

# 一类椭圆问题的全局Hölder估计

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**摘要:** 文章主要考虑一类椭圆方程 Dirichlet 边值问题的弱解的全局 Hölder 估计。文章首先证明了全局高阶可积性, 再给出证明主要结论的几个引理, 通过建立适合的比较估计, 利用 Young 不等式、Hölder 不等式等技巧, 以及覆盖迭代讨论, 获得了一类椭圆方程 Dirichlet 边值问题的弱解的全局 Hölder 估计。

**关键词:** 边值问题; 弱解; Hölder 估计

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## Global Hölder Estimates for a Class of Elliptic Problems

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**Abstract:** The global Hölder estimate of weak solutions for a class of Dirichlet boundary value problems of elliptic equations was considered. This article first proves the global higher order integrability, and then gives several lemmas that prove the main conclusions. By establishing suitable comparative estimates and using processing tools such as the Young inequality and the Hölder inequality, the global Hölder estimate of weak solutions for a class of Dirichlet boundary value problems of elliptic equations was obtained through covering iterative discussions.

**Key words:** boundary value problem; weak solution; Hölder estimate

### 0 引言

具有非标准增长条件的偏微分方程和变分问题源自弹性力学、电流变流体动力学<sup>[1-3]</sup>和图像处理<sup>[4-5]</sup>, 在近年来引起了许多专家学者的关注。Acerbi 和 Mingione<sup>[1]</sup>研究了具有  $p(x)$  增长条件的积分泛函

$$\mathcal{F}(u, \Omega) := \int_{\Omega} f(x, \nabla u) dx, \quad |z|^{\rho(x)} \leq f(x, z) \leq L(1 + |z|^{\rho(x)})$$

的极小化的  $C^{1,\alpha}$  正则性, 其中  $p(x)$  满足 Hölder 连续的条件。它的一个特例就是 Zhikov-Kozlov-Oleinik 在研究均匀化问题<sup>[6]</sup>时首次引入的一种积分泛函  $\mathcal{F}(u, \Omega) := \int_{\Omega} (\mu^2 + D(u)^2)^{p(x)/2} dx$ 。它为各种数学物理问题提供变分模型。Rajagopal 和 Růžička<sup>[2-3]</sup>详细阐述了一种电流变液的模型, 其稳态情

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况下的模型为

$$-\operatorname{div} \mathbf{S}(x, \varepsilon(v)) = g(x, v, Dv), \quad \operatorname{div} v = 0,$$

其中  $v$  表示流体速度,  $\varepsilon(v)$  是梯度  $Dv$  的对称部分, 附加应力张量  $\mathbf{S}$  满足具有  $p(x)$  增长的单调性条件。此外, 对于粘度受温度影响的流体<sup>[7]</sup>, 也会出现这种类型的模型。热敏电阻问题的微分系统<sup>[7-9]</sup> 包含的方程  $-\operatorname{div}(p(x)|Du|^{\rho(x)-2}Du) = 0$  的解在  $u = 0$  时对应于  $\mathcal{F}$  的极小值。关于变指数 Lebesgue 空间和 Sobolev 空间的一般理论及其相关应用, 可参考文献 [10-13]。

设  $\Omega$  为  $\mathbf{R}^n$  中具有  $C^{1,\gamma}$  边界的有界区域,  $\gamma \geq 1, n \geq 2$ 。本文考虑具有可变指数的  $p(x)$ -Laplace 方程

$$\operatorname{div} A(x, \nabla u) = \operatorname{div}(|f|^{\rho(x)-2}f), \quad x \in \Omega, \tag{1}$$

其 Dirichlet 边界条件为

$$u = u_0, \quad x \in \partial\Omega, \tag{2}$$

这里可变指数  $p(x)$  满足

$$1 < \gamma_1 = \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) = \gamma_2 < \infty, \tag{3}$$

并在  $\bar{\Omega}$  上 Hölder 连续,

$$p(x) \in C^{0,\alpha_1}(\bar{\Omega}), \tag{4}$$

也就是说, 存在正常数  $L$  和指数  $\alpha_1 \in (0, 1)$ , 使得  $\forall x_1, x_2 \in \bar{\Omega}$ , 有  $|p(x_1) - p(x_2)| \leq L|x_1 - x_2|^{\alpha_1}$ 。

$f = (f^1, \dots, f^n)$  是给定的向量, 满足

$$f^i(x) \in C^{0,\alpha_2}(\bar{\Omega}), \quad 1 \leq i \leq n, \tag{5}$$

其中常数  $\alpha_2 \in (0, 1)$ 。另外, 算子  $A = A(x, \xi): \bar{\Omega} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ , 对所有  $\xi, \eta \in \mathbf{R}^n, x, y \in \bar{\Omega}$ , 满足如下结构性条件:

$$[A(x, \xi) - A(x, \eta)] \cdot (\xi - \eta) \geq \begin{cases} C_1 |\xi - \eta|^{\rho(x)}, & \rho(x) \geq 2, \\ C_1 (|\xi|^2 + |\eta|^2)^{\frac{\rho(x)-2}{2}} |\xi - \eta|^2, & 1 < \rho(x) < 2, \end{cases} \tag{6}$$

$$|A(x, \xi)| \leq C_2 (1 + |\xi|^2)^{\frac{\rho(x)-1}{2}}, \tag{7}$$

$$A(x, \xi) \cdot \xi \geq C_3 (\mu^2 + |\xi|^2)^{\frac{\rho(x)}{2}} - C_4, \tag{8}$$

$$|A(x, \xi) - A(y, \xi)| \leq C_5 |x - y|^{\alpha_3} \log(\mu^2 + |\xi|^2) (\mu^2 + |\xi|^2)^{\frac{\rho(x)-1}{2}}, \tag{9}$$

其中  $\mu, \alpha_3$  为正常数,  $C_i (i = 1, 2, 3, 4, 5)$  为正常数。

注 特别地, 当  $A(x, \xi) = |\xi|^{\rho(x)-2}\xi$  且  $f = 0$ , (1) 即为  $p(x)$ -Laplace 方程  $\operatorname{div}(|\nabla u|^{\rho(x)-2}\nabla u) = 0$ 。

本文主要考虑具有可变指数的椭圆问题 (1) — (2) 的弱解的全局 Hölder 连续性。关于具有可变指数的椭圆问题的正则性已经有许多结论<sup>[14-19]</sup>。在 Hölder 连续性方面, Zhang 和 Zhou<sup>[20]</sup> 获得了强  $p(x)$ -Laplace 方程

$$\operatorname{div}(|\nabla u|^{\rho(x)-2}\nabla u) + |\nabla u|^{\rho(x)-2} \log(|\nabla u|) \nabla u \cdot \nabla p = 0$$

弱解梯度的局部 Hölder 连续性; Yao<sup>[21]</sup> 获得了  $p(x)$ -Laplace 方程

$$\operatorname{div}\left(\left(A\nabla u \cdot \nabla u\right)^{\frac{\rho(x)-2}{2}} A\nabla u\right) = \operatorname{div}\left(|f|^{\rho(x)-2}f\right)$$

弱解的局部Hölder正则性,并将其推广<sup>[22]</sup>到方程

$$\operatorname{div} A(x, \nabla u) = \operatorname{div} \left( |f|^{\rho(x)-2} f \right);$$

近期<sup>[23]</sup>又获得了带有对数函数的椭圆 $\rho(x)$ -Laplace方程

$$\operatorname{div} \left( (A \nabla u \cdot \nabla u)^{\frac{\rho(x)-2}{2}} \ln(e + (A \nabla u \cdot \nabla u)^{1/2}) A \nabla u \right) = \operatorname{div} \left( |f|^{\rho(x)-2} \ln(e + |f|) f \right)$$

弱解的局部Hölder连续性;佟玉霞等<sup>[24]</sup>获得了变指数 $A$ -调和方程

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u)$$

弱解的梯度的局部Hölder连续性。关于椭圆方程的更多研究具体参见文献[25-27]。本文主要考虑具有可变指数的椭圆方程的弱解 $u \in W^{1,\rho(x)}(\Omega)$ 的全局Hölder估计。先证明所给方程的全局高阶可积性,再与参考方程进行比较估计,最后应用覆盖迭代讨论证出全局Hölder估计。首先给出如下定义。

**定义1** 假设 $f \in L^{\rho(x)}(\Omega)$ 。函数 $u \in W^{1,\rho(x)}(\Omega)$ 称为Dirichlet边值问题(1)–(2)的弱解,若 $u - u_0 \in W_0^{1,\rho(x)}(\Omega)$ ,且对任给 $\varphi \in W_0^{1,\rho(x)}(\Omega)$ ,有

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} |f|^{\rho(x)-2} f \cdot \nabla \varphi \, dx. \quad (10)$$

下面是本文的主要结论。

**定理1** 设 $\Omega$ 为 $\mathbf{R}^n$ 中具有 $C^{1,\gamma}$ 边界的有界区域, $\gamma \geq 1, n \geq 2$ 。若 $u \in W^{1,\rho(x)}(\Omega)$ 为椭圆方程Dirichlet边值问题(1)–(2)的弱解,满足条件(3)–(9), $u_0 \in C^{1,\gamma}(\partial\Omega)$ ,则存在常数 $\alpha \in (0, 1)$ ,使得

$$u \in C^{1,\alpha}(\bar{\Omega}), \text{ 其中 } \alpha = \frac{(\beta - n\kappa - \lambda)\gamma_1}{(1 + \kappa)\gamma_2}, \beta = \frac{\alpha_3}{2}, \kappa = \frac{\beta}{n + \alpha_4}, \alpha_3, \alpha_4 \in (0, 1).$$

## 1 预备知识和引理

具有可变指数的Lebesgue-Sobolev空间 $L^{\rho(x)}(\Omega) = \left\{ f: \Omega \rightarrow \mathbf{R} \mid f \text{ 可测, 且 } \int_{\Omega} |f|^{\rho(x)} \, dx < \infty \right\}$ ,其上定义范数

$$\|f\|_{L^{\rho(x)}(\Omega)} = \inf \left\{ \lambda > 0: \int_{\Omega} \left| \frac{f}{\lambda} \right|^{\rho(x)} \, dx \leq 1 \right\},$$

此时 $L^{\rho(x)}(\Omega)$ 为Banach空间。 $W^{1,\rho(x)}(\Omega)$ 空间定义为 $W^{1,\rho(x)}(\Omega) = \{u \in L^{\rho(x)}(\Omega): \nabla u \in L^{\rho(x)}(\Omega)\}$ ,其上定义范数 $\|u\|_{W^{1,\rho(x)}(\Omega)} = \|u\|_{L^{\rho(x)}(\Omega)} + \|\nabla u\|_{L^{\rho(x)}(\Omega)}$ ,此时 $W^{1,\rho(x)}(\Omega)$ 为Banach空间。

对于任意给定的 $x_0 \in \bar{\Omega}$ ,常数 $R > 0$ ,开球 $B(x_0, R) = \{x: |x - x_0| < R\}$ 。 $\Omega(x_0, R) = B(x_0, R) \cap \Omega$ 。为方便记,常数 $C$ 在下文中不一定相等,仅表示出与 $C$ 有关的参数。

下面给出边值问题(1)–(2)的高阶可积性结论。

**引理1** 设 $\Omega$ 为 $\mathbf{R}^n$ 中具有 $C^{1,\gamma}$ 边界的有界区域, $\gamma \geq 1, n \geq 2$ 。若 $u \in W^{1,\rho(x)}(\Omega)$ 为椭圆问题(1)–(2)的弱解,满足条件(3)–(9), $u_0 \in W^{1,\infty}(\Omega), \|u_0\|_{W^{1,\infty}(\Omega)} \leq G$ 。则存在正常数 $\sigma_0 < \frac{1}{n}, R_0 < 1$ 与 $n, \gamma_1, \gamma_2, L, \alpha_i (i = 1, 2, \dots, 5), C_i (i = 1, 2, \dots, 5)$ 有关的常数 $C$ ,使得 $\nabla u \in L^{\rho(x)(1+\sigma_0)}(\Omega)$ 。并且对所有 $x \in \bar{\Omega}, R \in (0, R_0), \sigma \in (0, \sigma_0]$ ,有

$$\int_{\Omega(x,R)} |\nabla u|^{\rho(x)(1+\sigma)} \, dx \leq C \left( \int_{\Omega(x,2R)} |\nabla u|^{\rho(x)} \, dx \right)^{1+\sigma} + C \left( \int_{\Omega(x,2R)} (1 + |f|^{\rho(x)(1+\sigma)}) \, dx \right). \quad (11)$$

**证明** 首先延拓 $u_0$ 使得 $u_0 \in W^{1,\infty}(\mathbf{R}^n), \|u_0\|_{W^{1,\infty}(\Omega)} \leq G$ 。并且在 $\mathbf{R}^n \setminus \Omega$ 上增加定义 $u = u_0$ 。由于

$p(x) \in C^{0,\alpha_1}(\bar{\Omega})$ , 可延拓  $p(x)$  使得  $p(x) \in C^{0,\alpha_1}(\mathbf{R}^n)$ , 且满足

$$1 < \gamma_1 = \inf_{\mathbf{R}^n} p(x) \leq \sup_{\mathbf{R}^n} p(x) = \gamma_2 < \infty.$$

令  $x_0 \in \partial\Omega$ , 考虑球  $B(x_0, 2R_1)$ , 这里假设  $R_1$  充分小, 使得

$$\int_{B(x_0, 2R_1)} |\nabla u|^{\rho(x)} dx \leq 1, \int_{B(x_0, 2R_1)} |\nabla u_0|^{\rho(x)} dx \leq 1.$$

下面证明存在正常数  $R_0 \leq R_1, \sigma \in (0, \sigma_0], \sigma_0 \in (0, \frac{1}{n})$  和  $C$ , 使得对所有满足  $R \leq R_0$  的球  $B(x, R) =: B_R \subset B_{2R} \subset B(x_0, 2R_1)$ , 有

$$\int_{B_R} |\nabla u - \nabla u_0|^{\rho(x)} dx \leq C \left( \int_{B_{2R}} |\nabla u - \nabla u_0|^{\frac{\rho(x)}{1+\sigma}} dx \right)^{1+\sigma} + C \int_{B_R} (1 + |f|^{\rho(x)}) dx. \tag{12}$$

为此需分别考虑下述三种情形:

(1) 当  $B_{\frac{3}{2}R} \subset B(x_0, 2R_1) \cap \Omega$ , 此时由文献[14]中定理5的证明过程中的结论(见第128页的第11行—14行), 有

$$\begin{aligned} \int_{B_R} |\nabla u - \nabla u_0|^{\rho(x)} dx &\leq C \int_{B_{2R}} |\nabla u|^{\rho(x)} dx + C \int_{B_{2R}} |\nabla u_0|^{\rho(x)} dx \leq \\ &C \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1+\sigma}} dx \right)^{1+\sigma} + C \int_{B_{2R}} (1 + |f|^{\rho(x)}) dx + C \int_{B_{2R}} |\nabla u_0|^{\rho(x)} dx \leq \\ &C \left( \int_{B_{2R}} |\nabla u - \nabla u_0|^{\frac{\rho(x)}{1+\sigma}} dx \right)^{1+\sigma} + C \int_{B_{2R}} (1 + |f|^{\rho(x)}) dx, \end{aligned} \tag{13}$$

这就获得了(12)式。

(2) 当  $B_{\frac{3}{2}R} \subset B(x_0, 2R_1) \setminus \Omega$ , 此时(12)式左侧等于零, 于是(12)式显然成立。

(3) 当  $B_{\frac{3}{2}R} \cap \partial\Omega \neq \emptyset$ , 定义

$$p_1 = \min_{B_{2R}} p(x), \quad p_2 = \max_{B_{2R}} p(x).$$

任给常数  $\sigma \in (0, 1)$ , 取  $R_0$  足够小, 满足  $L(4R_0)^{\alpha_1} \leq \sigma$ . 于是有

$$p_2 - p_1 \leq L(4R)^{\alpha_1} \leq L(4R_0)^{\alpha_1} \leq \sigma.$$

于是由  $\frac{p_2 - p_1}{p_1} \leq \frac{\sigma}{p_1} \leq \sigma$ , 可知  $\frac{p_2}{p_1} \leq 1 + \sigma$ . 令截断函数  $\eta \in C_0^\infty(B_{2R}), 0 \leq \eta \leq 1, |\nabla \eta| \leq C/R$ , 在  $B_R$  上  $\eta \equiv 1$ . 在定义1中取  $\varphi = \eta^{p_2}(u - u_0)$ , 则

$$\nabla \varphi = p_2 \eta^{p_2-1}(u - u_0) \nabla \eta + \eta^{p_2} \nabla(u - u_0). \tag{14}$$

于是由定义1, 有

$$\begin{aligned} \int_{\Omega} p_2 \eta^{p_2-1}(u - u_0) A(x, \nabla u) \cdot \nabla \eta dx + \int_{\Omega} \eta^{p_2} A(x, \nabla u) \cdot \nabla(u - u_0) dx = \\ \int_{\Omega} p_2 \eta^{p_2-1}(u - u_0) |f|^{\rho(x)-2} f \cdot \nabla \eta dx + \int_{\Omega} \eta^{p_2} |f|^{\rho(x)-2} f \cdot \nabla(u - u_0) dx, \end{aligned} \tag{15}$$

从而

$$\begin{aligned} \int_{B_{2R}} \eta^{p_2} A(x, \nabla u) \cdot \nabla u dx \leq p_2 \int_{B_{2R}} \eta^{p_2-1} |u - u_0| |A(x, \nabla u)| |\nabla \eta| dx + \int_{B_{2R}} \eta^{p_2} A(x, \nabla u) \cdot \nabla u_0 dx + \\ p_2 \int_{B_{2R}} \eta^{p_2-1} |u - u_0| |f|^{\rho(x)-1} |\nabla \eta| dx + \int_{B_{2R}} \eta^{p_2} |f|^{\rho(x)-2} f \cdot \nabla(u - u_0) dx. \end{aligned} \tag{16}$$

下面分别估计上式两侧的积分。由条件(8),

$$\begin{aligned} \int_{B_{2R}} \eta^{\rho_2} A(x, \nabla u) \cdot \nabla u dx &\geq C_3 \int_{B_{2R}} \eta^{\rho_2} (\mu^2 + |\nabla u|^2)^{\frac{\rho(x)}{2}} dx - C_4 \int_{B_{2R}} \eta^{\rho_2} dx \geq \\ &C_3 \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx - C_4 \int_{B_{2R}} \eta^{\rho_2} dx. \end{aligned} \quad (17)$$

由条件(7)和 Young 不等式, 并注意到  $|\nabla \eta| \leq C/R$ , 且  $\rho(x) \geq \frac{\rho_2}{\rho_2 - 1} [\rho(x) - 1]$ , 于是有

$$\begin{aligned} &\rho_2 \int_{B_{2R}} \eta^{\rho_2 - 1} |u - u_0| |A(x, \nabla u)| |\nabla \eta| dx \leq \\ &C \rho_2 \int_{B_{2R}} \eta^{\rho_2 - 1} (1 + |\nabla u|^{\rho(x) - 1}) \left| \frac{u - u_0}{R} \right| dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} \eta^{\rho_2} (1 + |\nabla u|^{\rho(x) - 1})^{\frac{\rho_2}{\rho_2 - 1}} dx + C(\tau) \rho_2 \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} \eta^{\rho_2} (|\nabla u|^{\rho(x)} + 1) dx + C(\tau) \rho_2 \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + C(\tau) \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx + C|B_R|. \end{aligned} \quad (18)$$

由条件(7)和 Young 不等式,

$$\begin{aligned} &\int_{B_{2R}} \eta^{\rho_2} A(x, \nabla u) \cdot \nabla u_0 dx \leq \\ &C_2 \int_{B_{2R}} \eta^{\rho_2} (1 + |\nabla u|^{\rho(x) - 1}) |\nabla u_0| dx \leq \\ &C_2 \int_{B_{2R}} \eta^{\rho_2} |\nabla u_0| dx + C_2 \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + C(\tau) \int_{B_{2R}} \eta^{\rho_2} |\nabla u_0|^{\rho(x)} dx \leq \\ &C_2 \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + C(\tau) |B_R|. \end{aligned} \quad (19)$$

由  $|\nabla \eta| \leq C/R$  和 Young 不等式, 有

$$\begin{aligned} &\rho_2 \int_{B_{2R}} \eta^{\rho_2 - 1} |u - u_0| |f|^{\rho(x) - 1} |\nabla \eta| dx \leq C \rho_2 \int_{B_{2R}} \eta^{\rho_2 - 1} |f|^{\rho(x) - 1} \left| \frac{u - u_0}{R} \right| dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} \eta^{\rho_2} |f|^{\frac{\rho_2}{\rho_2 - 1} [\rho(x) - 1]} dx + C(\tau) \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} \eta^{\rho_2} (|f|^{\rho(x)} + 1) dx + C(\tau) \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx \leq \\ &C \rho_2 \tau \int_{B_{2R}} |f|^{\rho(x)} dx + C(\tau) \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx + C|B_R|. \end{aligned} \quad (20)$$

由 Young 不等式, 有

$$\begin{aligned} &\int_{B_{2R}} \eta^{\rho_2} |f|^{\rho(x) - 2} f \cdot \nabla(u - u_0) dx \leq \int_{B_{2R}} \eta^{\rho_2} |f|^{\rho(x) - 1} |\nabla u - \nabla u_0| dx \leq \\ &C(\tau) \int_{B_{2R}} \eta^{\rho_2} |f|^{\rho(x)} dx + \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u - \nabla u_0|^{\rho(x)} dx \leq \\ &C(\tau) \int_{B_{2R}} |f|^{\rho(x)} dx + \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u_0|^{\rho(x)} dx \leq \\ &C(\tau) \int_{B_{2R}} |f|^{\rho(x)} dx + \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + C|B_R|. \end{aligned} \quad (21)$$

联立(16)式—(21)式, 可得

$$\begin{aligned} &C_3 \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx \leq (C \rho_2 + C_2 + 1) \tau \int_{B_{2R}} \eta^{\rho_2} |\nabla u|^{\rho(x)} dx + \\ &C(\tau) \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx + C(\tau) \int_{B_{2R}} |f|^{\rho(x)} dx + C|B_R|. \end{aligned} \quad (22)$$

利用截断函数  $\eta$  的定义, 并取  $\tau = \frac{C_3}{2(Cp_2 + C_2 + 1)}$ , 即取  $\tau$  足够小, 使得  $(Cp_2 + C_2 + 1)\tau < C_3$ , 从而

而有

$$\int_{B_R} |\nabla u|^{\rho(x)} dx \leq C \int_{B_{2R}} \left| \frac{u - u_0}{R} \right|^{\rho_2} dx + C \int_{B_{2R}} |f|^{\rho(x)} dx + C |B_R|_0 \tag{23}$$

注意到  $\partial\Omega$  为  $C^{1,\gamma}$  边界, 利用 Sobolev-Poincaré 不等式, 将上式中含  $|u - u_0|$  的积分用  $|\nabla(u - u_0)|$  来估计, 并注意到  $\rho_2 > \rho_1 > \frac{\rho_1}{1 + \sigma}$ , 于是有

$$\begin{aligned} \int_{B_R} |\nabla u|^{\rho(x)} dx &\leq C \left( \int_{B_{2R}} |\nabla u - \nabla u_0|^{\frac{\rho_1}{1 + \sigma}} dx \right)^{\frac{(1 + \sigma)\rho_2}{\rho_1}} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C \leq \\ &C \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho_1}{1 + \sigma}} dx \right)^{\frac{(1 + \sigma)\rho_2}{\rho_1}} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C. \end{aligned} \tag{24}$$

又因为  $\rho_2 - \rho_1 \leq L(4R)^{\alpha_1}$ , 从而  $\frac{(1 + \sigma)\rho_2}{\rho_1} \leq \frac{(1 + \sigma)L(4R)^{\alpha_1}}{\rho_1} + (1 + \sigma)$ , 并利用  $\frac{\rho_1}{1 + \sigma} \leq \frac{\rho(x)}{1 + \sigma}$  和

$\frac{\rho_2}{\rho_1} \leq 1 + \sigma$ , 由上式可得

$$\begin{aligned} \int_{B_R} |\nabla u|^{\rho(x)} dx &\leq \\ &C \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{\frac{(1 + \sigma)\rho_2}{\rho_1}} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C \leq \\ &C \left( \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{\frac{(1 + \sigma)L(4R)^{\alpha_1}}{\rho_1}} + 1 \right) \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C = \\ &C \left( (2R)^{\frac{-n(1 + \sigma)L(4R)^{\alpha_1}}{\rho_1}} \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{\frac{(1 + \sigma)L(4R)^{\alpha_1}}{\rho_1}} + 1 \right) \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C \leq \\ &C \left( \left( (2R)^{-(4R)^{\alpha_1}} \right)^{\frac{n(1 + \sigma)L}{\gamma_1}} \left( \int_{B_{2R}} (1 + |\nabla u|^{\rho(x)}) dx \right)^{\frac{(1 + \sigma)4L(R_0)^{\alpha_1}}{\gamma_1}} + 1 \right) \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + \\ &C \int_{B_{2R}} |f|^{\rho(x)} dx + C \leq C \left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C, \end{aligned} \tag{25}$$

这里使用了  $\int_{B(x_0, 2R_1)} |\nabla u|^{\rho(x)} dx \leq 1$ , 并利用了当  $0 < R \leq R_0 < 1$  时  $(2R)^{-(4R)^{\alpha_1}} \leq C$ . 于是有

$$\begin{aligned} \int_{B_R} |\nabla u - \nabla u_0|^{\rho(x)} dx &\leq C \int_{B_R} |\nabla u|^{\rho(x)} dx + C \int_{B_R} |\nabla u_0|^{\rho(x)} dx \leq \\ &\left( \int_{B_{2R}} |\nabla u|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C \leq \\ &C \left( \int_{B_{2R}} |\nabla u - \nabla u_0|^{\frac{\rho(x)}{1 + \sigma}} dx \right)^{1 + \sigma} + C \int_{B_{2R}} |f|^{\rho(x)} dx + C, \end{aligned} \tag{26}$$

这就获得了(12)式。

由 Gehring 引理, (12)式暗示了存在  $\sigma_0 > 0$  使得  $|\nabla u - \nabla u_0| \in L_{loc}^{\rho(x)(1 + \sigma_0)}(B(x_0, 2R_1))$ , 且有

$$\left(\int_{B_R} |\nabla u - \nabla u_0|^{\rho(x)(1+\sigma)} dx\right)^{\frac{1}{1+\sigma}} \leq C \int_{B_{2R}} |\nabla u - \nabla u_0|^{\rho(x)} dx + C \left(\int_{B_{2R}} |f|^{\rho(x)(1+\sigma)} dx\right)^{\frac{1}{1+\sigma}} + C, \quad (27)$$

这里正常数  $R_0 \leq R_1, \sigma \in (0, \sigma_0], \sigma_0 \in \left(0, \frac{1}{n}\right), B(x, R) =: B_R \subset B_{2R} \subset B(x_0, 2R_1)$ 。

令  $x \in B(x_0, R_1) \cap \bar{\Omega}, R \leq R_0, B(x, 2R) \subset B(x_0, 2R_1)$ 。于是，注意到对常数  $\epsilon \in (0, 1), \epsilon |B(x, R)| \leq |\Omega(x, R)| \leq |B(x, R)|$  且  $u_0 \in W^{1,\infty}(\Omega)$ ，由 (27) 式可得 (12) 式。由  $\bar{\Omega}$  的紧性，可知引理 1 的结论是正确的。

注1 由引理 1 的证明易知，条件  $u_0 \in W^{1,\infty}(\Omega)$  可以减弱为

$$u_0 \in W^{1,\rho(x)(1+\sigma_1)}(\Omega) \cap C^{0,\gamma}(\partial\Omega)。$$

令  $\Omega_R = \Omega(x, R) = B(x, R) \cap \Omega, \bar{\Omega}_R = \bar{\Omega}(x, R) = B(x, R) \cap \bar{\Omega}$ 。令  $x_0 \in \partial\Omega$ 。令  $R_1$  足够小，使得  $\int_{\Omega(x_0, 2R_1)} |\nabla u|^{\rho(x)} dx \leq 1$ ，且对任给  $x, y \in \Omega(x_0, 2R_1)$ ，有

$$|p(x) - p(y)| \leq C_1 |x - y|^{\alpha_1} \leq C_1 (4R_1)^{\alpha_1} \leq \frac{\sigma_0 \gamma_1}{\sigma_0 + 2}, \quad (28)$$

其中  $\sigma_0$  为引理 1 中的正常数。

令  $x_c \in \bar{\Omega}(x_0, 2R_1), \Omega(x_c, R) = \Omega_R \subset \Omega_{2R} \subset \Omega(x_0, 2R_1)$ 。

定义  $p^*(R) = \sup_{\Omega_{2R}} p(x)$ ，令  $x^* \in \bar{\Omega}_{2R}$  满足  $p(x^*) = p^*(R) = p^*$ 。定义  $\bar{A}(\eta) = A(x^*, \eta)$ ，考虑边值问题

$$\begin{cases} \operatorname{div} \bar{A}(\nabla v) = 0, & x \in \Omega_R, \\ v = u, & x \in \partial\Omega_{R^0} \end{cases} \quad (29)$$

由于  $\operatorname{div} \left( |f(x^*)|^{\rho^*-2} f(x^*) \right) = 0$ ，故不妨将边值问题 (29) 写作

$$\begin{cases} \operatorname{div} \bar{A}(\nabla v) = \operatorname{div} \left( |f(x^*)|^{\rho^*-2} f(x^*) \right), & x \in \Omega_R, \\ v = u, & x \in \partial\Omega_{R^0} \end{cases} \quad (30)$$

下面的结论取自文献 [28]，是证明定理 1 的基础。

引理 2<sup>[16, 28]</sup> 设  $\Omega$  为  $\mathbf{R}^n$  中具有  $C^{1,\gamma}$  边界的有界区域， $u \in C^{1,\gamma}(\partial\Omega)$ 。椭圆问题 (30) 存在唯一解  $v$ ，满足  $v \in C^{1,\alpha_4}(\bar{\Omega}_{R/2})$ ，且有

$$\sup_{\Omega_{R/2}} |\nabla v|^{\rho^*} \leq c_1 \left( R^{-n} \int_{\Omega_R} |\nabla v|^{\rho^*} dx + G^{\rho^*} \right), \quad (31)$$

$$\operatorname{osc}_{\Omega_\rho} \nabla v \leq c_2 \left( \frac{\rho}{R} \right)^{\alpha_4} \left( \sup_{\Omega_{R/2}} |\nabla v| + GR^\gamma \right), \quad 0 < \rho < \frac{R}{2}, \quad (32)$$

$$\int_{\Omega_R} |\nabla v|^{\rho^*} dx \leq c_3 \int_{\Omega_R} (1 + |\nabla u|^{\rho^*}) dx, \quad (33)$$

$$\sup_{\Omega_R} |u - v| \leq \operatorname{osc}_{\Omega_R} u. \quad (34)$$

这里  $\alpha_4 \in (0, 1), c_1, c_2, c_3$  为正常数，仅与  $p^*, n, C_i (i = 1, 2, \dots, 4), \gamma$  有关。

引理 3 若  $u$  为椭圆问题 (1) — (2) 满足条件 (3) — (9) 的弱解， $v$  是 (30) 的弱解，则存在仅依赖于  $n, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \sigma_0$  的正常数  $\beta$ ，使得

$$\int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx \leq CR^\beta \int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx, \quad (35)$$

其中  $C$  与  $n, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, C_i (i = 1, 2, \dots, 5)$  有关。

证明 若  $u$  为椭圆问题 (1) — (2) 满足条件 (3) — (9) 的弱解， $v$  是 (30) 的弱解，在弱解的定义中

取试验函数  $\varphi = u - v$ , 有

$$\int_{\Omega_R} A(x, \nabla u) \cdot \nabla(u - v) dx = \int_{\Omega_R} |f(x)|^{\rho(x)-2} f(x) \cdot \nabla(u - v) dx, \quad (36)$$

$$\int_{\Omega_R} \bar{A}(\nabla v) \cdot \nabla(u - v) dx = \int_{\Omega_R} |f(x^*)|^{\rho^*-2} f(x^*) \cdot \nabla(u - v) dx, \quad (37)$$

于是有

$$\begin{aligned} & \int_{\Omega_R} [\bar{A}(\nabla u) - \bar{A}(\nabla v)] \cdot \nabla(u - v) dx = \\ & \int_{\Omega_R} \bar{A}(\nabla u) \cdot \nabla(u - v) dx - \int_{\Omega_R} |f(x^*)|^{\rho^*-2} f(x^*) \cdot \nabla(u - v) dx = \\ & \int_{\Omega_R} [\bar{A}(\nabla u) - A(x, \nabla u)] \cdot \nabla(u - v) dx + \int_{\Omega_R} \left( |f(x)|^{\rho(x)-2} f(x) - |f(x^*)|^{\rho(x)-2} f(x^*) \right) \cdot \nabla(u - v) dx + \\ & \int_{\Omega_R} \left( |f(x^*)|^{\rho(x)-2} f(x^*) - |f(x^*)|^{\rho^*-2} f(x^*) \right) \cdot \nabla(u - v) dx. \end{aligned} \quad (38)$$

记上式为

$$I = I_1 + I_2 + I_3.$$

估计  $I_1$ : 由(9)式, 结合使用带  $\tau$  的 Young 不等式和(33)式, 对  $\forall \delta > 0$ , 有

$$\begin{aligned} I_1 &= \int_{\Omega_R} [\bar{A}(\nabla u) - A(x, \nabla u)] \cdot \nabla(u - v) dx \leq \\ & C_5 \int_{\Omega_R} |x - x^*|^{\alpha_3} \left| \log(\mu^2 + |\nabla u|^2) \right| \left( \mu^2 + |\nabla u|^2 \right)^{\frac{\rho(x)-1}{2}} |\nabla(u - v)| dx \leq \\ & CR^{\alpha_3} \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho^*-1+\delta} \right) |\nabla(u - v)| dx \leq \\ & C(\tau) R^{\alpha_3} \int_{\Omega_R} \left( |\nabla u - \nabla v|^{\rho^*} \right) dx + CR^{\alpha_3} \tau \int_{\Omega_R} \left( 1 + |\nabla u|^{\frac{(\rho^*-1+\delta)\rho^*}{\rho^*-1}} \right) dx \leq \\ & C(\tau) R^{\alpha_3} \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho^*} \right) dx + CR^{\alpha_3} \tau \int_{\Omega_R} \left( 1 + |\nabla u|^{\frac{(\rho^*-1+\delta)\rho^*}{\rho^*-1}} \right) dx. \end{aligned} \quad (39)$$

因  $p_* := \inf_{\Omega_{2R}} p(x) \geq \gamma_1 > 1$ , 由(28)式有

$$p^* = p_* + (p^* - p_*) \leq p_* + \frac{\sigma_0 \gamma_1}{\sigma_0 + 2}, \quad (40)$$

于是有

$$p^* \leq p^* \left( 1 + \frac{\sigma_0}{2} \right) \leq \left( p_* + \frac{\sigma_0 \gamma_1}{\sigma_0 + 2} \right) \left( 1 + \frac{\sigma_0}{2} \right) \leq p_*(1 + \sigma_0) \leq p(x)(1 + \sigma_0). \quad (41)$$

令  $\delta$  满足  $0 < \delta < \frac{\sigma_0(\gamma_1 - 1)}{2}$ , 于是有

$$\frac{(\rho^* - 1 + \delta)\rho^*}{\rho^* - 1} = p^* \left( 1 + \frac{\delta}{\rho^* - 1} \right) \leq p^* \left( 1 + \frac{\delta}{\gamma_1 - 1} \right) \leq p^* \left( 1 + \frac{\sigma_0}{2} \right) \leq p(x)(1 + \sigma_0). \quad (42)$$

由式(39)和式(42), 可得

$$I_1 \leq C(\tau) R^{\alpha_3} \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho^*} \right) dx + CR^{\alpha_3} \tau \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho(x)(1+\sigma_0)} \right) dx. \quad (43)$$

由引理1, 并注意到  $\{f^i\} \subset C^{0, \alpha_2}(\bar{\Omega})$ , 于是有

$$\int_{\Omega_R} |\nabla u|^{\rho(x)(1+\sigma_0)} dx \leq C \left( \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx \right)^{1+\sigma_0} + C. \quad (44)$$

注意到  $\partial\Omega$  为  $C^{1, \gamma}$  边界, 且  $R_1$  的选取足够小, 使得  $\int_{\Omega(x_0, 2R_1)} |\nabla u|^{\rho(x)} dx \leq 1$ , 于是有

$$\begin{aligned}
\int_{\Omega_R} |\nabla u|^{\rho(x)(1+\sigma_0)} dx &\leq C \left( \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx \right)^{1+\sigma_0} + C = \\
&C \left( \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx \right)^{\sigma_0} \left( \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx \right) + C = \\
&\frac{C}{|\Omega_{2R}|^{\sigma_0}} \left( \int_{\Omega_R} |\nabla u|^{\rho(x)} dx \right)^{\sigma_0} \left( \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx \right) + C \leq \\
&\frac{C}{|\Omega_{2R}|^{\sigma_0}} \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx + C \leq \\
&CR^{-n\sigma_0} \int_{\Omega_{2R}} |\nabla u|^{\rho(x)} dx + C \leq \\
&CR^{-n\sigma_0} \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + C.
\end{aligned} \tag{45}$$

即

$$\int_{\Omega_R} |\nabla u|^{\rho(x)(1+\sigma_0)} dx \leq CR^{-n\sigma_0} \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + CR^n. \tag{46}$$

于是将上式代入(43)式,可得

$$I_1 \leq C(\tau)R^{\alpha_3} \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + CR^{\alpha_3 - n\sigma_0} \tau \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + CR^{n+\alpha_3} \tau. \tag{47}$$

估计  $I_2$ : 分两种情况。当  $\rho(x) \geq 2$ , 使用基本不等式

$$\left| |\xi|^{\rho-2} \xi - |\eta|^{\rho-2} \eta \right| \leq C(|\xi| + |\eta|)^{\rho-2} |\xi - \eta|, \quad \forall \rho \geq 2, \xi, \eta \in \mathbf{R}^n, C = C(\rho),$$

和 Young 不等式, 以及(33)式, 并注意到  $\{f^i\} \subset C^{0, \alpha_2}(\bar{\Omega})$ , 于是有

$$\begin{aligned}
I_2 &= \int_{\Omega_R} \left( |f(x)|^{\rho(x)-2} f(x) - |f(x^*)|^{\rho(x)-2} f(x^*) \right) \cdot \nabla(u-v) dx \leq \\
&\int_{\Omega_R} \left| |f(x)|^{\rho(x)-2} f(x) - |f(x^*)|^{\rho(x)-2} f(x^*) \right| |\nabla(u-v)| dx \leq \\
&C \int_{\Omega_R} (|f(x)| + |f(x^*)|)^{\rho(x)-2} |f(x) - f(x^*)| |\nabla(u-v)| dx \leq \\
&CR^{\alpha_2} \int_{\Omega_R} |\nabla u - \nabla v| dx \leq \\
&CR^{\alpha_2} \tau \int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx + C(\tau)R^{\alpha_2} \int_{\Omega_R} 1 dx \leq \\
&CR^{\alpha_2} \tau \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + C(\tau)R^{n+\alpha_2}.
\end{aligned} \tag{48}$$

当  $1 < \rho(x) < 2$ , 使用基本不等式

$$\left| |\xi|^{\rho-2} \xi - |\eta|^{\rho-2} \eta \right| \leq C|\xi - \eta|^{\rho-1}, \quad \forall \rho \in (1, 2), \xi, \eta \in \mathbf{R}^n, C = C(\rho),$$

和 Young 不等式, 以及(33)式, 并注意到  $\{f^i\} \subset C^{0, \alpha_2}(\bar{\Omega})$ , 于是有

$$\begin{aligned}
I_2 &= \int_{\Omega_R} \left( |f(x)|^{\rho(x)-2} f(x) - |f(x^*)|^{\rho(x)-2} f(x^*) \right) \cdot \nabla(u-v) dx \leq \\
&C \int_{\Omega_R} \left| |f(x)| - |f(x^*)| \right|^{\rho(x)-1} |\nabla(u-v)| dx \leq \\
&C \int_{\Omega_R} R^{\alpha_2(\rho(x)-1)} |\nabla u - \nabla v| dx \leq \\
&CR^{\alpha_2(\gamma_1-1)} \int_{\Omega_R} |\nabla u - \nabla v| dx \leq \\
&CR^{\alpha_2(\gamma_1-1)} \tau \int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx + C(\tau)R^{\alpha_2(\gamma_1-1)} \int_{\Omega_R} 1 dx \leq \\
&CR^{\alpha_2(\gamma_1-1)} \tau \int_{\Omega_{2R}} (1 + |\nabla u|^{\rho^*}) dx + C(\tau)R^{n+\alpha_2(\gamma_1-1)}.
\end{aligned} \tag{49}$$

估计  $I_3$ : 使用中值定理, Young 不等式, 以及 (33) 式, 并注意到  $p(x) \in C^{0, \alpha_1}(\bar{\Omega})$  和  $\{f^i\} \subset C^{0, \alpha_2}(\bar{\Omega})$ , 有

$$\begin{aligned}
 I_3 &= \int_{\Omega_R} \left( |f(x^*)|^{\rho(x)-2} f(x^*) - |f(x^*)|^{\rho^*-2} f(x^*) \right) \cdot \nabla(u-v) dx \leq \\
 &\int_{\Omega_R} \left| |f(x^*)|^{\rho(x)-2} - |f(x^*)|^{\rho^*-2} \right| |f(x^*)| |\nabla(u-v)| dx \leq \\
 &C \int_{\Omega_R} |p(x) - \rho^*| \left( 1 + |f(x^*)|^{\rho^*-1} \right) |\ln(|f(x^*)|)| |\nabla u - \nabla v| dx \leq \\
 &CR^{\alpha_1} \int_{\Omega_R} |\nabla u - \nabla v| dx \leq \\
 &CR^{\alpha_1} \tau \int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx + C(\tau) R^{\alpha_1} \int_{\Omega_R} 1 dx \leq \\
 &CR^{\alpha_1} \tau \int_{\Omega_{2R}} \left( 1 + |\nabla u|^{\rho^*} \right) dx + C(\tau) R^{n+\alpha_1}.
 \end{aligned} \tag{50}$$

联立式 (47) 一式 (50), 可得

$$\begin{aligned}
 I &\leq C \left( R^{\alpha_3} + R^{\alpha_3 - n\sigma_0} \tau + R^{\alpha_2} \tau + R^{\alpha_2(\gamma_1 - 1)} \tau + R^{\alpha_1} \tau \right) \int_{\Omega_{2R}} \left( 1 + |\nabla u|^{\rho^*} \right) dx + \\
 &C \left( R^{n+\alpha_3} \tau + R^{n+\alpha_2} + R^{n+\alpha_2(\gamma_1 - 1)} + R^{n+\alpha_1} \right).
 \end{aligned} \tag{51}$$

令  $\tau = R < 1$ , 注意到  $\sigma_0 < \frac{1}{n}$ , 故  $\alpha_3 - n\sigma_0 + 1 > \alpha_3$ , 从而  $R^{\alpha_3 - n\sigma_0} \tau = R^{\alpha_3 - n\sigma_0 + 1} < R^{\alpha_3}$ . 取  $\tilde{\beta} := \min \{ \alpha_3, \alpha_2 + 1, \alpha_2(\gamma_1 - 1) + 1, \alpha_1 + 1 \}$ , 则上式即为

$$I \leq CR^{\tilde{\beta}} \int_{\Omega_{2R}} \left( 1 + |\nabla u|^{\rho^*} \right) dx. \tag{52}$$

当  $\rho^* \geq 2$  时, 由 (6) 式可知

$$I = \int_{\Omega_R} [\bar{A}(\nabla u) - \bar{A}(\nabla v)] \cdot \nabla(u-v) dx \geq C_1 \int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx. \tag{53}$$

于是由 (52) 式和 (53) 式, 可得引理 3 的结论。当  $1 < \rho^* < 2$  时, 由 (6) 式可知

$$I = \int_{\Omega_R} [\bar{A}(\nabla u) - \bar{A}(\nabla v)] \cdot \nabla(u-v) dx \geq C_1 \int_{\Omega_R} \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{\rho^*-2}{2}} |\nabla u - \nabla v|^2 dx. \tag{54}$$

于是结合上式和 Hölder 不等式, 并利用 (33) 式, 可得

$$\begin{aligned}
 &\int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx = \\
 &\int_{\Omega_R} \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{\rho^*-2}{4}} |\nabla u - \nabla v| \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{2-\rho^*}{4}} |\nabla u - \nabla v|^{\rho^*-1} dx \leq \\
 &\left( \int_{\Omega_R} \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{\rho^*-2}{2}} |\nabla u - \nabla v|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega_R} \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{2-\rho^*}{2}} |\nabla u - \nabla v|^{2\rho^*-2} dx \right)^{\frac{1}{2}} \leq \\
 &\left( \frac{I}{C_1} \right)^{\frac{1}{2}} \left( \int_{\Omega_R} \left( |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{\rho^*}{2}} dx \right)^{\frac{1}{2}} \leq CI^{\frac{1}{2}} \left( \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho^*} \right) dx \right)^{\frac{1}{2}}.
 \end{aligned} \tag{55}$$

上式代入 (52) 式, 可得

$$\int_{\Omega_R} |\nabla u - \nabla v|^{\rho^*} dx \leq C \left( R^{\tilde{\beta}} \int_{\Omega_{2R}} \left( 1 + |\nabla u|^{\rho^*} \right) dx \right)^{\frac{1}{2}} \left( \int_{\Omega_R} \left( 1 + |\nabla u|^{\rho^*} \right) dx \right)^{\frac{1}{2}} \leq CR^{\tilde{\beta}/2} \int_{\Omega_{2R}} \left( 1 + |\nabla u|^{\rho^*} \right) dx. \tag{56}$$

令  $\beta = \tilde{\beta}/2$  即可。引理 3 证毕。

**引理 4** 令  $B(x_0, 2R_1)$  如上定义。若  $u$  为 (1) — (2) 满足条件 (3) — (9) 的弱解, 则对任给  $\lambda \in (0, n)$ , 存在正常数  $R_\lambda < R_1/16$  和仅与  $n, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \lambda$  有关的正常数  $C$ , 使得  $\forall x_c \in \bar{\Omega}(x_0, R_1/2)$ ,  $\forall \rho: 0 < \rho < R_\lambda$ , 有

$$\int_{\Omega(x_c, \rho)} |\nabla u|^{\rho^*(\rho)} dx \leq C\rho^{-\lambda}, \quad (57)$$

其中  $\rho^*(\rho) = \sup_{\Omega(x_c, \rho)} p(x)$ 。

**证明** 令  $x_c \in \bar{\Omega}(x_0, R_1/2)$  和  $\rho < R_1/16$ , 令  $R > 0$  满足  $\Omega(x_c, \rho) = \Omega_\rho \subset \Omega_{R/2} \subset \Omega_{8R} \subset \Omega(x_0, 2R_1)$ 。若  $v$  为式 (30) 的弱解, 则由式 (31) 和式 (33), 有

$$\begin{aligned} \int_{\Omega_\rho} |\nabla u|^{\rho^*(\rho)} dx &\leq C \int_{\Omega_\rho} |\nabla u - \nabla v|^{\rho^*(\rho)} dx + C \int_{\Omega_\rho} |\nabla v|^{\rho^*(\rho)} dx \leq \\ &C \int_{\Omega_R} (1 + |\nabla u - \nabla v|^{\rho^*(R)}) dx + C \int_{\Omega_\rho} (1 + |\nabla v|^{\rho^*(R)}) dx \leq \\ CR^n + CR^\beta \int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx + C\rho^n + C|\Omega_\rho| &\left( c_1 R^{-n} \int_{\Omega_R} |\nabla v|^{\rho^*} dx + G^{\rho^*} \right) \leq \\ CR^n + CR^\beta \int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx + C\rho^n + C \left( \frac{\rho}{R} \right)^n &\int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx \leq \\ C \left( R^\beta + \left( \frac{\rho}{R} \right)^n \right) \int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx + CR^n. \end{aligned} \quad (58)$$

于是由覆盖迭代讨论 (见文献 [1] 引理 3.2) 和类似 (46) 式的估计, 对每个  $\lambda \in (0, n)$ , 存在正常数  $R_\lambda < R_1/16$  和  $C > 0$ , 有

$$\int_{\Omega} |\nabla u|^{\rho^*(\rho)} dx \leq C\rho^{-\lambda}. \quad (59)$$

## 2 主要结论的证明

**定理 1 的证明** 令  $2\rho < R < 2R < R_\lambda$ , 且  $\Omega(x_c, \rho) = \Omega_\rho \subset \Omega_{R/2} \subset \Omega_{32R} \subset \Omega(x_0, 2R_1)$ 。令  $v$  是问题 (30) 的唯一解, 由式 (31) 一式 (33) 可得

$$\begin{aligned} \int_{\Omega_\rho} |\nabla v - (\nabla v)_{\Omega_\rho}|^{\rho^*} dx &\leq C \left( \frac{\rho}{R} \right)^{\alpha_1} \int_{\Omega_{R/2}} (\sup |\nabla v|^{\rho^*} + G^{\rho^*} R^{\gamma \rho^*}) dx \leq \\ C \left( \frac{\rho}{R} \right)^{\alpha_1} R^{-n} \int_{\Omega_R} (|\nabla v|^{\rho^*} + G^{\rho^*}) dx &\leq \\ C \left( \frac{\rho}{R} \right)^{\alpha_1} \int_{\Omega_R} (1 + |\nabla v|^{\rho^*}) dx &\leq \\ C \left( \frac{\rho}{R} \right)^{\alpha_1} \int_{\Omega_R} (1 + |\nabla u|^{\rho^*}) dx. \end{aligned} \quad (60)$$

根据引理 3, 式 (60) 和引理 4 可得

$$\begin{aligned} \int_{\Omega} |\nabla u - (\nabla u)_{\Omega_\rho}|^{\rho^*} dx &\leq C \int_{\Omega_\rho} |\nabla u - (\nabla v)_{\Omega_\rho}|^{\rho^*} dx \leq \\ C \int_{\Omega_\rho} |\nabla u - \nabla v|^{\rho^*} dx + C \int_{\Omega_\rho} |\nabla v - (\nabla v)_{\Omega_\rho}|^{\rho^*} dx &\leq \\ CR^\beta \int_{\Omega_{2R}} (|\nabla u|^{\rho^*} + 1) dx + C \left( \frac{\rho}{R} \right)^{\alpha_1} \rho^n \int_{\Omega_R} (1 + |\nabla u|^{\rho^*}) dx &\leq \\ CR^{n+\beta} \int_{\Omega_{2R}} |\nabla u|^{\rho^*} dx + CR^{n+\beta} + C \left( \frac{\rho}{R} \right)^{\alpha_1} \rho^n R^{-\lambda} + C \left( \frac{\rho}{R} \right)^{\alpha_1} \rho^n &\leq \\ CR^{n+\beta-\lambda} + CR^{n+\beta} + C \left( \frac{\rho}{R} \right)^{\alpha_1} \rho^n R^{-\lambda} + C \left( \frac{\rho}{R} \right)^{\alpha_1} \rho^n. \end{aligned} \quad (61)$$

令  $\rho = \frac{R^{1+\kappa}}{2}$ , 其中  $\kappa = \frac{\beta}{n+\alpha_4} > 0$ ,  $\alpha_4 \in (0, 1)$  是引理 2 中的常数, 于是有  $\left(\frac{\rho}{R}\right)^{\alpha_4} \rho^n < R^{n+\beta}$ . 注意到  $R < 1$ , 则上述不等式右侧的  $R$  有相同的指数  $n + \beta - \lambda$ ,

$$\int_{\Omega_r} |\nabla u - (\nabla u)_{\Omega_r}|^{p^*} dx \leq CR^{n+\beta-\lambda} + CR^{n+\beta} + CR^{n+\beta-\lambda} + CR^{n+\beta} \leq CR^{n+\beta-\lambda} \leq C\rho^{\frac{n+\beta-\lambda}{1+\kappa}} = C\rho^{n+\frac{\beta-n\mu-\lambda}{1+\kappa}}. \quad (62)$$

令  $\lambda = \frac{\beta\alpha_4}{2(n+\alpha_4)} > 0$ , 则有  $\beta - n\mu - \lambda > 0$ . 上式即为

$$\int_{\Omega_r} |\nabla u - (\nabla u)_{\Omega_r}|^{p^*} dx \leq C\rho^{\frac{\beta-n\mu-\lambda}{1+\kappa}}. \quad (63)$$

最后, 由 Hölder 不等式可得

$$\left(\int_{\Omega_r} |\nabla u - (\nabla u)_{\Omega_r}|^{\gamma_1} dx\right)^{\frac{1}{\gamma_1}} \leq \left(\int_{\Omega_r} |\nabla u - (\nabla u)_{\Omega_r}|^{p^*} dx\right)^{\frac{1}{p^*}} \left(\int_{\Omega_r} 1 dx\right)^{\frac{p^*-\gamma_1}{p^*\gamma_1}} \leq C\rho^{\frac{\beta-n\mu-\lambda}{(1+\kappa)p^*}}, \quad (64)$$

即

$$\int_{\Omega_r} |\nabla u - (\nabla u)_{\Omega_r}|^{\gamma_1} dx \leq C\rho^{\frac{(\beta-n\mu-\lambda)\gamma_1}{(1+\kappa)p^*}} = C\rho^\alpha, \quad (65)$$

其中  $\alpha = \frac{(\beta-n\mu-\lambda)\gamma_1}{(1+\kappa)p^*} \in (0, 1)$ . 定理 1 证毕。

### 3 结论

本文主要将 Yao 在文献 [22] 中建立的局部 Hölder 连续性, 使用 Acerbi 和 Mingione 在文献 [14] 中建立逆 Hölder 不等式的方法, 并利用比较估计和迭代法等技巧, 获得了变指数  $p(x)$ -Laplace 方程

$$\operatorname{div} A(x, \nabla u) = \operatorname{div} \left( |f|^{\rho(x)-2} f \right), \quad x \in \Omega,$$

在 Dirichlet 边界条件

$$u = u_0, \quad x \in \partial\Omega$$

下, 弱解的全局 Hölder 估计, 其中算子  $A = A(x, \xi)$  满足文章要求的结构性条件。

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