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具有两羊群间运输的布鲁氏菌病模型分析

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摘　要: 本文提出了考虑环境因素且具有两羊群间运输的多组易感-暴露-感染-免疫(SEIVW)动力学模型,旨在研究运输及交叉感染对布鲁氏菌病传播的影响。首先计算出基本再生数R_0,全局动力学完全由R_0决定。当$R_0<1$时,无病平衡点是全局渐近稳定的;当$R_0>1$时,存在唯一的地方病平衡点是全局渐近稳定的。对基本再生数进行了参数敏感性分析,证实了及时清除染病羊群的尸体,增加疫苗覆盖率是有效的羊布鲁氏菌病的控制策略。此外,合理控制羊群之间的运输以及实现羊群的自给自足是控制羊布鲁氏菌病的另一有效措施。

关键词: 布鲁氏菌病;基本再生数;全局动力学;防控措施
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Analysis of a Brucellosis Model with Transportation Between Two Flocks

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Abstract: This paper proposes a multi-group susceptible-exposed-infected-vaccinated (SEIVW) dynamic model with transportation between two flocks considering environmental factors, in order to propose a multi-group SEIVW dynamical model with two flocks' transport. Firstly, the basic reproduction number R_0 is obtained, and the global dynamics are completely determined by R_0. When $R_0<1$, the disease-free equilibrium is globally asymptotically stable; when $R_0>1$, there is a unique local equilibrium that is globally asymptotically stable. The sensitivity analysis of the parameters of the basic reproduction number is carried out and confirmed that timely removal of the bodies of infected sheep and increasing vaccine coverage were effective control strategies for sheep brucellosis. In addition, reasonably reducing the transportation between flocks and achieving self-sufficiency of flocks is another effective control strategy for controlling brucellosis.

Key words: brucellosis; the basic reproduction number; global dynamics; prevention and control strategy

0　引言

布鲁氏菌病(简称布病)是一种由布鲁氏菌引起的急性人畜共患病[1]。它主要通过直接或间接接触受感染的动物或其分泌物而传播,其历史可以追溯到1886年[2]。布鲁氏菌存在6种类型[3],这些细菌在受污染的土壤和水中可存活1至4个月,在牛奶和肉类中则为2个月。布病虽然历史悠久,但通过合适的预防方法可以有效控制其传播。

近年来,世界范围内的数学家都在高度关注布鲁氏菌病的传播规律,提出并执行了许多数学

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模型来研究其动力学行为[4-7]。数学工具使研究人员能够获得有关感染、传播和敏感参数的基本预测,以控制或尽量减少传播。布病存在于多个种群间,并存在相互作用。由于影响种间关系的因素甚多,为了深入研究,DENG[8]基于感染牛-羊对人的直接感染和环境因素对人的间接感染,建立了一类易感-感染-康复(SIRWSsIsRsScIc)的布病传播模型,根据内蒙古布病病例报告数据进行数值模拟并对其发展趋势进行预测,表明对牛羊进行隔离喂养和环境消杀可以一定程度上降低患病人数。梁桂珍等[9]根据布病高发地区的羊-人传播的特征,建立了具有时滞的布病传播模型。提出对易感绵羊接种疫苗、扑杀感染绵羊和宣传教育很好地控制了布病的传播。Ma等[10]利用后向欧拉方法构建离散时间人羊耦合的易感羊群-暴露羊群-染病羊群-易感人群-急性染病人群-慢性染病人群($SEIWS_hI_hC_h$)布病传播模型,调查不同控制措施(包括消毒和宣传教育)对吉林省布病传播的影响,深入探讨不同控制措施对布传播的影响,证实了吉林省目前防控措施的有效性。

基于对自然生物以及畜牧业相关领域研究的分析可知,牲畜的买卖与迁徙在造成不同区域同一种群之间的相互作用的同时,也会使区域内的种间关系发生变化。例如:Zhang等[11]通过构建一个易感-暴露-染病-免疫(SEIV)模型得出北方地区奶牛的输入可能导致浙江省感染布病的奶牛数量波动较大,因此鼓励奶牛场坚持奶牛自给自足的生产。胡佳欣[12]研究了非正规活禽调运对H7N9的影响,结果表明非正规活禽调运出现时,虽然高危地区患病人数略有减少,但两个地区的总感染人数增加,并且随着调运率的增加染病人数不断上升。Yousef[13]看到2019新型冠状病毒流行期间的全面封锁是管理和控制感染的重要阶段,说明两地点之间与交通相关的感染以及封锁期的必要性。对于布病而言,在我国北方以畜牧业为主的地区依然流行。这主要是因为这些地区存在大量的羊群运输,并且在运输过程中潜伏期和染病期的羊群也被运输。羊群在不同地区间的运输必然会加剧布病的流行。

尽管已有对布病模型全面的研究,然而羊群间运输对布病流行的影响还不清晰。基于上述分析可知当下研究羊群间的运输对布病传播至关重要。解决此类问题的关键在于要建立适当的数学模型来描述羊群间的运输对于布病流行的影响,从而揭示其传播规律,进而提供相应的预防和控制策略。受上述思想启发,本文在考虑两羊群间运输、羊对人的直接感染和环境因素对人的间接感染等多个因素的前提下,建立一类多组易感-暴露-感染-免疫(SEIVW)布病传播动力学模型,利用微分方程定性理论分析模型的动态问题,通过敏感性分析发现控制羊群的有向运输是控制布病流行的重要措施,及时捕杀患病羊群是控制布病流行的有效方法。

1 模型的建立

1.1 模型公式

根据布病在羊群与人之间的主要传播机理,建立了一个羊、环境与人之间的布病传播模型。模型包含14个仓室。其中,在t时刻易感羊群数量为$S_{Ai}(t)$,潜伏羊群数量为$E_{Ai}(t)$,染病羊群数量为$I_{Ai}(t)$,免疫羊群数量为$V_{Ai}(t)$,则羊群总数量为$N_{Ai}(t)=S_{Ai}(t)+E_{Ai}(t)+I_{Ai}(t)+V_{Ai}(t)$;而易感人群数量为$S_H(t)$,潜伏人群数量为$E_H(t)$,染病人群数量为$I_H(t)$,免疫人群数量为$V_H(t)$,则人口总数量为$N_H(t)=S_H(t)+E_H(t)+I_H(t)+V_H(t)$;在$t$时刻环境中的病毒的数量为$W_i$。为保证模型的生物学意义,所有参数非负,其中各参数的描述见表1。

表1 各参数变量的表示和描述

Table 1 Representation and description of each parameter variable

参数	描述	参数	描述
Λ_{Ai}	易感羊群i的出生率	a_{ij}	羊群j到i的运输率($i\neq j$)
λ_{Ai}	免疫羊群i的免疫失效率	a_{ji}	羊群i到j的运输率($i\neq j$)

续表1 各参数变量的表示和描述

Continued Table 1 Representation and description of each parameter variable

参数	描述	参数	描述
d_{Ai}	羊群i的自然死亡率	Λ_H	易感人群的出生率
β_{Ai}	感染羊群i对易感羊群i的感染率	λ_H	免疫人群的免疫失效率
μ_{Ai}	感染羊群i因病死亡及捕杀率	d_H	人口的自然死亡率
α_i	环境i对易感羊群i的感染率	β_H	羊群i对易感人群的感染率($i=1,2$)
γ_{Ai}	易感羊群i的免疫覆盖率	μ_H	患病人群因病死亡率
δ_{Ai}	潜伏羊群i的发病率	c	环境i对于易感人群的感染率
r_{Ai}	潜伏羊群i和患病羊群i单位时间内向环境排出的布鲁氏菌数	γ_H	易感人群的免疫覆盖率
e_i	环境i中布鲁氏菌的衰退率	δ_H	潜伏人群的发病率

根据以上假设,建立如下布病传播的微分方程模型

$$\begin{cases}\dfrac{\mathrm{d}S_{Ai}}{\mathrm{d}t}=\Lambda_{Ai}-(d_{Ai}+\gamma_{Ai})S_{Ai}+\lambda_{Ai}V_{Ai}-\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})-\alpha_i S_{Ai}W_i-S_{Ai}\sum_{j=1}^{2}a_{ji}+S_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}E_{Ai}}{\mathrm{d}t}=\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})+\alpha_i S_{Ai}W_i-(d_{Ai}+\delta_{Ai})E_{Ai}-E_{Ai}\sum_{j=1}^{2}a_{ji}+E_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}I_{Ai}}{\mathrm{d}t}=\delta_{Ai}E_{Ai}-(d_{Ai}+\mu_{Ai})I_{Ai}-I_{Ai}\sum_{j=1}^{2}a_{ji}+I_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}V_{Ai}}{\mathrm{d}t}=\gamma_{Ai}S_{Ai}-(d_{Ai}+\lambda_{Ai})V_{Ai}-V_{Ai}\sum_{j=1}^{2}a_{ji}+V_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}W_i}{\mathrm{d}t}=r_{A_i}(E_{A_i}+I_{A_i})-e_iW_i-W_i\sum_{j=1}^{2}a_{ji}+W_j\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}S_H}{\mathrm{d}t}=\Lambda_H+\lambda_H V_H-(d_H+\gamma_H)S_H-\beta_H S_H(E_{Ai}+I_{Ai})-c_iS_HW_i,\\ \dfrac{\mathrm{d}E_H}{\mathrm{d}t}=\beta_H S_H(E_{Ai}+I_{Ai})+cS_HW_1-(d_H+\delta_H)E_H,\\ \dfrac{\mathrm{d}I_H}{\mathrm{d}t}=\delta_H E_H-(d_H+\mu_H)I_H,\\ \dfrac{\mathrm{d}V_H}{\mathrm{d}t}=\gamma_H S_H-(d_H+\lambda_H)V_H。\end{cases} \tag{1}$$

其中$i=1,2$;$j=1,2$;$i\neq j$。

定理1 当$S_{Ai}(0)\geqslant 0$, $E_{Ai}(0)\geqslant 0$, $I_{Ai}(0)\geqslant 0$, $V_{Ai}(0)\geqslant 0$, $W_i(0)\geqslant 0$, $S_H(0)\geqslant 0$, $E_H(0)\geqslant 0$, $I_H(0)\geqslant 0$, $V_H(0)\geqslant 0$时,系统(1)的解满足$S_{Ai}(t)\geqslant 0$, $E_{Ai}(t)\geqslant 0$, $I_{Ai}(t)\geqslant 0$, $V_{Ai}(t)\geqslant 0$, $W_i(t)\geqslant 0$, $S_H(t)\geqslant 0$, $E_H(t)\geqslant 0$, $I_H(t)\geqslant 0$, $V_H(t)\geqslant 0$。

证明 根据系统(1)有,

$$\frac{\mathrm{d}S_{Ai}}{S_{Ai}\mathrm{d}t}=\frac{\Lambda_{Ai}-(d_{Ai}+\gamma_{Ai})S_{Ai}+\lambda_{Ai}V_{Ai}-\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})-\alpha_i S_{Ai}W_i-S_{Ai}\sum_{j=1}^{2}a_{ji}+S_{Aj}\sum_{j=1}^{n}a_{ij}}{S_{Ai}},$$

计算可得$S_{Ai}(t)=S_{Ai}(0)\mathrm{e}^{\int_0^t \Lambda_{Ai}-(d_{Ai}+\gamma_{Ai})S_{Ai}+\lambda_{Ai}V_{Ai}-\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})-\alpha_i S_{Ai}W_i-S_{Ai}\sum_{j=1}^{n}a_{ji}+S_{Aj}\sum_{j=1}^{n}a_{ij}}\geqslant 0$,同理可得当$E_{Ai}(0)\geqslant 0$, $I_{Ai}\geqslant 0$, $V_{Ai}(0)\geqslant 0$, $W_i(0)\geqslant 0$, $S_H(0)\geqslant 0$, $E_H(0)\geqslant 0$, $I_H(0)\geqslant 0$, $V_H(0)\geqslant 0$时,$E_{Ai}(t)\geqslant 0$, $I_{Ai}(t)\geqslant 0$, $V_{Ai}(t)\geqslant 0$, $W_i(t)\geqslant 0$, $S_H(t)\geqslant 0$, $E_H(t)\geqslant 0$, $I_H(t)\geqslant 0$, $V_H(t)\geqslant 0$。定理得证。

1.2 正向不变集

对系统(1)的前5个方程进行动力学分析

$$\begin{cases}\dfrac{\mathrm{d}S_{Ai}}{\mathrm{d}t}=\Lambda_{Ai}-(d_{Ai}+\gamma_{Ai})S_{Ai}+\lambda_{Ai}V_{Ai}-\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})-\alpha_i S_{Ai}W_i-S_{Ai}\sum_{j=1}^{2}a_{ji}+S_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}E_{Ai}}{\mathrm{d}t}=\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})+\alpha_i S_{Ai}W_i-(d_{Ai}+\delta_{Ai})E_{Ai}-E_{Ai}\sum_{j=1}^{2}a_{ji}+E_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}I_{Ai}}{\mathrm{d}t}=\delta_{Ai}E_{Ai}-(d_{Ai}+\mu_{Ai})I_{Ai}-I_{Ai}\sum_{j=1}^{2}a_{ji}+I_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}V_{Ai}}{\mathrm{d}t}=\gamma_{Ai}S_{Ai}-(d_{Ai}+\lambda_{Ai})V_{Ai}-V_{Ai}\sum_{j=1}^{2}a_{ji}+V_{Aj}\sum_{j=1}^{2}a_{ij},\\ \dfrac{\mathrm{d}W_i}{\mathrm{d}t}=r_{A_i}(E_{A_i}+I_{A_i})-e_iW_i-W_i\sum_{j=1}^{2}a_{ji}+W_j\sum_{j=1}^{2}a_{ij}。\end{cases} \tag{2}$$

将系统(2)前4个方程相加

$$(S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai})'=$$
$$\Lambda_{Ai}-d_{Ai}(S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai})-\mu_{Ai}I_{Ai}\leqslant,$$
$$\Lambda_{Ai}-d_{Ai}(S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai})$$

则满足式子:

$$\limsup_{t\to\infty}(S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai})\leqslant\frac{\Lambda_{Ai}}{d_{Ai}},$$
$$\limsup_{t\to\infty}W_i\leqslant\frac{r_{Ai}\Lambda_{Ai}}{d_{Ai}e_i}。$$

由此得到系统(2)的可行域范围

$$\Omega=\Big\{(S_{Ai},E_{Ai},I_{Ai},V_{Ai},W_i)\in\mathbb{R}_+^{10}|S_{Ai},E_{Ai},I_{Ai},V_{Ai},W_i\geqslant0,$$
$$0\leqslant S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai}\leqslant\frac{\Lambda_{Ai}}{d_{Ai}},W_i\leqslant\frac{r_{Ai}\Lambda_{Ai}}{d_{Ai}e_i}\Big\},$$

其中Ω为系统(2)的正向不变集。对于系统(1)的后4个方程,

$$\begin{cases}\dfrac{\mathrm{d}S_H}{\mathrm{d}t}=\Lambda_H+\lambda_HV_H-(d_H+\gamma_H)S_H-\beta_HS_H(E_H+I_H)-c_iS_HW_i,\\ \dfrac{\mathrm{d}E_H}{\mathrm{d}t}=\beta_HS_H(E_H+I_H)+c_iS_HW_1-(d_H+\delta_H)E_H,\\ \dfrac{\mathrm{d}I_H}{\mathrm{d}t}=\delta_HE_H-(d_H+\mu_H)I_H,\\ \dfrac{\mathrm{d}V_H}{\mathrm{d}t}=\gamma_HS_H-(d_H+\lambda_H)V_H。\end{cases} \tag{3}$$

则有

$$\limsup_{t\to\infty}(S_H+E_H+I_H+V_H)=\frac{\Lambda_H}{d_H},$$

可得系统(3)的正向不变集为

$$\Gamma=\Big\{(S_H,E_H,I_H,V_H)\in\mathbb{R}_+^4|S_H+E_H+I_H+V_H\leqslant\frac{\Lambda_H}{d_H}\Big\}。$$

2 平衡点的性质

2.1 无病平衡点

将系统(1)中前5个方程,改写为

$$
\begin{cases}
\dfrac{\mathrm{d}S_{A1}}{\mathrm{d}t}=\Lambda_{A1}-(d_{A1}+\gamma_{A1})S_{A1}+\lambda_{A1}V_{A1}-\beta_{A1}S_{A1}(E_{A1}+I_{A1})-\alpha_1 S_{A1}W_1-a_{21}S_{A1}+a_{12}S_{A2},\\
\dfrac{\mathrm{d}E_{A1}}{\mathrm{d}t}=\beta_{A1}S_{A1}(E_{A1}+I_{A1})+\alpha_1 S_{A1}W_1-(d_{A1}+\delta_{A1})E_{A1}-a_{21}E_{A1}+a_{12}E_{A2},\\
\dfrac{\mathrm{d}I_{A1}}{\mathrm{d}t}=\delta_{A1}E_{A1}-(d_{A1}+\mu_{A1})I_{A1}-a_{21}I_{A1}+a_{12}I_{A2},\\
\dfrac{\mathrm{d}V_{A1}}{\mathrm{d}t}=\gamma_{A1}S_{A1}-(d_{A1}+\lambda_{A1})V_{A1}-a_{21}V_{A1}+a_{12}V_{A2},\\
\dfrac{\mathrm{d}W_1}{\mathrm{d}t}=r_{A1}(E_{A1}+I_{A1})-e_1W_1-a_{21}W_1+a_{12}W_2,\\
\dfrac{\mathrm{d}S_{A2}}{\mathrm{d}t}=\Lambda_{A2}-(d_{A2}+\gamma_{A2})S_{A2}+\lambda_{A2}V_{A2}-\beta_{A2}S_{A2}(E_{A2}+I_{A2})-\alpha_2 S_{A2}W_2-a_{12}S_{A2}+a_{21}S_{A1},\\
\dfrac{\mathrm{d}E_{A2}}{\mathrm{d}t}=\beta_{A2}S_{A2}(E_{A2}+I_{A2})+\alpha_2 S_{A2}W_2-(d_{A2}+\delta_{A2})E_{A2}-a_{12}E_{A2}+a_{21}E_{A1},\\
\dfrac{\mathrm{d}I_{A2}}{\mathrm{d}t}=\delta_{A2}E_{A2}-(d_{A2}+\mu_{A2})I_{A2}-a_{12}I_{A2}+a_{21}I_{A1},\\
\dfrac{\mathrm{d}V_2}{\mathrm{d}t}=\gamma_{A2}S_{A2}-(d_{A2}+\lambda_{A2})V_{A2}-a_{12}V_{A2}+a_{21}V_{A1},\\
\dfrac{\mathrm{d}W_2}{\mathrm{d}t}=r_{A2}(E_{A2}+I_{A2})-e_2W_2-a_{12}W_2+a_{21}W_1。
\end{cases}
\tag{4}
$$

设$f_i=(S_{Ai},E_{Ai},I_{Ai},V_{Ai},W_i)$, $g=(S_H,E_H,I_H,V_H)$, $i=1,2$,则系统(2)的无病平衡点P_A^0为:$P_A^0=(f_1^0,f_2^0)=(S_1^0,0,0,V_1^0,0,S_2^0,0,0,V_2^0,0)$,其中

$$S_1^0=\frac{a_{12}(\Lambda_{A2}+\lambda_{A2}A_2)+(\Lambda_{A1}+\lambda_{A1}A_1)(\lambda_{A2}+h_2+\gamma_{A2})}{-a_{12}a_{21}+(\lambda_{A1}+h_1+\gamma_{A1})(\lambda_{A2}+h_2+\gamma_{A2})},$$

$$S_2^0=\frac{a_{21}(\Lambda_{A1}+\lambda_{A1}A_1)+(\Lambda_{A2}+\lambda_{A2}A_2)(\lambda_{A1}+h_1+\gamma_{A1})}{-a_{12}a_{21}+(\lambda_{A1}+h_1+\gamma_{A1})(\lambda_{A2}+h_2+\gamma_{A2})},$$

$$V_1^0=\frac{a_{12}\gamma_{A2}(\Lambda_{A1}a_{21}+\Lambda_{A2}h_1)+\gamma_{A1}(\Lambda_{A2}a_{12}+\Lambda_{A1}h_2)(\lambda_{A2}+h_2+\gamma_{A2})}{[-a_{12}a_{21}+(\lambda_{A1}+h_1+\gamma_{A1})(\lambda_{A2}+h_2+\gamma_{A2})](h_1h_2-a_{12}a_{21})},$$

$$V_2^0=\frac{a_{21}\gamma_{A1}(\Lambda_{A2}a_{12}+\Lambda_{A1}h_2)+\gamma_{A2}(\Lambda_{A1}a_{21}+\Lambda_{A2}h_1)(\lambda_{A1}+h_1+\gamma_{A1})}{[-a_{12}a_{21}+(\lambda_{A1}+h_1+\gamma_{A1})(\lambda_{A2}+h_2+\gamma_{A2})](h_1h_2-a_{12}a_{21})},$$

$$h_1=d_{A1}+a_{21},$$

$$h_2=d_{A2}+a_{12},$$

$$A_1=\frac{\Lambda_{A1}h_2+\Lambda_{A2}a_{12}}{h_1h_2-a_{12}a_{21}},$$

$$A_2=\frac{\Lambda_{A2}h_1+\Lambda_{A1}a_{21}}{h_1h_2-a_{12}a_{21}}。$$

下证P_A^0的非负性。

证明 已知所有参数非负,在S_1^0,S_2^0,V_1^0,V_2^0中有:

$$-a_{12}a_{21}+(\lambda_{A1}+h_1+\gamma_{A1})(\lambda_{A2}+h_2+\gamma_{A2})=$$
$$\lambda_{A1}\lambda_{A2}+a_{12}(\lambda_{A1}+\gamma_{A1}+d_{A1})+a_{21}(\lambda_{A2}+\gamma_{A2}+d_{A2})+d_{A1}(\lambda_{A1}+\gamma_{A1})+d_{A2}(\lambda_{A2}+\gamma_{A2})\geqslant 0。$$

此外,在V_1^0,V_2^0中,

$$h_1h_2-a_{12}a_{21}=d_{A1}d_{A2}+d_{A1}a_{12}+d_{A2}a_{21}\geqslant 0。$$

由此可得$S_1^0\geqslant 0$,$S_2^0\geqslant 0$,$V_1^0\geqslant 0$,$V_2^0\geqslant 0$,平衡点P_A^0非负。

定义$x=(E_{A1},E_{A2},I_{A1},I_{A2},W_1,W_2)$,其中$x$只包含系统(4)的致病项,考虑以下辅助系统

$$\begin{cases}\dfrac{\mathrm{d}E_{A1}}{\mathrm{d}t}=\beta_{A1}S_{A1}(E_{A1}+I_{A1})+\alpha_1S_{A1}W_1-(d_{A1}+\delta_{A1})E_{A1}-a_{21}E_{A1}+a_{12}E_{A2},\\ \dfrac{\mathrm{d}E_{A2}}{\mathrm{d}t}=\beta_{A2}S_{A2}(E_{A2}+I_{A2})+\alpha_2S_{A2}W_2-(d_{A2}+\delta_{A2})E_{A2}-a_{12}E_{A2}+a_{21}E_{A1},\\ \dfrac{\mathrm{d}I_{A1}}{\mathrm{d}t}=\delta_{A1}E_{A1}-(d_{A1}+\mu_{A1})I_{A1}-a_{21}I_{A1}+a_{12}I_{A2},\\ \dfrac{\mathrm{d}I_{A2}}{\mathrm{d}t}=\delta_{A2}E_{A2}-(d_{A2}+\mu_{A2})I_{A2}-a_{12}I_{A2}+a_{21}I_{A1},\\ \dfrac{\mathrm{d}W_1}{\mathrm{d}t}=r_{A1}(E_{A1}+I_{A1})-e_1W_1-a_{21}W_1+a_{12}W_2,\\ \dfrac{\mathrm{d}W_2}{\mathrm{d}t}=r_{A2}(E_{A2}+I_{A2})-e_2W_2-a_{12}W_2+a_{21}W_1。\end{cases} \tag{5}$$

根据下一代矩阵法[14-15]可得

$$\mathcal{F}=\begin{pmatrix}\beta_{A1}S_{A1}(E_{A1}+I_{A1})+\alpha_1S_{A1}W_1\\ \beta_{A2}S_{A2}(E_{A2}+I_{A2})+\alpha_2S_{A2}W_2\\0\\0\\0\\0\end{pmatrix},\ \mathcal{V}=\begin{pmatrix}(d_{A1}+\delta_{A1}+a_{21})E_{A1}-a_{12}E_{A2}\\(d_{A2}+\delta_{A2}+a_{12})E_{A2}-a_{21}E_{A1}\\-\delta_{A1}E_{A1}+(d_{A1}+\mu_{A1}+a_{21})I_{A1}-a_{12}I_{A2}\\-\delta_{A2}E_{A2}+(d_{A2}+\mu_{A2}+a_{12})I_{A2}-a_{21}I_{A1}\\-r_{A1}(E_{A1}+I_{A1})+(e_1+a_{21})W_1-a_{12}W_2\\-r_{A2}(E_{A2}+I_{A2})+(e_2+a_{12})W_2-a_{21}W_1\end{pmatrix},$$

$$\boldsymbol{F}=\begin{pmatrix}\beta_{A1}S_{A1}&0&\beta_{A1}S_{A1}&0&\alpha_1S_{A1}&0\\0&\beta_{A2}S_{A2}&0&\beta_{A2}S_{A2}&0&\alpha_2S_{A2}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\boldsymbol{V}=\begin{pmatrix}d_{A1}+\delta_{A1}+a_{21}&-a_{12}&0&0&0&0\\-a_{21}&d_{A2}+\delta_{A2}+a_{12}&0&0&0&0\\-\delta_{A1}&0&d_{A1}+\mu_{A1}+a_{21}&-a_{12}&0&0\\0&-\delta_{A2}&-a_{21}&d_{A2}+\mu_{A2}+a_{12}&0&0\\-r_{A1}&0&-r_{A1}&0&e_1+a_{21}&-a_{12}\\0&-r_{A2}&0&-r_{A2}&-a_{21}&e_2+a_{12}\end{pmatrix},$$

定义

$$y_1=d_{A1}+\delta_{A1}+a_{21},y_2=d_{A2}+\delta_{A2}+a_{12},$$

$$y_3=d_{A1}+\mu_{A1}+a_{21},y_4=d_{A2}+\mu_{A2}+a_{12},$$

$$y_5=e_1+a_{21},y_6=e_2+a_{12},$$

$$b_{11}=\frac{y_2\delta_{A1}y_4+a_{12}\delta_{A2}a_{21}}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_3y_4)},b_{12}=\frac{y_2\delta_{A1}a_{12}+a_{12}\delta_{A1}y_3}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_3y_4)},$$

$$b_{13}=\frac{y_2r_{A1}y_6+a_{12}r_{A1}a_{21}}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_5y_6)},b_{14}=\frac{y_2r_{A1}a_{12}+a_{12}r_{A2}y_5}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_5y_6)},$$

$$b_{21}=\frac{a_{21}\delta_{A1}y_4+y_1\delta_{A2}a_{21}}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_3y_4)},b_{22}=\frac{a_{21}\delta_{A1}a_{12}+y_1\delta_{A1}y_3}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_3y_4)},$$

$$b_{23}=\frac{a_{21}r_{A1}y_6+y_1r_{A2}a_{21}}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_5y_6)},b_{24}=\frac{a_{21}r_{A1}a_{12}+y_1r_{A1}y_5}{(a_{12}a_{21}-y_1y_2)(a_{12}a_{21}-y_5y_6)},$$

$$A=-\frac{\beta_{A1}S_{A1}^{0}y_2}{(a_{12}a_{21}-y_1y_2)}+\beta_{A1}S_{A1}^{0}b_{11}+\alpha_1S_{A1}^{0}\left(-\frac{b_{11}r_{A1}y_6+b_{12}r_{A2}a_{21}}{a_{12}a_{21}-y_5y_6}+b_{13}\right),$$

$$B=-\frac{\beta_{A1}S_{A1}^{0}a_{12}}{(a_{12}a_{21}-y_1y_2)}+\beta_{A1}S_{A1}^{0}b_{12}+\alpha_1S_{A1}^{0}\left(-\frac{b_{11}r_{A1}a_{12}+b_{12}r_{A2}y_5}{a_{12}a_{21}-y_5y_6}+b_{14}\right),$$

$$C=-\frac{\beta_{A2}S_{A2}^{0}a_{21}}{(a_{12}a_{21}-y_1y_2)}+\beta_{A2}S_{A2}^{0}b_{21}+\alpha_2S_{A2}^{0}\left(-\frac{b_{21}r_{A1}y_6+b_{22}r_{A2}a_{21}}{a_{12}a_{21}-y_5y_6}+b_{23}\right),$$

$$D=-\frac{\beta_{A2}S_{A2}^{0}y_1}{(a_{12}a_{21}-y_1y_2)}+\beta_{A2}S_{A2}^{0}b_{22}+\alpha_2S_{A2}^{0}\left(-\frac{b_{21}r_{A1}a_{12}+b_{22}r_{A2}y_5}{a_{12}a_{21}-y_5y_6}+b_{24}\right)。$$

则基本再生数可表示为

$$R_0=\rho(FV^{-1})=\frac{A+D+\sqrt{(A+D)^2-4(AD-BC)}}{2},$$

其中ρ表示矩阵的谱半径。根据系统(1)定义$y=(E_{A1},E_{A2},I_{A1},I_{A2},W_1,W_2,E_H,I_H)$,求得其Jacobian矩阵为

$$\begin{pmatrix} \beta_{A1}S_{A1}-k_1 & a_{12} & \beta_{A1}S_{A1} & 0 & \alpha_1S_{A1} & 0 & 0 & 0\\ a_{21} & \beta_{A2}S_{A2}-k_2 & 0 & \beta_{A2}S_{A2} & 0 & \alpha_2S_{A2} & 0 & 0\\ \delta_{A1} & 0 & -k_1 & a_{12} & 0 & 0 & 0 & 0\\ 0 & \delta_{A2} & a_{21} & -k_2 & 0 & 0 & 0 & 0\\ r_{A1} & 0 & r_{A1} & 0 & -(e_1+a_{21}) & a_{12} & 0 & 0\\ 0 & r_{A2} & 0 & r_{A2} & a_{21} & -(e_2+a_{12}) & 0 & 0\\ 0 & 0 & 0 & 0 & cS_H & 0 & k_3 & \beta_HS_H\\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_H & -(d_H+\mu_H) \end{pmatrix}, \tag{6}$$

其中$k_1=(d_{A1}+\delta_{A1})+a_{21}$,$k_2=(d_{A2}+\delta_{A2})+a_{12}$,$k_3=\beta_HS_H-(d_H+\delta_H)$。根据矩阵的性质易得,系统(5)的Jacobian矩阵与(6)式相等,所以系统(1)的前5个方程与后4个方程相独立,则系统(2)和系统(1)有相同的基本再生数,即系统(1)的基本再生数也为R_0。R_0为系统(1)的关键阈值参数,表示了该系统的全局动态。依据系统(2)假设$N_{Ai}=S_{Ai}+E_{Ai}+I_{Ai}+V_{Ai}(i=1,2)$,$V_{Ai}(i=1,2)$,令$a_{21}+d_{A1}=k_1$,$a_{12}+d_{A2}=k_2$。把(3)中的前5个式子和后5个式分别相加得到

$$\begin{cases}\dfrac{\mathrm{d}N_{A1}(t)}{\mathrm{d}t}=\Lambda_{A1}-\mu_{A1}I_{A1}(t)-k_1N_1(t)+a_{12}N_2(t),\\ \dfrac{\mathrm{d}N_{A2}(t)}{\mathrm{d}t}=\Lambda_{A2}-\mu_{A2}I_{A2}(t)-k_2N_2(t)+a_{21}N_1(t),\end{cases} \tag{7}$$

其中N_i表示第i群羊的总数,考虑系统(4)对应的辅助线性系统有

$$\begin{cases}\dfrac{\mathrm{d}\hat{N}_1(t)}{\mathrm{d}t}=\Lambda_{A1}-k_1\hat{N}_1(t)+a_{12}\hat{N}_2(t),\\ \dfrac{\mathrm{d}\hat{N}_2(t)}{\mathrm{d}t}=\Lambda_{A2}-k_2\hat{N}_2(t)+a_{21}\hat{N}_1(t),\end{cases} \tag{8}$$

则系统(5)对应一个平衡点$P^0=(X_1,X_2)$,其对应的特征方程为

$$\lambda^2+(h_1+h_2)\lambda-a_{12}a_{21}=0。 \tag{9}$$

因为$h_1>0,h_2>0$,所以有$h_1+h_2>0$,$h_1h_2-a_{12}a_{21}=d_{A1}d_{A2}+a_{12}d_{A1}+a_{21}d_{A2}>0$,即在(7)式中的所有复根都为负实部,因此$P^0$局部渐近稳定。由此可得$\lim\limits_{t\to\infty}(\hat{N}_1(t),\hat{N}_2(t))=(X_1,X_2)$。由于线性系统的局部稳定性可以等价于全局稳定性,且系统(5)合作不可约。根据比较原理可得对于任意$\varepsilon>0$,存在$t\to+\infty$使得$N_i(t)\leqslant X_i+\varepsilon,(i=1,2)$。这表明,在初值非负的条件下,系统(2)全部的解都将趋于正不变集Ω,即随时间不断增大,系统达到稳定状态。令

$$G=F-V=\begin{pmatrix} \beta_{A1}S_{A1}-h_1 & a_{12} & \beta_{A1}S_{A1} & 0 & \alpha_1 S_{A1} & 0 \\ a_{21} & \beta_{A2}S_{A2}-h_2 & 0 & \beta_{A2}S_{A2} & 0 & \alpha_2 S_{A2} \\ \delta_{A1} & 0 & -h_1 & a_{12} & 0 & 0 \\ 0 & \delta_{A2} & a_{21} & -h_2 & 0 & 0 \\ r_{A1} & 0 & r_{A1} & 0 & e_1+a_{21} & a_{12} \\ 0 & r_{A2} & 0 & r_{A2} & a_{21} & e_2+a_{12} \end{pmatrix},$$

其中 $h_1=d_{A1}+\delta_{A1}+a_{21}$, $h_2=d_{A2}+\delta_{A2}+a_{12}$ 定义 $s(G)=\max\{\mathrm{Re}\lambda\}$,其中 λ 为 G 的特征值。所以 $s(G)$ 是 G 的一个简单特征值[16],并且其 $R_0>1\Leftrightarrow s(G)>0$ 向量均为正。根据 Van den Driessche 和 Watmough 的定理2[14]得到等价条件:$R_0<1\Leftrightarrow s(G)<0$。

定理2 $R_0<1$ 时系统(2)的无病平衡点 P_A^0 是局部渐近稳定的。

证明 在证明系统(2)的无病平衡点 P_A^0 局部稳定性的过程中,首先验证了 van den Driessche 和 Watmough 在(A1)—(A5)的假设[14]。

(A1) 如果 $x\geqslant 0$,则 $F_i, V_i^+, V_i^-\geqslant 0, i=1,\cdots,m$。

(A2) 如果 $x_i=0$,则 $V_i^-=0$。尤其地如果 $x\in X_s$,则有 $V_i^-=0, i=1,\cdots,m$。

(A3) 如果 $i>m$,则 $F_i=0$。

(A4) 如果 $x\in X_s$,则 $F_i=0$ 且 $V_i^+=0, i=1,\cdots,m$。

(A5) 如果 $F(x)$ 设为零,那么 $DF(x_0)$ 的所有特征值均为负实部。

其中假设(A1)—(A4)易证。对于假设(A5),系统(2)在 P_A^0 点的 10×10Jacobian 矩阵为 $J(P_A^0)=\begin{pmatrix} G & 0 \\ J_3 & J_4 \end{pmatrix}$,当 $J_3=-F$,

$$J_4=\begin{pmatrix} -(d_{A1}+\gamma_{A1}+\alpha_1 W_1) & 0 & \lambda_{A1} & 0 \\ 0 & -(d_{A2}+\gamma_{A2}+\alpha_2 W_2) & 0 & \lambda_{A2} \\ \gamma_{A1} & 0 & -(d_{A1}+\gamma_{A1}+a_{21}) & 0 \\ 0 & \gamma_{A2} & 0 & -(d_{A2}+\gamma_{A2}+a_{12}) \end{pmatrix},$$

$J(P_A^0)$ 有负实部。计算 J_4 的特征值有

$$s(J_4)=\max\{-d_{A1},-d_{A2},-(d_{A1}+\gamma_{A1}+\alpha_1 W_1),-(d_{A2}+\gamma_{A2}+\alpha_2 W_2)\}<0。$$

如果 $R_0<1$,那么 $s(G)<0$ 和 $s(J_4)<0$,则无病平衡点 P_A^0 是局部渐近稳定的。

定理3 当 $R_0<1$ 时系统(2)的无病平衡点 P_A^0 在 Ω 内是全局渐近稳定的。

证明 对于系统(2)的第4式有

$$\frac{\mathrm{d}V_{Ai}}{\mathrm{d}t}=\gamma_{Ai}S_{Ai}-(d_{Ai}+\lambda_{Ai})V_{Ai}-V_{Ai}\sum_{j=1}^{n}a_{ji}+V_{Aj}\sum_{j=1}^{n}a_{ij}=$$

$$\gamma_{Ai}(N_i-(E_{Ai}+I_{Ai}+V_{Ai}))-(d_{Ai}+\lambda_{Ai})V_{Ai}-V_{Ai}\sum_{j=1}^{n}a_{ji}+V_{Aj}\sum_{j=1}^{n}a_{ij}\leqslant$$

$$\gamma_{Ai}\frac{\Lambda_{Ai}}{d_{Ai}}-(d_{Ai}+\lambda_{Ai}+\gamma_{Ai})V_{Ai}。$$

选择一个足够小的数 $\varepsilon_1>0$,存在 $t_i>0$,使得对于任意 $t>t_i$,有 $V_{Ai}\leqslant\dfrac{\gamma_{A1}\Lambda_{A1}}{d_{A1}+\lambda_{A1}+\gamma_{A1}}+\varepsilon_1=V_{Ai}^0+\varepsilon_1, i=1,2$。同样的,对于系统(2)的第1式有

$$\frac{\mathrm{d}S_{Ai}}{\mathrm{d}t}=\Lambda_{Ai}-(d_{Ai}+\gamma_{Ai})S_{Ai}+\lambda_{Ai}V_{Ai}-\beta_{Ai}S_{Ai}(E_{Ai}+I_{Ai})-\alpha_i S_{Ai}W_i-S_{Ai}\sum_{j=1}^{n}a_{ji}+S_{Aj}\sum_{j=1}^{n}a_{ij},$$

$$\leqslant\Lambda_{Ai}+\lambda_{Ai}(V_{Ai}^0+\varepsilon_1)-(d_{Ai}+\gamma_{Ai})S_{Ai}。$$

则 $\limsup\limits_{t\to\infty}S_{Ai}^0=\dfrac{\Lambda_{A1}+\lambda_{A1}(V_{A1}^0+\varepsilon_1)}{d_{A1}+\gamma_{A1}}=S_{Ai}^0+\varepsilon_2,(i=1,2)$。对于系统(2)有 $S_{Ai}\leqslant S_{Ai}^0+\varepsilon_2, t>t_i$,此外,

$$\begin{cases}\dfrac{\mathrm{d}E_{Ai}}{\mathrm{d}t}\leqslant(S_{Ai}^0+\varepsilon_2)(\beta_{Ai}(E_{Ai}+I_{Ai})+\alpha_iS_{Ai}W_i)-(d_{Ai}+\delta_{Ai})E_{Ai}-E_{Ai}\sum\limits_{j=1}^2a_{ji}+E_{Aj}\sum\limits_{j=1}^2a_{ij},\\ \dfrac{\mathrm{d}I_{Ai}}{\mathrm{d}t}=\delta_{Ai}E_{Ai}-(d_{Ai}+\mu_{Ai})I_{Ai}-I_{Ai}\sum\limits_{j=1}^2a_{ji}+I_{Aj}\sum\limits_{j=1}^2a_{ij}, \\ \dfrac{\mathrm{d}W_i}{\mathrm{d}t}=r_{A_i}(E_{A_i}+I_{A_i})-e_iW_i-W_i\sum\limits_{j=1}^2a_{ji}+W_j\sum\limits_{j=1}^2a_{ij},\end{cases}\quad i=1,2。\tag{10}$$

对于全部的 $t>t_i$ 考虑如下的辅助系统

$$\begin{cases}\dfrac{\mathrm{d}\hat{E}_{Ai}}{\mathrm{d}t}=(S_{Ai}^0+\varepsilon_2)(\beta_{Ai}(\hat{E}_{Ai}+\hat{I}_{Ai})+\alpha_i\hat{W}_i)-(d_{Ai}+\delta_{Ai})\hat{E}_{Ai}-\hat{E}_{Ai}\sum\limits_{j=1}^2a_{ji}+\hat{E}_{Aj}\sum\limits_{j=1}^2a_{ij},\\ \dfrac{\mathrm{d}\hat{I}_{Ai}}{\mathrm{d}t}=\delta_{Ai}\hat{E}_{Ai}-(d_{Ai}+\mu_{Ai})\hat{I}_{Ai}-\hat{I}_{Ai}\sum\limits_{j=1}^2a_{ji}+\hat{I}_{Aj}\sum\limits_{j=1}^2a_{ij}, \\ \dfrac{\mathrm{d}\hat{W}_i}{\mathrm{d}t}=r_{A_i}(\hat{E}_{A_i}+\hat{I}_{A_i})-e_i\hat{W}_i-\hat{W}_i\sum\limits_{j=1}^2a_{ji}+\hat{W}_j\sum\limits_{j=1}^2a_{ij},\end{cases}\quad i=1,2。\tag{11}$$

定义

$$\boldsymbol{G}_0=\begin{pmatrix}\beta_{A1}&0&\beta_{A1}&0&\alpha_1&0\\0&\beta_{A2}&0&\beta_{A2}&0&\alpha_2\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}。$$

设 $\boldsymbol{G}_1=\boldsymbol{G}+\varepsilon_2\boldsymbol{G}_0$。根据 Van den Driessche 和 Watmough 的定理 2[14] 推导可得,当且仅当 $s(\boldsymbol{G})<0$ 时 $R_0<1$。因此存在足够小的 $\varepsilon_2>0$ 使得 $s(\boldsymbol{G})<0$。根据 Perron-Frobenius 定理当 $s(\boldsymbol{G}_1)<0$ 时,矩阵 $\boldsymbol{G}_1$ 的所有特征值均有负实部。因此当 $t\to+\infty$ 时 $(\hat{E}_{A1}(t),\hat{E}_{A2}(t),\hat{I}_{A1}(t),\hat{I}_{A2}(t),\hat{W}_1(t),\hat{W}_2(t))\to(0,0,0,0,0,0)$,这表明系统(2)的零解是全局渐近稳定的。

根据 Smith 和 Waltman[17] 的比较原理可知当 $t\to+\infty$ 时 $(E_{A1}(t),E_{A2}(t),I_{A1}(t),I_{A2}(t),W_1(t),W_2(t))\to(0,0,0,0,0,0)$。同时由 Thieme 的渐近自治系统理论[16] 可知当 $t\to+\infty$ 时 $(S_{A1},S_{A2},V_{A1},V_{A2})\to(S_{A1}^0,S_{A2}^0,V_{A1}^0,V_{A2}^0)$。因此当 $R_0<1$ 时 P_A^0 是全局吸引的,则当 $R_0<1$ 时,系统(2)的无病平衡点 P_A^0 是全局渐近稳定的。定理3得证。

对于系统(3),计算得无病平衡点为 $P_H^0=(g^0)=(S_H^0,0,0,V_H^0)$,其中

$$V_H=\frac{\Lambda_H\gamma_H}{d_H(\gamma_H+d_H+\lambda_H)},\ S_H=\frac{\Lambda_Hd_H(\gamma_H+d_H+\lambda_H)+\gamma_H\lambda_H\Lambda_H}{d_H(\gamma_H+d_H+\lambda_H)(\gamma_H+d_H)}。$$

根据定理 2,当 $t\to+\infty$ 时,$E_{Ai}(t)\to0$ 和 $I_{Ai}(t)\to0$。同样的证明方法可得,当 $t\to+\infty$ 时 $E_H(t)\to0$ 和 $I_H(t)\to0$,则系统(3)的极限系统为

$$\begin{cases}\dfrac{\mathrm{d}S_H}{\mathrm{d}t}=\Lambda_H+\lambda_HV_H-(d_H+\gamma_H)S_H,\\ \dfrac{\mathrm{d}V_H}{\mathrm{d}t}=\gamma_HS_H-(d_H+\lambda_H)V_H。\end{cases}\tag{12}$$

根据 Hurwitz 判据[18]可知其平衡点 (S_H^0,V_H^0) 是局部渐近稳定的。由于系统(12)是一个线性系统,则 (S_H^0,V_H^0) 是全局渐近稳定的。根据渐近自治系统理论[16],当 $R_0<1$ 时,P_H^0 是全局渐近稳定的。综上 P^0 是全局渐近稳定的。

2.2 地方病平衡点

本节给出系统(1)地方病平衡点的性质。

定理4 假设$[a_{ij}]$和$[a_{ji}]$是不可约的,则当$R_0>1$时,地方病平衡点 $P^*=(S_{Ai}^*,E_{Ai}^*,I_{Ai}^*,V_{Ai}^*,V_{Ai}^*,W_i^*,S_H^*,$

$E_H^*, I_H^*, V_H^*,), i=1,2$是全局渐近稳定的。

证明 系统(1)可被转化为如下形式:

$$
\begin{cases}
\dfrac{\mathrm{d}S_{Ai}}{\mathrm{d}t}=S_{Ai}\left(\dfrac{\Lambda_{Ai}}{S_{Ai}^*}\left(\dfrac{S_{Ai}^*}{S_{Ai}}-1\right)+\dfrac{\lambda_{Ai}V_{Ai}^*}{S_{Ai}^*}\left(\dfrac{V_{Ai}S_{Ai}^*}{V_{Ai}^*S_{Ai}}-1\right)-\alpha_i W_i{}^*\left(\dfrac{W_i}{W_i{}^*}-1\right)\right. \\
\qquad\left.-\beta_{Ai}\left(E_{Ai}^*\left(\dfrac{E_{Ai}}{E_{Ai}^*}-1\right)+I_{Ai}^*\left(\dfrac{I_{Ai}}{I_{Ai}^*}-1\right)\right)+\sum_{j=1}^n a_{ij}\dfrac{S_{Aj}^*}{S_{Ai}^*}\left(\dfrac{S_{Aj}S_{Ai}^*}{S_{Aj}^*S_{Ai}}-1\right)\right), \\
\dfrac{\mathrm{d}E_{Ai}}{\mathrm{d}t}=\dfrac{E_{Ai}S_{Ai}^*}{E_{Ai}^*}\left(\beta_{Ai}E_{Ai}^*\left(\dfrac{S_{Ai}^*}{S_{Ai}}-1\right)+\beta_{Ai}I_{Ai}^*\left(\dfrac{S_{Ai}I_{Ai}E_{Ai}^*}{S_{Ai}^*I_{Ai}^*E_{Ai}^*}-1\right)+\alpha_i W_i^*\left(\dfrac{S_{Ai}W_iE_{Ai}^*}{S_{Ai}^*W_i^*E_{Ai}^*}-1\right)+\sum_{j=1}^n a_{ij}\left(E_{Aj}^*\left(\dfrac{S_{Ai}}{S_{Ai}^*}-1\right)\right)\right), \\
\dfrac{\mathrm{d}I_{Ai}}{\mathrm{d}t}=\dfrac{\delta_{Ai}E_{Ai}^*I_{Ai}}{I_{Ai}^*}\left(\dfrac{E_{Ai}I_{Ai}^*}{E_{Ai}^*I_{Ai}}-1\right)+\dfrac{\sum_{j=1}^n a_{ij}I_{Aj}^*I_{Ai}}{I_{Ai}^*}\left(\dfrac{I_{Aj}I_{Ai}^*}{I_{Aj}^*I_{Ai}}-1\right), \\
\dfrac{\mathrm{d}V_{Ai}}{\mathrm{d}t}=\dfrac{\gamma_{Ai}S_{Ai}^*V_{Ai}}{V_{Ai}^*}\left(\dfrac{S_{Ai}V_{Ai}^*}{S_{Ai}^*V_{Ai}}-1\right)+\dfrac{\sum_{j=1}^n a_{ij}V_{Aj}^*V_{Ai}}{V_{Ai}^*}\left(\dfrac{V_{Aj}V_{Ai}^*}{V_{Aj}^*V_{Ai}}-1\right), \\
\dfrac{\mathrm{d}W_i}{\mathrm{d}t}=\dfrac{r_{A_i}W_i}{W_i^*}\left(E_{Ai}^*\left(\dfrac{E_{Ai}W_i^*}{E^*W_i}-1\right)+I_{Ai}^*\left(\dfrac{I_{Ai}W_i^*}{I^*W_i}-1\right)\right)+\dfrac{\sum_{j=1}^n a_{ij}W_i}{W_i^*}\left(W_j^*\left(\dfrac{W_jW_i^*}{W_j^*W_i}-1\right)\right), \\
\dfrac{\mathrm{d}S_H}{\mathrm{d}t}=S_H\left(\dfrac{\Lambda_H}{S_H^*}\left(\dfrac{S_H^*}{S_H}-1\right)+\dfrac{\lambda_HV_H^*}{S_H^*}\left(\dfrac{V_HS_H^*}{V_H^*S_H}-1\right)-c_iW_i{}^*\left(\dfrac{W_i}{W_i{}^*}-1\right)-\beta_H\left(E_H^*\left(\dfrac{E_H}{E_H^*}-1\right)+I_H^*\left(\dfrac{I_H}{I_H^*}-1\right)\right)\right), \\
\dfrac{\mathrm{d}E_H}{\mathrm{d}t}=\dfrac{E_HS_H^*}{E_H^*}\left(\beta_HE_H^*\left(\dfrac{S_H^*}{S_H}-1\right)+\beta_HI_H^*\left(\dfrac{S_HI_HE_H^*}{S_H^*I_H^*E_H^*}-1\right)+c_iW_i^*\left(\dfrac{S_HW_iE_H^*}{S_H^*W_i^*E_H^*}-1\right)\right), \\
\dfrac{\mathrm{d}I_H}{\mathrm{d}t}=\dfrac{\delta_HE_H^*I_H}{I_H^*}\left(\dfrac{E_HI_H^*}{E_H^*I_H}-1\right), \\
\dfrac{\mathrm{d}V_H}{\mathrm{d}t}=\dfrac{\gamma_HS_H^*V_H}{V_H^*}\left(\dfrac{S_HV_H^*}{S_H^*V_H}-1\right)。
\end{cases}
$$

定义Lyapunov函数为

$$
\begin{gathered}
L=S_{Ai}-S_{Ai}^*-S_{Ai}^*\ln\frac{S_{Ai}}{S_{Ai}^*}+V_{Ai}-V_{Ai}^*-V_{Ai}^*\ln\frac{V_{Ai}}{V_{Ai}^*}+E_{Ai}-E_{Ai}^*-E_{Ai}^*\ln\frac{E_{Ai}}{E_{Ai}^*}+ \\
M_1\left(I_{Ai}-I_{Ai}^*-I_{Ai}^*\ln\frac{I_{Ai}}{I_{Ai}^*}\right)+N\left(W_i-W_i^*-W_i^*\ln\frac{W_i}{W_i^*}\right)+S_H-S_H^*-S_H^*\ln\frac{S_H}{S_H^*}+V_H-V_H^*- \\
V_H^*\ln\frac{V_H}{V_H^*}+E_H-E_H^*-E_H^*\ln\frac{E_H}{E_H^*}+M_2\left(I_H-I_H^*-I_H^*\ln\frac{I_H}{I_H^*}\right),
\end{gathered}
$$

则L的导数为

$$
\begin{gathered}
\frac{\mathrm{d}L}{\mathrm{d}t}=\left(1-\frac{S_{Ai}^*}{S_{Ai}}\right)S_{Ai}'+\left(1-\frac{V_{Ai}^*}{V_{Ai}}\right)V_{Ai}'+\left(1-\frac{E_{Ai}^*}{E_{Ai}}\right)E_{Ai}'+M_1\left(1-\frac{I_{Ai}^*}{I_{Ai}}\right)I_{Ai}'+N\left(1-\frac{W_i^*}{W_i}\right)W_i'+ \\
\left(1-\frac{S_H^*}{S_H}\right)S_H'+\left(1-\frac{V_H^*}{V_H}\right)V_H'+\left(1-\frac{E_H^*}{E_H}\right)E_H'+M_2\left(1-\frac{I_H^*}{I_H}\right)I_{Ai}'= \\
(S_{Ai}-S_{Ai}^*)\left(\frac{\Lambda_{Ai}}{S_{Ai}^*}\left(\frac{S_{Ai}^*}{S_{Ai}}-1\right)+\frac{\lambda_{Ai}V_{Ai}^*}{S_{Ai}^*}\left(\frac{V_{Ai}S_{Ai}^*}{V_{Ai}^*S_{Ai}}-1\right)-\beta_{Ai}\left(E_{Ai}^*\left(\frac{E_{Ai}}{E_{Ai}^*}-1\right)+I_{Ai}^*\left(\frac{I_{Ai}}{I_{Ai}^*}-1\right)\right)-\right. \\
\left.\alpha_iW_i{}^*\left(\frac{W_i}{W_i{}^*}-1\right)+\sum_{j=1}^n a_{ij}\frac{S_{Aj}^*}{S_{Ai}^*}\left(\frac{S_{Aj}S_{Ai}^*}{S_{Aj}^*S_{Ai}}-1\right)\right)+ \\
(V_{Ai}-V_{Ai}^*)\left(\frac{\gamma_{Ai}S_{Ai}^*}{V_{Ai}^*}\left(\frac{S_{Ai}V_{Ai}^*}{S_{Ai}^*V_{Ai}}-1\right)+\frac{\sum_{j=1}^n a_{ij}V_{Aj}^*}{V_{Ai}^*}\left(\frac{V_{Aj}V_{Ai}^*}{V_{Aj}^*V_{Ai}}-1\right)\right)+
\end{gathered}
$$

$$M_1(I_{Ai}-I_{Ai}^*)\left(\frac{\delta_{Ai}E_{Ai}^*}{I_{Ai}^*}\left(\frac{E_{Ai}I_{Ai}^*}{E_{Ai}^*I_{Ai}}-1\right)+\frac{\sum_{j=1}^n a_{ij}I_{Aj}^*}{I_{Ai}^*}\left(\frac{I_{Aj}I_{Ai}^*}{I_{Aj}^*I_{Ai}}-1\right)\right)+\frac{S_{Ai}^*}{E_{Ai}^*}(E_{Ai}-E_{Ai}^*)$$

$$\left(\beta_{Ai}E_{Ai}^*\left(\frac{S_{Ai}^*}{S_{Ai}}-1\right)+\beta_{Ai}I_{Ai}^*\left(\frac{S_{Ai}I_{Ai}E_{Ai}^*}{S_{Ai}^*I_{Ai}^*E_{Ai}^*}-1\right)+\alpha_iW_i^*\left(\frac{S_{Ai}W_iE_{Ai}^*}{S_{Ai}^*W_i^*E_{Ai}^*}-1\right)+\sum_{j=1}^n a_{ij}\left(E_{Aj}^*\left(\frac{S_{Ai}}{S_{Ai}^*}-1\right)\right)\right)+$$

$$N(W-W_i^*)\left(\frac{r_{Ai}E_{Ai}^*}{W_i^*}\left(\frac{E_{Ai}W_i^*}{E^*W_i}-1\right)+I_{Ai}^*\left(\frac{I_{Ai}W_i^*}{I^*W_i}-1\right)+\frac{\sum_{j=1}^n a_{ij}}{W_i^*}\left(W_j^*\left(\frac{W_jW_i^*}{W_j^*W_i}-1\right)\right)\right)+$$

$$(S_H-S_H^*)\left(\frac{\Lambda_H}{S_H^*}\left(\frac{S_H^*}{S_H}-1\right)+\frac{\lambda_HV_H^*}{S_H^*}\left(\frac{V_HS_H^*}{V_H^*S_H}-1\right)-\beta_H\left(E_H^*\left(\frac{E_H}{E_H^*}-1\right)+I_H^*\left(\frac{I_H}{I_H^*}-1\right)\right)-c_iW_i^*\left(\frac{W_i}{W_i^*}-1\right)\right)+$$

$$(V_H-V_H^*)\left(\frac{\gamma_HS_H^*}{V_H^*}\left(\frac{S_HV_H^*}{S_H^*V_H}-1\right)\right)+M_2(I_H-I_H^*)\left(\frac{\delta_HE_H^*}{I_H^*}\left(\frac{E_HI_H^*}{E_H^*I_H}-1\right)\right)+$$

$$\frac{S_H^*}{E_H^*}(E_H-E_H^*)\left(\beta_HE_H^*\left(\frac{S_H^*}{S_H}-1\right)+\beta_HI_H^*\left(\frac{S_HI_HE_H^*}{S_H^*I_H^*E_H^*}-1\right)+c_iW_i^*\left(\frac{S_HW_iE_H^*}{S_H^*W_i^*E_H^*}-1\right)\right)=B'+C',$$

$$B'=\Lambda_{Ai}(S_{Ai}-S_{Ai}^*)\left(\frac{1}{S_{Ai}}-\frac{1}{S_{Ai}^*}\right)+\lambda_{Ai}(S_{Ai}-S_{Ai}^*)\left(\frac{V_{Ai}}{S_{Ai}}-\frac{V_{Ai}^*}{S_{Ai}^*}\right)+$$

$$\gamma_{Ai}(S_{Ai}-S_{Ai}^*)\left(\frac{S_{Ai}}{V_{Ai}}-\frac{S_{Ai}^*}{V_{Ai}^*}\right)+\sum_{j=1}^n a_{ij}(S_{Ai}-S_{Ai}^*)\left(\frac{S_{Aj}}{S_{Ai}}-\frac{S_{Aj}^*}{S_{Ai}^*}\right)+$$

$$\sum_{j=1}^n a_{ij}(V_{Ai}-V_{Ai}^*)\left(\frac{V_{Aj}}{V_{Ai}}-\frac{V_{Ai}^*}{V_{Aj}^*}\right)+\Lambda_H(S_H-S_H^*)\left(\frac{1}{S_H}-\frac{1}{S_H^*}\right)+$$

$$\lambda_H(S_H-S_H^*)\left(\frac{V_H}{S_H}-\frac{V_H^*}{S_H^*}\right)+\gamma_H(S_H-S_H^*)\left(\frac{S_H}{V_H}-\frac{S_H^*}{V_H^*}\right)=$$

$$\lambda_{Ai}V_{Ai}^*\left(2-\frac{S_{Ai}^*V_{Ai}}{S_{Ai}V_{Ai}^*}-\frac{S_{Ai}V_{Ai}^*}{S_{Ai}^*V_{Ai}}\right)+(d_{Ai}-\lambda_{Ai})V_{Ai}{}^*\left(3-\frac{S_{Ai}^*}{S_{Ai}}-\frac{V_{Ai}}{V_{Ai}^*}-\frac{S_{Ai}V_{Ai}^*}{S_{Ai}^*V_{Ai}}\right)+$$

$$((d_{Ai}-\lambda_{Ai})S_{Ai}^*+\beta_{Ai}S_{Ai}^*(E_{Ai}^*+I_{Ai}^*)+\alpha_iS_{Ai}^*W_i^*)\left(2-\frac{S_{Ai}^*}{S_{Ai}}-\frac{S_{Ai}}{S_{Ai}^*}\right)+$$

$$\lambda_HV_H^*\left(2-\frac{S_H^*V_H}{S_HV_H^*}-\frac{S_HV_H^*}{S_H^*V_H}\right)+(d_H-\lambda_H)V_H^*\left(3-\frac{S_H^*}{S_H}-\frac{V_H}{V_H^*}-\frac{S_HV_H^*}{S_H^*V_H}\right)+$$

$$((d_H-\lambda_H)S_H^*+\beta_HS_H^*(E_H^*+I_H^*)+c_iS_H^*W_i^*)\left(2-\frac{S_H^*}{S_H}-\frac{S_H}{S_H^*}\right)+$$

$$\sum_{j=1}^n a_{ij}(S_{Ai}-S_{Ai}^*)\left(\frac{S_{Aj}}{S_{Ai}}-\frac{S_{Aj}^*}{S_{Ai}^*}\right)+\sum_{j=1}^n a_{ij}(V_{Ai}-V_{Ai}^*)\left(\frac{V_{Aj}}{V_{Ai}}-\frac{V_{Ai}^*}{V_{Aj}^*}\right),$$

$$C'=N_1\left(r_{Ai}E_{Ai}^*\left(1+\frac{E_{Ai}}{E_{Ai}^*}-\frac{W_i}{W_i^*}-\frac{E_{Ai}W_i^*}{E_{Ai}^*W_i}\right)+r_{Ai}I_{Ai}^*\left(1+\frac{I_{Ai}}{I_{Ai}^*}-\frac{W_i}{W_i^*}-\frac{I_{Ai}W_i^*}{I_{Ai}^*W_i}\right)\right)+$$

$$N_2\left(\sum_{j=1}^n a_{ij}I_{Ai}^*\left(1+\frac{W_j}{W_j^*}-\frac{W_i}{W_i^*}-\frac{W_jW_i^*}{W_j^*W_i}\right)\right)+M_1\delta_{Ai}E_{Ai}^*\left(1+\frac{E_{Ai}}{E_{Ai}^*}-\frac{I_{Ai}}{I_{Ai}^*}-\frac{E_{Ai}I_{Ai}^*}{E_{Ai}^*I_{Ai}}\right)+$$

$$M_3\sum_{j=1}^n a_{ij}I_{Aj}^*\left(1+\frac{I_{Aj}}{I_{Aj}^*}-\frac{I_{Ai}}{I_{Ai}^*}-\frac{I_{Aj}I_{Ai}^*}{I_{Aj}^*I_{Ai}}\right)+M_2\delta_HE_H^*\left(1+\frac{E_H}{E_H^*}-\frac{I_H}{I_H^*}-\frac{E_HI_H^*}{E_H^*I_H}\right)+$$

$$\beta_{Ai}S_{Ai}^*I_{Ai}^*\left(\frac{I_{Ai}}{I_{Ai}^*}+\frac{S_{Ai}}{S_{Ai}^*}-\frac{E_{Ai}}{E_{Ai}^*}-\frac{S_{Ai}I_{Ai}E_{Ai}^*}{S_{Ai}^*I_{Ai}^*E_{Ai}}\right)+\alpha_iS_{Ai}^*W_i^*\left(\frac{W_i}{W_i^*}+\frac{S_{Ai}}{S_{Ai}^*}-\frac{E_{Ai}}{E_{Ai}^*}-\frac{S_{Ai}W_iE_{Ai}^*}{S_{Ai}^*W_i^*E_{Ai}}\right)+$$

$$\sum_{j=1}^n a_{ij}\left(E_{Aj}^*\left(\frac{S_{Ai}}{S_{Ai}^*}-1\right)\right)+\beta_HS_H^*I_H^*\left(\frac{I_H}{I_H^*}+\frac{S_H}{S_H^*}-\frac{E_H}{E_H^*}-\frac{S_HI_HE_H^*}{S_H^*I_H^*E_H}\right)+c_iS_H^*W_i^*\left(\frac{W_i}{W_i^*}+\frac{S_H}{S_H^*}-\frac{E_H}{E_H^*}-\frac{S_HW_iE_H^*}{S_H^*W_i^*E_H}\right)。$$

考虑如下等式

$$
\begin{cases}
\alpha_i S_{Ai}^* W_i^* - r_{Ai} N\left(E_{Ai}^* + I_{Ai}^*\right) \dfrac{W_i}{W_i^{\,*}} = 0, \\
\left(\beta_{Ai} S_{Ai}^* I_{Ai}^* - M_1 \delta_{Ai} E_{Ai}^* + r_{Ai} N I_{Ai}^*\right) \dfrac{I_{Ai}}{I_{Ai}^*} = 0, \\
\left(\left(M_1 \delta_{Ai} E_{Ai}^* + r_{Ai} N I_{Ai}^*\right) - \alpha_i S_{Ai}^* V_i^* - \beta_{Ai} S_{Ai}^* V_{Ai}^*\right) \dfrac{E_{Ai}}{E_{Ai}^*} = 0, \\
\left(\beta_H S_H^* I_H^* - M_2 \delta_H E_H^* + r_H N I_H^*\right) \dfrac{I_H}{I_H^*} = 0, \\
\left(\left(M_2 \delta_H E_H^* + r_H N I_H^*\right) - c_i S_H^* V_H^* - \beta_H S_H^* V_H^*\right) \dfrac{E_H}{E_H^*} = 0。
\end{cases} \tag{13}
$$

则有

$$
N_1 = \frac{\alpha_i S_{Ai}^* W_i^{\,*}}{r_{Ai}\left(E_{Ai}^* + I_{Ai}^*\right)}, N_2 = \frac{\alpha_i S_{Ai}^* W_i^{\,*}}{\sum_{j=1}^n a_{ij}\left(E_{Ai}^* + I_{Ai}^*\right)}, M_1 = \frac{S_{Ai}^* I_{Ai}^*\left(\beta_{Ai}\left(E_{Ai}^* + I_{Ai}^*\right)\right) + \alpha_i W_i^{\,*}}{\delta_{Ai} E_{Ai}^*\left(E_{Ai}^* + I_{Ai}^*\right)},
$$

$$
M_2 = \frac{S_H^* I_H^*\left(\beta_H\left(E_H^* + I_H^*\right)\right) + c_i W_i^{\,*}}{\delta_H E_H^*\left(E_H^* + I_H^*\right)}, M_3 = \frac{S_{Ai}^* I_{Ai}^*\left(\beta_{Ai}\left(E_{Ai}^* + I_{Ai}^*\right)\right) + \alpha_i W_i^{\,*}}{\sum_{j=1}^n a_{ij} I_{Aj}^*\left(E_{Ai}^* + I_{Ai}^*\right)}。
$$

把 M_1, M_2, M_3, N_1, N_2 代入 C' 得

$$
C' = \beta_{Ai} S_{Ai}^* I_{Ai}^* \left(4 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} I_{Ai}^*}{E_{Ai}^* I_{Ai}} - \frac{S_{Ai} I_{Ai} E_{Ai}^*}{S_{Ai}^* I_{Ai}^* E_{Ai}} - \frac{I_{Aj}}{I_{Aj}^*} - \frac{I_{Ai}}{I_{Ai}^*} - \frac{I_{Aj} I_{Ai}^*}{I_{Aj}^* I_{Ai}}\right) +
$$

$$
\frac{\alpha_i S_{Ai}^* W_i^* E_{Ai}^*}{E_{Ai}^* + I_{Ai}^*}\left(3 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} W_i^*}{E_{Ai}^* W_i} - \frac{S_{Ai} W_i E_{Ai}^*}{S_{Ai}^* W_i^* E_{Ai}}\right) + \frac{\alpha_i S_{Ai}^* W_i^* I_{Aj}^*}{E_{Ai}^* + I_{Ai}^*}\left(3 - \frac{W_j}{W_j^*} - \frac{W_i}{W_i^*} - \frac{W_j W_i^*}{W_j^* W_i}\right) +
$$

$$
\frac{\alpha_i S_{Ai}^* W_i^* I_{Ai}^*}{E_{Ai}^* + I_{Ai}^*}\left(4 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} I_{Ai}^*}{E_{Ai}^* I_{Ai}} - \frac{I_{Ai} W_i^*}{I_{Ai}^* W_i} - \frac{S_{Ai} W_i E_{Ai}^*}{S_{Ai}^* W_i^* E_{Ai}}\right) - \left(\beta_{Ai} S_{Ai}^* I_{Ai}^* + \alpha_i S_{Ai}^* W_i^*\right)\left(2 - \frac{S_{Ai}^*}{S_{Ai}} - \frac{S_{Ai}}{S_{Ai}^*}\right)。
$$

因此,L 沿系统(13)解的导数为

$$
\frac{\mathrm{d}L}{\mathrm{d}t} = B' + C' = \lambda_{Ai} V_{Ai}^{\,*}\left(2 - \frac{S_{Ai}^* V_{Ai}}{S_{Ai} V_{Ai}^*} - \frac{S_{Ai} V_{Ai}^*}{S_{Ai}^* V_{Ai}}\right) + (d_{Ai} - \lambda_{Ai}) V_{Ai}^{\,*}\left(3 - \frac{S_{Ai}^*}{S_{Ai}} - \frac{V_{Ai}}{V_{Ai}^*} - \frac{S_{Ai} V_{Ai}^*}{S_{Ai}^* V_{Ai}}\right) +
$$

$$
\left((d_{Ai} - \lambda_{Ai}) S_{Ai}^* + \beta_{Ai} S_{Ai}^*\left(E_{Ai}^* + I_{Ai}^*\right) + \alpha_i S_{Ai}^* W_i^*\right)\left(2 - \frac{S_{Ai}^*}{S_{Ai}} - \frac{S_{Ai}}{S_{Ai}^*}\right) + \lambda_H V_H^{\,*}\left(2 - \frac{S_H^* V_H}{S_H V_H^*} - \frac{S_H V_H^*}{S_H^* V_H}\right) +
$$

$$
(d_H - \lambda_H) V_H^{\,*}\left(3 - \frac{S_H^*}{S_H} - \frac{V_H}{V_H^*} - \frac{S_H V_H^*}{S_H^* V_H}\right) + \left((d_H - \lambda_H) S_H^* + \beta_H S_H^*\left(E_H^* + I_H^*\right) + c_i S_H^* W_i^*\right)\left(2 - \frac{S_H^*}{S_H} - \frac{S_H}{S_H^*}\right) +
$$

$$
\sum_{j=1}^n a_{ij}\left(S_{Ai} - S_{Ai}^*\right)\left(\frac{S_{Aj}}{S_{Ai}} - \frac{S_{Aj}^*}{S_{Ai}^*}\right) + \sum_{j=1}^n a_{ij}\left(V_{Ai} - V_{Ai}^*\right)\left(\frac{V_{Aj}}{V_{Ai}} - \frac{V_{Ai}^*}{V_{Aj}^*}\right) +
$$

$$
\beta_{Ai} S_{Ai}^* I_{Ai}^*\left(4 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} I_{Ai}^*}{E_{Ai}^* I_{Ai}} - \frac{S_{Ai} I_{Ai} E_{Ai}^*}{S_{Ai}^* I_{Ai}^* E_{Ai}} - \frac{I_{Aj}}{I_{Aj}^*} - \frac{I_{Ai}}{I_{Ai}^*} - \frac{I_{Aj} I_{Ai}^*}{I_{Aj}^* I_{Ai}}\right) +
$$

$$
\frac{\alpha_i S_{Ai}^* W_i^* E_{Ai}^*}{E_{Ai}^* + I_{Ai}^*}\left(3 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} W_i^*}{E_{Ai}^* W_i} - \frac{S_{Ai} W_i E_{Ai}^*}{S_{Ai}^* W_i^* E_{Ai}}\right) + \frac{\alpha_i S_{Ai}^* W_i^* I_{Aj}^*}{E_{Ai}^* + I_{Ai}^*}\left(3 - \frac{W_j}{W_j^*} - \frac{W_i}{W_i^*} - \frac{W_j W_i^*}{W_j^* W_i}\right) +
$$

$$
\frac{\alpha_i S_{Ai}^* W_i^* I_{Ai}^*}{E_{Ai}^* + I_{Ai}^*}\left(4 - \frac{S_{Ai}}{S_{Ai}^*} - \frac{E_{Ai} I_{Ai}^*}{E_{Ai}^* I_{Ai}} - \frac{I_{Ai} W_i^*}{I_{Ai}^* W_i} - \frac{S_{Ai} W_i E_{Ai}^*}{S_{Ai}^* W_i^* E_{Ai}}\right) - \left(\beta_{Ai} S_{Ai}^* I_{Ai}^* + \alpha_i S_{Ai}^* W_i^*\right)\left(2 - \frac{S_{Ai}^*}{S_{Ai}} - \frac{S_{Ai}}{S_{Ai}^*}\right) \leqslant 0。
$$

当 $R_0 > 1$ 时,表达式 $\frac{\mathrm{d}L}{\mathrm{d}t} < 0$。当且仅当 $S_{Ai} = S_{Ai}^*$, $E_{Ai} = E_{Ai}^*$, $I_{Ai} = I_{Ai}^*$, $V_{Ai} = V_{Ai}^*$, $W_i = W_i^*$, $S_H = S_H^*$, $E_H = E_H^*$, $I_H = I_H^*$, $V_H = V_H^*$ 时,$\frac{\mathrm{d}L}{\mathrm{d}t} = 0$。根据 LaSalle 的不变集原理[19],当 $R_0 > 1$ 时,P^* 是

全局渐近稳定的。

在文献[20-23]中使用了相似的Lyapunov函数模型。本节参照文献[20]中的Lyapunov函数模型,证明了具有两羊群间运输的人羊混合交叉感染模型的地方病平衡的全局稳定性。

3 数值模拟

在流行病模型的计算中,疾病动态的阈值由基本再生数R_0表示,主要目的是通过使基本再生数R_0小于1来达到控制疾病的目的,所以了解基本再生数是如何依赖于模型中的参数的尤为重要。在下面的结果中,通过数值模拟来证明基本再生数R_0是疾病灭绝和持续的全局阈值参数。

本节中使用系统(2)的解并结合内蒙古自治区为例来做参数估计。所有参数都以年为单位,参数值的解释如下:

根据内蒙古自治区统计局2010—2020年的数据[24]得到人口年平均出生率为$\Lambda_H=3\,480$,平均死亡率为$d_H=0.007\,23$。根据兴安盟统计局2015年至2019年的数据[25]计算羊的平均死亡率和平均出生率,得到两组扎赉特和乌兰浩特羊群平均出生率$\Lambda_{A1}=489\,992$,$\Lambda_{A2}=135\,776$和$\Lambda_{A1}=762\,578$,$\Lambda_{A2}=234\,213$。羊群的自然死亡率为$d_{A1}=0.335\,8$,$d_{A2}=0.487\,3$。根据《兴安盟统计年鉴》[25],计算从扎赉特运输到乌兰浩特的羊的调运系数为$a_{21}=0.409\,2$,从乌兰浩特到扎赉特的羊的调运系数为$a_{12}=0.507\,1$。

文献[26]指出,几乎所有患布病羊只在一年内死亡,由此设置死亡率$\mu_{A1}=\mu_{A2}=1$。文献[27]提到疫苗的保护期为三年,设置免疫失效率$\lambda_{A1}=0.3$,$\lambda_{A2}=0.3$,$\lambda_H=0.3$。根据米景川等[28]的研究,疫苗的免疫覆盖率为0.316,则有$\gamma_{A1}=0.316$,$\gamma_{A2}=0.316$,$\gamma_H=0.316$。根据文献[29],设$\delta_{A1}=1$, $\delta_{A2}=1$, $\beta_{A1}=1\times10^{-7}$, $\beta_{A2}=1\times10^{-7}$, $\alpha_1=6\times10^{-8}$, $\alpha_2=6\times10^{-8}$, $c=1.745\,3\times10^{-12}$, $e_1=3.6$,$e_2=3.6$,$r_{Ai}=15$,$r_{A2}=15$。由于人类感染布鲁氏菌的潜伏期一般为两周,人们通常在潜伏期内没有症状。因此,大多数人不能及时接受治疗进入感染期,故$\delta_H=1$。最后通过拉丁超立方抽样和马尔可夫链蒙特卡罗模拟估计出$\beta_H=3.164\,7\times10^{-7}$。

根据以上参数,使用python求解ODE方程,分别求得$R_0=0.356<1$和$R_0=2.439>1$,数值模拟出P^0(见图1)和P^*(见图2)的时间序列图以及$S(t)$和$I(t)$在不同初始条件下的相图(图3、图4)。

图5分别模拟了六组参数对R_0的影响,由(a)—(e)可知易感羊群i的出生率Λ_{Ai},感染羊群i对

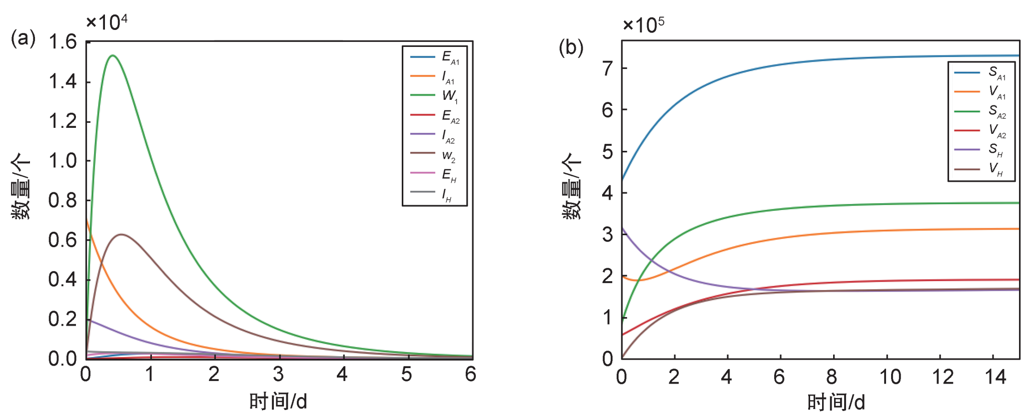

图1 $R_0=0.356<1$时,系统(1)的时间序列图,此时为无病平衡点

(a)图表示所有潜伏仓室,感染仓室以及环境仓室的数量都趋于零,(b)图表示所有的易感仓室和免疫仓室分别趋于不为零的常数,这表明疾病会随着时间推移消失。

Fig. 1 Time series diagram of system (1) with $R_0=0.356<1$, which is the disease-free equilibrium

(a) shows that the number of all latent compartments, infected compartments and environmental compartments tends to zero. (b) shows that all susceptible compartments and immune compartments tend to nonzero constants. This indicates that the disease will disappear over time.

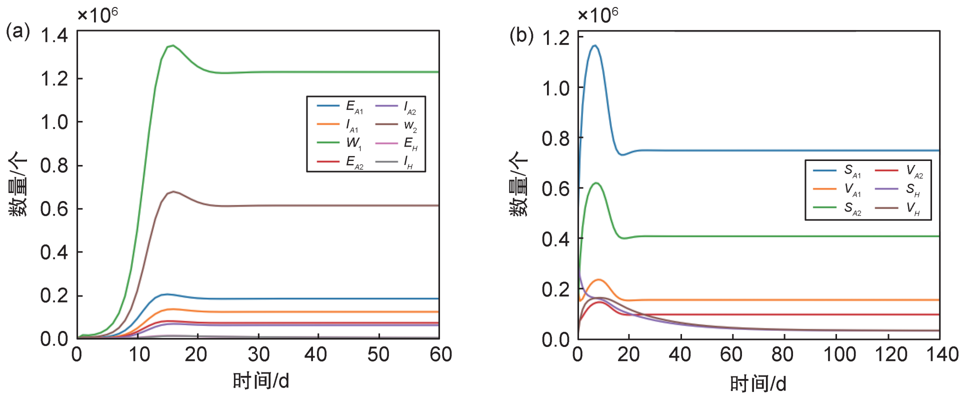

图2 $R_0=2.439>1$时,系统(1)的时间序列图,此时为地方病平衡点

(a)图表示所有潜伏仓室,感染仓室以及环境仓室的数量分别趋于不为零的常数,(b)图表示所有的易感仓室和免疫仓室的数量也分别趋于不为零的常数。这表明疾病随着时间的推移最终成为地方性流行病。

Fig. 2 Time series diagram of system (1) with $R_0=2.439>1$, which is the endemic equilibrium

(a) shows that the number of all latent compartments, infected compartments and environmental compartments tends to be non-zero constants, respectively. (b) shows that all susceptible compartments and immune compartments tend to nonzero constants, respectively. This indicates that the disease eventually becomes an endemic epidemic over time.

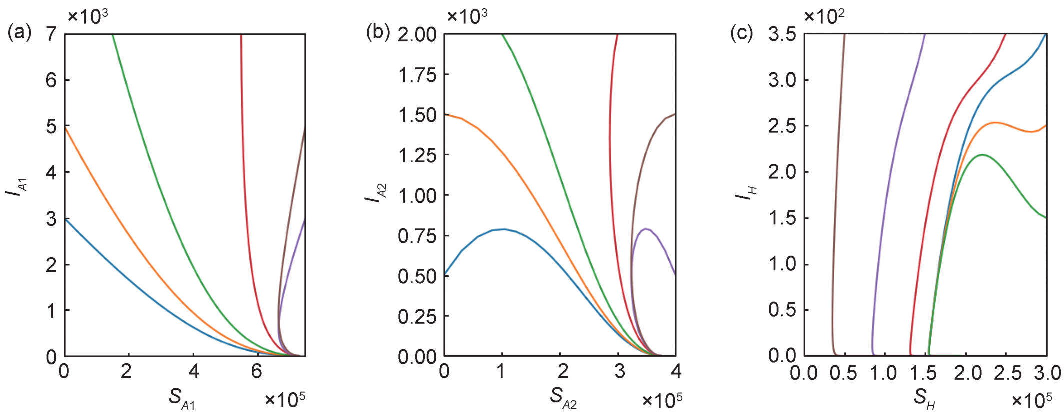

图3 $R_0=0.356<1$时,系统(1)的相图

在不同的初值下,所有轨迹都趋于一个固定点,系统最终达到稳定平衡状态。

(a) S_{A1}和I_{A1}的相图;(b)S_{A2}和I_{A2}的相图;(c)S_H和I_H的相图。

Fig. 3 Phase diagram of system (1) when $R_0=0.356<1$

Under different initial values, all trajectories tend to a fixed point, and the system eventually reaches a stable equilibrium state.

(a) Phase diagrams of S_{A1} and I_{A1}; (b) Phase diagrams of S_{A2} and I_{A2}; (c) Phase diagrams of s_H and I_H.

易感羊群i的感染率β_{Ai},潜伏羊群i和患病羊群i单位时间内向环境排出的布鲁氏菌数r_{Ai}与R_0呈正相关;易感羊群i的免疫覆盖率γ_{Ai},感染羊群i因病死亡及捕杀率μ_{Ai}与R_0呈负相关。不难看出,提高免疫覆盖率和及时捕杀患病羊群可以有效控制R_0。而图(d)和(e)比较而言,R_0对μ_{Ai}的敏感性远大于γ_{Ai},因此,及时捕杀患病羊群的效果远大于疫苗接种的效果。当这些控制策略达到一定程度时,会使$R_0<1$,布病不再流行。对于图2(f),当两区之间羊的运输量相同时,始终存在$R_0<1$,且当a_{21}的运输量接近0时,只有a_{12}存在不足以使疾病流行,反则疾病持续存在。这表明避免羊群基数大的地区之间进行互相买卖和运输可以有效控制布病的传播。

4 结论

羊的布鲁氏菌病是中国部分地区最大的公共卫生威胁之一。尽管人们认识到它是一个重要

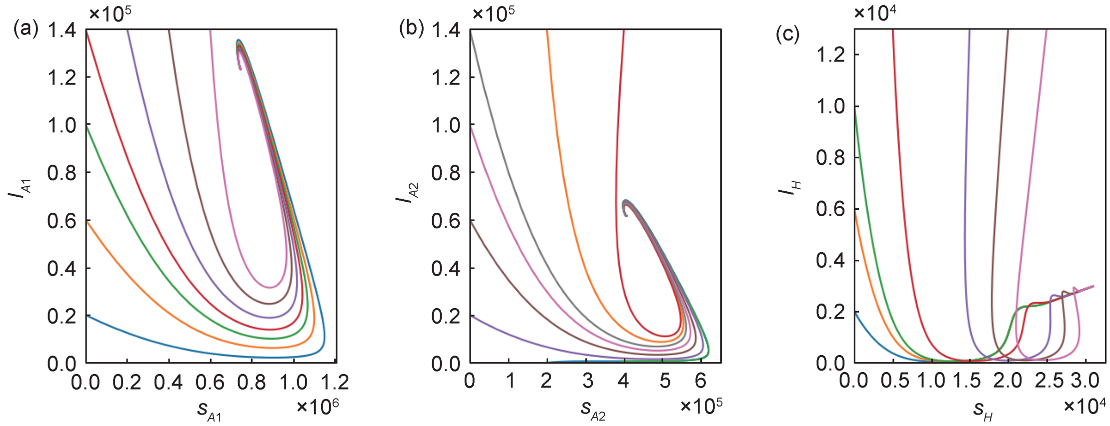

图4　$R_0=2.439>1$时,系统(1)的相图

在不同的初值下,所有轨迹都趋于一个固定点,系统最终达到稳定平衡状态。

(a) S_{A1}和I_{A1}的相图;(b)S_{A2}和I_{A2}的相图;(c)S_H和I_H的相图。

Fig. 4　Phase diagram of system (1) when $R_0=2.439>1$.

Under different initial values, all trajectories tend to a fixed point, and the system eventually reaches a stable equilibrium state.

(a) Phase diagrams of S_{A1} and I_{A1}; (b) Phase diagrams of S_{A2} and I_{A2}; (c) Phase diagrams of s_H and I_H.

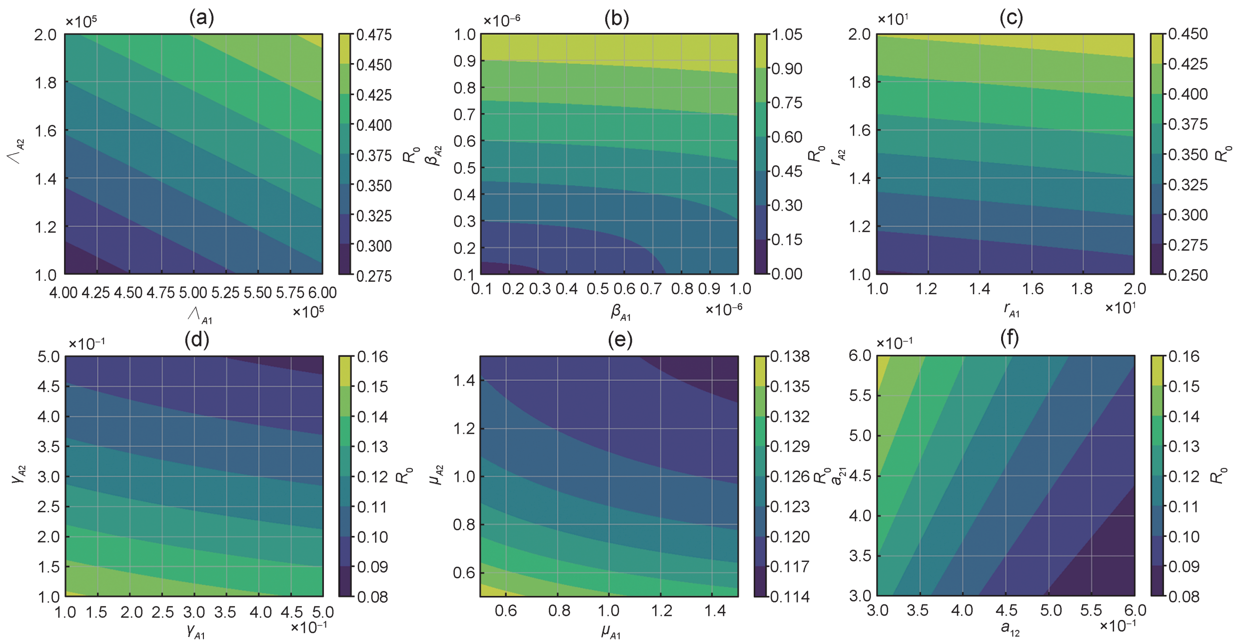

图5　不同参数的敏感度分析

(a)以Λ_{Ai}表示R_0;(b)以β_{Ai}表示R_0;(c)以r_{Ai}表示R_0;(d)以γ_{Ai}表示R_0;(e)以μ_{Ai}表示R_0;(f)以a_{ij}表示R_0。通过改变参数值使得$R_0<1$,从而控制疾病传播。

Fig. 5　Sensitivity analysis of different parameters

(a) R_0 in terms of Λ_{Ai}; (b) R_0 in terms of β_{Ai}; (c) R_0 in terms of r_{Ai}; (d) R_0 in terms of γ_{Ai}; (e) R_0 in terms of μ_{Ai}; (f) R_0 in terms of a_{ij}. By changing the parameter value to make $R_0<1$, the spread of the disease is controlled.

的经济和健康问题,并且有行之有效的控制措施,但它仍然以相对较高的频率发生。因此,中央与地方政府一直在寻求减少布鲁氏菌病传播的有力方法。本文采用多组SEIVW模型研究了具有两羊群间运输的布鲁氏菌病传播的影响。首先给出系统(1)的基本再生数R_0,此系统的动力学行为完全由基本再生数R_0的大小决定。系统具有两个非负平衡点,即无病平衡点P^0和地方病平衡点P^*。无病平衡点始终存在,当$R_0<1$时是全局渐近稳定的,而$R_0>1$是不稳定的;地方病平衡点在$R_0>1$时存在且是全局渐近稳定的。通过对基本再生数R_0参数的敏感性分析,证实了及时消除染

病尸体是羊布鲁氏菌病有效的控制策略。此外羊群间的运输极大程度影响着疾病的流行。与文献[28]的控制策略相比,认为避免羊群基数大的地区进行互相买卖和运输,以实现羊群自给自足是控制羊布鲁氏菌病的另一有效控制策略。

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