^p\mathrm{d}t\leqslant\\&\frac{1}{2}\left(\int_0^T\Big(\big|{}_0D_t^\alpha x(t)\big|+\big|{}_0D_t^\alpha y(t)\big|\Big)^p\mathrm{d}t\right)^{\frac{2-p}{2}}\left(\int_0^T\frac{\big|{}_0D_t^\alpha x(t)-{}_0D_t^\alpha y(t)\big|^2}{\Big(\big|{}_0D_t^\alpha x(t)\big|+\big|{}_0D_t^\alpha y(t)\big|\Big)^{2-p}}\mathrm{d}t\right)^{\frac{p}{2}}\leqslant\\&\frac{K}{2}\big(\|x\|^p+\|y\|^p\big)^{\frac{2-p}{2}}\left(\int_0^T\Big[\Phi_p\big({}_0D_t^\alpha x(t)\big)-\Phi_p\big({}_0D_t^\alpha y(t)\big)\Big]\,{}_tD_T^\alpha\big(x(t)-y(t)\big)\mathrm{d}t\right)^{\frac{p}{2}}。\end{aligned}$$

同理可得

$$\frac{1}{2}\int_0^T\big|{}_tD_T^\alpha x(t)-{}_tD_T^\alpha y(t)\big|^p\mathrm{d}t\leqslant\frac{K}{2}\big(\|x\|^p+\|y\|^p\big)^{\frac{2-p}{2}}\left(\int_0^T\Big[\Phi_p\big({}_tD_T^\alpha x(t)\big)-\Phi_p\big({}_tD_T^\alpha y(t)\big)\Big]\,{}_0D_t^\alpha\big(x(t)-y(t)\big)\mathrm{d}t\right)^{\frac{p}{2}},$$

其中$K=2^{\frac{(p-1)(2-p)}{2}}$,因此对于任意的$x,\ y\in X,\quad x\neq y$,有

$$\begin{aligned}&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_0D_t^\alpha x(t)\big)-\Phi_p\big({}_0D_t^\alpha y(t)\big)\Big]\,{}_tD_T^\alpha\big(x(t)-y(t)\big)\mathrm{d}t+\\&\frac{1}{2}\int_0^T\Big[\Phi_p\big({}_tD_T^\alpha x(t)\big)-\Phi_p\big({}_tD_T^\alpha y(t)\big)\Big]\,{}_0D_t^\alpha\big(x(t)-y(t)\big)\mathrm{d}t\geqslant K^{-\frac{2}{P}}\big(\|x\|^p+\|y\|^p\big)^{\frac{P-2}{P}}\|x-y\|^2>0,\end{aligned}$$

当$p\geqslant 2$,对于任意的$x, y\in X$, $x\neq y$时,结合(17)式和(18)式,我们有

$$\left(\varphi'(x)-\varphi'(y)\right)(x-y)\geqslant\frac{1}{2}\int_0^T\left|{}_0D_t^\alpha x(t)-{}_0D_t^\alpha y(t)\right|^p\mathrm{d}t+\frac{1}{2}\int_0^T\left|{}_tD_T^\alpha x(t)-{}_tD_T^\alpha y(t)\right|^p\mathrm{d}t>0。$$

因此,对任意的$p>1$,在上述讨论的基础上,我们得到$\left(\varphi'(x)-\varphi'(y)\right)(x-y)>0$,即$\varphi'$是严格单调算子。由文献[19]中的定理26可知,在X^*上存在φ'的逆,且该逆是连续的。由(11)式,我们有$\varphi(u)=-\frac{1}{p}\int_0^T\left({}_0D_t^\alpha u(t),\ {}_tD_T^\alpha u(t)\right)\mathrm{d}t\geqslant\frac{1}{p\left|\cos(\pi\alpha)\right|}\|u\|_\alpha^p$,这表明当$\|u\|\to\infty$时,$\varphi(u)\to\infty$。因此,$\varphi$是强制的。类似于文献[11],我们可以得到$\varphi$是序列弱下半连续的且属于$\Gamma_X$。

另一方面,由于F_j和I_j的连续性,很容易证明φ和ω的导数是紧的。另外,存在泛函φ的严格局部极小值0,使得$\varphi(0)=\omega(0)=0$。

根据式(15)知,存在$\varepsilon_1, \varepsilon_2>0$,使得对于任意的$t\in[0,T]$, $|x|\in(0,\varepsilon_1)\cup(\varepsilon_2,\infty)$,有

$$F_j(t,x(t))\leqslant k_j\left|x(t)\right|^p, \tag{19}$$

由于F_j具有连续性,故存在$r>0$, $\sigma>p$,使得对于任意的$t\in[0,T]$,$x\in\mathbb{R}$,有

$$F_j(t,x(t))\leqslant k_j\left|x(t)\right|^p+r\left|x(t)\right|^\sigma。\tag{20}$$

记$k^*=\max\limits_{0\leqslant j\leqslant m}\{k_j\}$,则基于(20)式和引理1,我们有

$$\omega(x)=\sum_{j=0}^m\int_{s_j}^{t_{j+1}}F_j\left(t,\ x(t)\right)\mathrm{d}t\leqslant k^*TB^p\|x\|^p+rTB^\sigma\|x\|^\sigma,$$

因此对于任意的$\sigma>p$,有

$$\limsup_{x\to0}\frac{\omega(x)}{\varphi(x)}\leqslant\limsup_{x\to0}\frac{k^*TB^p\|x\|^p+rTB^\sigma\|x\|^\sigma}{\dfrac{1}{p\left|\cos(\pi\alpha)\right|}\|u\|_\alpha^p}=p\left|\cos(\pi\alpha)\right|k^*TB^p\leqslant pk^*TB^p, \tag{21}$$

此外,由(19)式可得

$$\begin{aligned}&\limsup_{x\to0}\frac{\omega(x)}{\varphi(x)}\leqslant\\&\limsup_{x\to\infty}\frac{\sum_{j=0}^m\int_{|x|\leqslant\varepsilon_2}F_j\left(t,x(t)\right)\mathrm{d}t}{\dfrac{1}{p\left|\cos(\pi\alpha)\right|}\|u\|_\alpha^p}+\limsup_{x\to\infty}\frac{\sum_{j=0}^m\int_{|x|>\varepsilon_2}F_j\left(t,x(t)\right)\mathrm{d}t}{\dfrac{1}{p\left|\cos(\pi\alpha)\right|}\|u\|_\alpha^p}\leqslant\\&\limsup_{x\to\infty}\frac{\sum_{j=0}^m\int_{|x|>\varepsilon_2}F_j\left(t,x(t)\right)\mathrm{d}t}{\dfrac{1}{p\left|\cos(\pi\alpha)\right|}\|u\|_\alpha^p}\leqslant pk^*TB^P。\end{aligned}\tag{22}$$

因此,结合(21)式和(22)式,我们有$\rho_1=\max\left\{0,\ \limsup\limits_{\|u\|\to+\infty}\dfrac{\omega(u)}{\varphi(u)},\ \limsup\limits_{u\to0}\dfrac{\omega(u)}{\varphi(u)}\right\}\leqslant pk^*TB^p$。

此外,利用(16)式,我们有

$$\rho_2=\sup_{x\in\varphi^{-1}((0,+\infty))}\frac{\omega(x)}{\varphi(x)}=\sup_{x\in X\setminus\{0\}}\frac{\omega(x)}{\varphi(x)}\geqslant\frac{\sum_{j=0}^m\int_{s_j}^{t_{j+1}}F_j\left(t,\eta(t)\right)\mathrm{d}t}{\dfrac{1}{p}\int_0^T\left({}_0D_t^\alpha\eta(t),\ {}_tD_T^\alpha\eta(t)\right)\mathrm{d}t}>pk^*TB^P\geqslant\rho_1。$$

由定理1可知,定理2成立。

例 证明下列边值问题解的存在性

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{1}{2}\left({}_{0}^{c}D_{t}^{\frac{3}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{\frac{3}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)\right]+\lambda\nabla F(t,u(t))=0, & t\in\left(s_{j},t_{j+1}\right],\ j=0,1,\cdots,m,\\ \Delta\left({}_{0}^{c}D_{t}^{-\frac{1}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{-\frac{1}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)(t_{j})=\mu I_{j}u(t_{j}), & j=1,2,\cdots,m,\\ \left({}_{0}^{c}D_{t}^{-\frac{1}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{-\frac{1}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)(t)= & \\ \quad\left({}_{0}^{c}D_{t}^{-\frac{1}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{-\frac{1}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)(t_{j}^{+}), & t\in\left(t_{j},s_{j}\right],\ j=1,2,\cdots,m,\\ \left({}_{0}^{c}D_{t}^{-\frac{1}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{-\frac{1}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)(t_{j}^{-})= & \\ \quad\left({}_{0}^{c}D_{t}^{-\frac{1}{4}}\Phi_{p}\left({}_{0}D_{t}^{\frac{3}{4}}u(t)\right)-{}_{t}^{c}D_{T}^{-\frac{1}{4}}\Phi_{p}\left({}_{t}D_{T}^{\frac{3}{4}}u(t)\right)\right)(t_{j}^{+}), & j=1,2,\cdots,m;\ u(0)=u(T)=0,\end{cases}\tag{23}$$

其中 $0=s_0<t_1=\dfrac{1}{3}<s_1=\dfrac{2}{3}<t_2=1$。

证明 取 $F_j(t,u(t))=\mathrm{e}^{-|u(t)|}u^4(t)$ 且有

$$u(t)=\begin{cases}3t, & t\in\left[0,\dfrac{1}{3}\right),\\ 1, & t\in\left[\dfrac{1}{3},\dfrac{2}{3}\right],\\ 3(1-t), & t\in\left(\dfrac{2}{3},1\right].\end{cases}$$

显然,F_j 是关于 u 的 C^1 泛函且有 $F_j(t,0)=0$,满足 $\limsup\limits_{|u(t)|\to 0}\dfrac{F_j(t,u(t))}{|u(t)|^3}=\limsup\limits_{|u(t)|\to\infty}\dfrac{F_j(t,u(t))}{|u(t)|^3}=0$。通过直接计算,我们得到

$${}_{t}^{c}D_{T}^{\frac{3}{4}}u(t)={}_{0}^{c}D_{t}^{\frac{3}{4}}u(t)=\frac{1}{\Gamma\left(\frac{1}{4}\right)}\begin{cases}12t^{\frac{3}{4}}, & t\in\left[0,\dfrac{1}{3}\right),\\ 4\sqrt[4]{27}, & t\in\left[\dfrac{1}{3},\dfrac{2}{3}\right],\\ 4\sqrt[4]{27}-12\left(t-\dfrac{2}{3}\right)^{\frac{1}{4}}, & t\in\left(\dfrac{2}{3},1\right],\end{cases}$$

且 $\|u(t)\|\approx 8.483,\quad B^3=1.391\,2$。选取 $k^*=10^{-5}$,则有

$$\frac{\int_0^{\frac{1}{3}}F_j(t,u(t))\mathrm{d}t+\int_{\frac{1}{3}}^{\frac{2}{3}}F_j(t,u(t))\mathrm{d}t+\int_{\frac{2}{3}}^{1}F_j(t,u(t))\mathrm{d}t}{\|u(t)\|^3}\approx 2.8\times 10^{-4}>3k^*TB^3\approx 4.12\times 10^{-5}。$$

因此,利用定理2,我们得到:对于每一个紧区间 $[\theta_1,\theta_2]\subset(3.6\times 10^3,2.4\times 10^4)$,存在 $R>0$,且满足下列性质:对于每一个 $\lambda\in[\theta_1,\theta_2]$,存在 $\xi>0$,使得对于每一个 $\mu\in[0,\xi]$,问题(23)至少有三个范数小于 R 的解。

3 结论

在本文研究中,我们主要利用变分法建立问题(3)解的存在性。首先,从问题(2)出发,我们将

线性分数阶微分方程推广到区间$\left(s_j, t_{j+1}\right]$上的非线性分数阶微分方程。其次,我们给出问题(3)的弱解的定义,且使其解的多重性依赖于两个参数λ和μ。最后,我们得到当$p=2$和$\mu=\lambda=1$时,问题(3)退化为问题(2)。因此,我们补充和推广了现有文献[13]的结果。

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