一类分数阶差分方程组边值问题的正解*

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摘要 运用不动点指数理论研究一类分数阶差分方程组边值问题正解的存在性:1)将该问题转化为等价的和分方程,构造对应的算子方程;2)在非线性项合适的条件下获得算子正不动点的存在性,从而获得原问题的正解.

关键词 分数阶差分方程组;边值问题;不动点指数;正解
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0 引言

令 $[a,b]_{\mathbb{N}_a} := \{a, a+1, a+2, \cdots, b\}(b-a\in\mathbb{N}_1)$, $\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$, 本文运用不动点指数理论研究一类分数阶差分方程组边值问题正解的存在性.

$$\begin{cases} -\Delta_{v-3}^{v}x(t) = f_1(t+v-1, x(t+v-1), y(t+v-1)), \\ \qquad t\in[0,b+2]_{\mathbb{N}_0}, \\ -\Delta_{v-3}^{v}y(t) = f_2(t+v-1, x(t+v-1), y(t+v-1)), \\ \qquad t\in[0,b+2]_{\mathbb{N}_0}, \\ x(v-3) = (\Delta_{v-3}^{\alpha}x(t))\Big|_{t=v-\alpha-2} = \left(\Delta_{v-3}^{\beta}x(t)\right)\Big|_{t=v+b+2-\beta} = 0, \\ y(v-3) = (\Delta_{v-3}^{\alpha}y(t))\Big|_{t=v-\alpha-2} = \left(\Delta_{v-3}^{\beta}y(t)\right)\Big|_{t=v+b+2-\beta} = 0, \end{cases} \tag{1}$$

式中:$2<v\leqslant 3, 1<\beta<2, v-\beta>1, 0<\alpha<1, b>3(b\in\mathbb{N})$; Δ_{v-3}^{v}是离散型分数阶算子;非线性项 $f_i\in C([v-1,b+v+1]_{\mathbb{N}_{v-1}}\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+), i=1,2$.

近年来分数阶模型掀起了研究热潮[1-13],分数阶差分方程模型也引起关注[14-15].例如文献[1]中运用Guo-Krasnoselskii不动点定理研究了分数阶差分方程边值问题的正解,即

$$\begin{cases} -\Delta_{v-2}^{v}u(t) = f(t+v-1, u(t+v-1)), 1<v\leqslant 2, \\ u(v-2) = 0, \Delta_{v-1}^{v-1}u(v+N) = 0, \end{cases}$$

式中非线性项f满足:超线性和次线性条件$f_0=0$、$f_\infty=+\infty$,$f_0=+\infty$、$f_\infty=0$,且

$$f_0 = \lim_{u\to 0^+}\frac{f(t,u)}{u}, f_\infty = \lim_{u\to+\infty}\frac{f(t,u)}{u}, 对 t\in[v-1, v+N]_{N_{v-1}}$$

一致成立.文献[8]中借助该条件研究了分数阶差分方程边值问题的正解,即

$$\begin{cases} \Delta^{v}y(t) = \lambda f(t+v-1, y(t+v-1)), t\in[0,T]_Z, \\ y(v-1) = y(v+T) + \sum_{i=1}^{N} F(t_i, y(t_i)), \end{cases}$$

式中:f满足半正型$f(t,y)\geqslant -M, M>0$的条件.文献[6]中研究了分数阶差分方程组边值问题,即

$$\begin{cases} -\Delta^{v_1}y_1(t) = \lambda_1 f_1(t+v_1-1, y_1(t+v_1-1), y_2(t+v_2-1)), \\ \qquad t\in[1,b+1]_{\mathbb{N}}, \\ -\Delta^{v_2}y_2(t) = \lambda_2 f_2(t+v_2-1, y_1(t+v_1-1), y_2(t+v_2-1)), \\ \qquad t\in[1,b+1]_{\mathbb{N}}, \\ y_1(v_1-2) = y_1(v_1+b+1) = 0, \\ y_2(v_2-2) = y_2(v_2+b+1) = 0, \end{cases}$$

式中v_1、$v_2\in(1,2]$.作者运用Guo-Krasnoselskii不动点定理,参数λ_1、$\lambda_2>0$在足够小的情形下获得该问题正解的存在性.文献[5]中运用不动点指数研究了分数阶差分方程组对偶边值问题的正解,即

$$\begin{cases} -\Delta_{v-3}^{v}x(t) = f_1(t+v-1, x(t+v-1), y(t+v-1)), \\ \qquad t\in[0,T-1]_{\mathbb{N}_0}, \\ -\Delta_{v-3}^{v}y(t) = f_2(t+v-1, x(t+v-1), y(t+v-1)), \\ \qquad t\in[0,T-1]_{\mathbb{N}_0}, \\ x(v-3) = (\Delta_{v-3}^{\alpha}x(t))\Big|_{t=v-\alpha-2} = 0, \\ y(v-3) = (\Delta_{v-3}^{\alpha}y(t))\Big|_{t=v-\alpha-2} = 0, \\ x(T+v-1) = ay(\xi+v), \\ y(T+v-1) = bx(\eta+v), \end{cases}$$

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式中：$v\in(2,3],\alpha\in(0,1)$；$f_i(i=1,2)$满足类似于超线性或次线性增长的条件.

受已有成果的启发，本文运用不动点指数研究式(1)正解的存在性，将式(1)转化为与其等价的和分方程组，研究 Green 函数的性质，获得一些重要不等式，并借助于依赖未知函数幂次(次数可不为 1)的非线性项增长条件.

1 基础知识

定义 1 令$t^{\underline{v}}:=\dfrac{\Gamma(t+1)}{\Gamma(t+1-v)}$，如果$t+1-v$是伽马函数的一个奇点且$t+1$不是奇点，则$t^{\underline{v}}=0$.

定义 2 当$v>0$时，函数f的v阶和分为

$$\Delta_a^{-v}f(t)=\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{\underline{v-1}}f(s),\ t\in\mathbb{N}_{a+v}.$$

由此定义f的v阶差分为

$$\Delta_a^{v}f(t)=\Delta^{N}\Delta_a^{v-N}f(t),t\in\mathbb{N}_{a+N-v},$$

式中：$i'\in\mathbb{N}$，且$0\leqslant N-1<v\leqslant N$.

将式(1)转化为与其等价的和分方程，构造算子方程与合适的锥，然后运用锥上的不动点指数研究算子方程不动点的存在性，进而获得原问题的解. 为此，我们需要获得式(1)对应的 Green 函数.

引理 3[3] 令$h:[v-1,b+v+1]_{\mathbb{N}_{v-1}}\to\mathbb{R}$，则差分方程边值问题

$$\begin{cases}-\Delta_{v-3}^{v}y(t)=h(t+v-1),\quad t\in[0,b+2]_{\mathbb{N}_0},\\ y(v-3)=\left(\Delta_{v-3}^{\alpha}y(t)\right)\Big|_{t=v-\alpha-2}=\left(\Delta_{v-3}^{\beta}y(t)\right)\Big|_{t=v+b+2-\beta}=0\end{cases}$$

的解可表示为和分方程

$$y(t)=\sum_{s=0}^{b+2}G(t,s)h(s+v-1),\quad t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}},$$

式中

$$G(t,s)=\frac{1}{\Gamma(v)}\begin{cases}\dfrac{t^{\underline{v-1}}(v+b-\beta-s+1)^{\underline{v-\beta-1}}}{(v+b-\beta+2)^{\underline{v-\beta-1}}}-(t-s-1)^{\underline{v-1}},\\ 0\leqslant s<t-v+1\leqslant b+2,\\ \dfrac{t^{\underline{v-1}}(v+b-\beta-s+1)^{\underline{v-\beta-1}}}{(v+b-\beta+2)^{\underline{v-\beta-1}}},\\ 0\leqslant t-v+1\leqslant s\leqslant b+2.\end{cases}\tag{2}$$

引理 4[3] Green 函数(G)有如下性质：

性质 1 $G(t,s)>0,(t,s)\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}\times[0,b+2]_{\mathbb{N}_0}$；

性质 2 $q(t)G(b+v+1,s)\leqslant G(t,s)\leqslant G(b+v+1,s)$，

式中

$$q(t)=\frac{t^{\underline{v-1}}}{(b+v+1)^{\underline{v-1}}},(t,s)\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}\times[0,b+2]_{\mathbb{N}_0}.$$

引理 5 令$\varphi(t+v-1)=G(b+v+1,t),t\in[0,b+2]_{\mathbb{N}_0}$，则$G$满足不等式

$$\sum_{t=v-1}^{b+v+1}q(t)\varphi(t)\cdot\varphi(s+v-1)\leqslant\sum_{t=v-1}^{b+v+1}G(t,s)\varphi(t)\leqslant\sum_{t=v-1}^{b+v+1}\varphi(t)\cdot\varphi(s+v-1),s\in[0,b+2]_{\mathbb{N}_0},\tag{3}$$

式中：$\varphi(t)=G(b+v+1,t-v+1),\ t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}$.

证明 仅证一个不等式，另一个不等式可同理证明. 根据引理 4 性质 2 的不等式可得

$$\sum_{t=v-1}^{b+v+1}G(t,s)\varphi(t)\geqslant\sum_{t=v-1}^{b+v+1}q(t)G(b+v+1,s)\varphi(t)=\sum_{t=v-1}^{b+v+1}q(t)\varphi(t)\cdot\varphi(s+v-1),s\in[0,b+2]_{\mathbb{N}_0}.$$

此即证明了式(3)左侧的不等式.

为了后续使用方便，令

$$k_1=\sum_{t=v-1}^{b+v+1}q(t)\varphi(t)、k_2=\sum_{t=v-1}^{b+v+1}\varphi(t).$$

设E是所有从$[v-3,b+v+1]_{\mathbb{N}_{v-3}}$到$\vec{\psi}$的映射所构成的空间，并赋予最大模范数，则$(E,\|\cdot\|)$构成一个 Banach 空间. 若$P=\{y\in E:y(t)\geqslant0,t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}\}$，则$P$是$E$上的锥. 从而亦可知$F_i\times F_i$构成一个 Banach 空间，其范数$\|(u,v)\|=\max\{\|u\|,\|v\|\}$ $P\times P$是其上的锥. 本研究的正解是指：若存在$(x_0,y_0)\in P\times P\backslash\{(0,0)\}$满足式(1)的方程和边界条件，则称$(x_0,y_0)$是式(1)的正解.

定义算子

$$(By)(t)=\sum_{s=0}^{b+2}G(t,s)y(s+v-1),y\in P,t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}},$$

则可得如下引理.

引理 6 令：

$P_0=y\in P:y(t)\geqslant q(t)\|y,\quad t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}$， 则$B(P)\subset P_0$.

证明 根据引理 4 性质 2 的不等式，对任意的$y\in P$，有

$$\sum_{s=0}^{b+2}G(b+v+1,s)y(s+v-1)\geqslant(By)(t)=\sum_{s=0}^{b+2}G(t,s)y(s+v-1)\geqslant\sum_{s=0}^{b+2}q(t)G(b+v+1,s)y(s+v-1).$$

由此即可得结论.

引理 7[16]　设E是一个 Banach 空间, B_R是E中的有界开子集, P是E上的锥. 若$A:\overline{B_R}\cap P\to P$是一全连续算子,且$u\neq Au+\lambda\varphi$、$\lambda\geqslant 0,u\in\partial B_R\cap P$,则$i(A,B_R\cap P,P)=0$, 其中$i$是锥$P$的不动点指数.

引理 8[16]　设E是一个 Banach 空间, B_r是E中的有界开子集, P是E上的锥, $0\in B_r$.若$A:\overline{B_r}\cap P\to P$是一个全连续算子, 且$u\neq\lambda Au$、$\lambda\in[0,1]$、$u\in\partial B_r\cap P$, 则$i(A,B_r\cap P,P)=1$.

引理 9[17]　令$\theta>0$、$n\geqslant 1$、$a_i\geqslant 0(i=1,2,\cdots,n)$则

$$\left(\sum_{i=1}^{n}a_i\right)^{\theta}\leqslant n^{\theta-1}\sum_{i=1}^{n}a_i^{\theta},\quad\forall\theta\geqslant 1,$$

$$\left(\sum_{i=1}^{n}a_i\right)^{\theta}\geqslant n^{\theta-1}\sum_{i=1}^{n}a_i^{\theta},\quad\forall 0<\theta\leqslant 1.$$

2　主要结论

根据引理 3, 在锥P上定义算子

$$\begin{cases}A_1(x,y)(t)=\sum_{s=0}^{b+2}G(t,s)f_1(s+v-1,x(s+v-1),y(s+v-1)),\\ A_2(x,y)(t)=\sum_{s=0}^{b+2}G(t,s)f_2(s+v-1,x(s+v-1),y(s+v-1)),\\ A(x,y)(t)=(A_1,A_2)(x,y)(t),\end{cases}\tag{4}$$

式中: G的意义见式(2); $t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}$. 显然若存在$(x_0,y_0)\in P\times P\backslash\{(0,0)\}$使得$A(x_0,y_0)=(x_0,y_0)$,即$A_1(x_0,y_0)=x_0,A_2(x_0,y_0)=y_0$,则$(x_0,y_0)$是式(1)的正解.

定义集合$B_\rho=\{x\in P:\|x\|<\rho\},\rho>0$.则$B_\rho$是$P$中的开球, 且

$$\partial B_\rho=\{x\in P:\|x\|=\rho\},\overline{B}_\rho=\{x\in P:\|x\|\leqslant\rho\}.$$

为了获得本文的结论, 给出需要使用的假设条件:

H0) $f_i\in C([v-1,b+v+1]_{\mathbb{N}_{v-1}}\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+),i=1,2$;

H1) 存在d_1、d_2、$d_3>0$, c_1、$c_2>0$, $\gamma_1\in(0,1],\varDelta_1=\gamma_1^{-1}$, 使得

$$d_1k_1+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2>1,$$

$$f_1(t,x,y)\geqslant d_1x+d_2y^{\gamma_1}-c_1,f_2(t,x,y)\geqslant d_3x^{\varDelta_1}-c_2,$$
$$x,y\in\mathbb{R}^+,t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}},$$

式中$\mathcal{M}=\max\limits_{s\in[0,b+2]_{\mathbb{N}_0}}G(b+v+1,s)$;

H2) 存在$r_1>0$, d_4、d_5、$d_6>0$, $\gamma_2\geqslant 1$, $\varDelta_2=\gamma_2^{-1}$, 使得

$$d_4k_2+d_5d_6^2(\mathcal{M}(b+3))^{\gamma_2-1}k_2^2<1,$$

$$f_1(t,x,y)\leqslant d_4x+d_5y^{\gamma_2},f_2(t,x,y)\leqslant d_6x^{\varDelta_2},$$
$$x,y\in[0,r_1],t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}};$$

H3) 存在$r_2>0$, d_7、d_8、$d_9>0$, $\gamma_3\in(0,1],\varDelta_3=\gamma_3^{-1}$, 使得

$$d_7k_1+d_8d_9^{\gamma_3}(\mathcal{M}(b+3))^{\gamma_3-1}k_1^2>1,$$

$$f_1(t,x,y)\geqslant d_7x+d_8y^{\gamma_3},f_2(t,x,y)\geqslant d_9x^{\varDelta_3},$$
$$x,y\in[0,r_2],\ t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}};$$

H4) 存在d_{10}、d_{11}、$d_{12}>0$, c_3、$c_4>0$, $\gamma_4\geqslant 1$, $\varDelta_4=\gamma_4^{-1}$, 使得

$$d_{10}k_2+d_{11}d_{12}^4(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2<1,$$

$$f_1(t,x,y)\leqslant d_{10}x+d_{11}y^{\gamma_4}+c_3,f_2(t,x,y)\leqslant d_{12}x^{\varDelta_4}+c_4,$$
$$x,y\in\mathbb{R}^+,\ t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.$$

在定理证明过程中, 用$c_1,c_2,\cdots,c_n\cdots$表示不同的正常数, 不算出它们的具体值(不影响定理证明).

定理 1　若 H0) ~ H2) 存在, 则式(1)至少存在 1 个正解.

证明　定义集合$S=\{(x,y)\in P\times P:(x,y)=A(x,y)+\lambda(\varphi,\varphi),\lambda\geqslant 0\}$, $\varphi\in P_0$是一个固定元素. 求证集合S是$P\times P$中的有界集. 若存在$(x,y)\in P\times P$, 则

$$x(t)=A_1(x,y)(t)+\lambda\varphi(t),y(t)=A_2(x,y)(t)+\lambda\varphi(t),t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.\tag{5}$$

注意到 H0)、引理 6 以及$\varphi\in P_0$, 由式(5)可知x、$y\in P_0$. 因为$\lambda\geqslant 0$、$\varphi\in P_0$,再根据式(5)可得

$$x(t)\geqslant A_1(x,y)(t),y(t)\geqslant A_2(x,y)(t),\quad t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.\tag{6}$$

将 H1) 中关于f_2的条件不等式代入式(6)中右端不等式可得

$$y(t)\geqslant\sum_{s=0}^{b+2}G(t,s)\left(d_3(x(s+v-1))^{\varDelta_1}-c_2\right)\geqslant d_3\sum_{s=0}^{b+2}G(t,s)(x(s+v-1))^{\varDelta_1}-c_5.$$

从而运用引理 9 可得

$$(y(t))^{\gamma_1}+c_5^{\gamma_1}\geqslant(y(t)+c_5)^{\gamma_1}\geqslant\left(d_3\sum_{s=0}^{b+2}G(t,s)[x(s+v-1)]^{\varDelta_1}\right)^{\gamma_1}\geqslant d_3^{\gamma_1}(b+3)^{\gamma_1-1}\sum_{s=0}^{b+2}(G(t,s))^{\gamma_1}(x(s+v-1))^{\varDelta_1\gamma_1}=(d_3\mathcal{M})^{\gamma_1}(b+3)^{\gamma_1-1}\sum_{s=0}^{b+2}\left(\frac{G(t,s)}{\mathcal{M}}\right)^{\gamma_1}x(s+v-1)\geqslant(d_3\mathcal{M})^{\gamma_1}(b+3)^{\gamma_1-1}\sum_{s=0}^{b+2}\frac{G(t,s)}{\mathcal{M}}x(s+v-1).\tag{7}$$

结合式(7), 将 H1) 中关于f_1的条件不等式代入式

(6)左端不等式可得

$$x(t)\geqslant\sum_{s=0}^{b+2}G(t,s)(d_1x(s+v-1)+d_2(y(s+v-1))^{\gamma_1}-c_1)\geqslant$$
$$d_1\sum_{s=0}^{b+2}G(t,s)x(s+v-1)+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1^{-1}}\cdot$$
$$\sum_{s=0}^{b+2}G(t,s)\sum_{\tau=0}^{b+2}G(s+v-1,\tau)x(\tau+v-1)-c_6-d_2c_5^{\gamma_1}k_2. \tag{8}$$

在式(8)两端乘以$\varphi(t)$,在$[v-1,\ b+v+1]$上求和,并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\geqslant d_1\sum_{t=v-1}^{b+v+1}\varphi(t)\sum_{s=0}^{b+2}G(t,s)x(s+v-1)+$$
$$d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}\sum_{t=v-1}^{b+v+1}\varphi(t)\sum_{s=0}^{b+2}G(t,s)\cdot$$
$$\sum_{\tau=0}^{b+2}G(s+v-1,\tau)x(\tau+v-1)-c_6k_2-d_2c_5^{\gamma_1}k_2^2\geqslant$$
$$d_1k_1\sum_{s=0}^{b+2}x(s+v-1)\varphi(s+v-1)+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1\cdot$$
$$\sum_{s=0}^{b+2}\varphi(s+v-1)\sum_{\tau=0}^{b+2}G(s+v-1,\tau)x(\tau+v-1)-c_6k_2-d_2c_5^{\gamma_1}k_2^2=$$
$$d_1k_1\sum_{s=0}^{b+2}x(s+v-1)\varphi(s+v-1)+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1\sum_{t=v-1}^{b+v+1}\varphi(t)\cdot$$
$$\sum_{\tau=0}^{b+2}G(t,\tau)x(\tau+v-1)-c_6k_2-d_2c_5^{\gamma_1}k_2^2\geqslant$$
$$d_1k_1\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2\cdot$$
$$\sum_{\tau=0}^{b+2}\varphi(\tau+v-1)x(\tau+v-1)-c_6k_2-d_2c_5^{\gamma_1}k_2^2.$$

解不等式可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\leqslant\frac{c_6k_2+d_2c_5^{\gamma_1}k_2^2}{d_1k_1+d_2d_3^{1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2-1}, \tag{9}$$

式中$x\in P_0$. 从而有

$$\|x\|\sum_{t=v-1}^{b+v+1}q(t)\varphi(t)\leqslant\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\leqslant\frac{c_6k_2+d_2c_5^{\gamma_1}k_2^2}{d_1k_1+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2-1},$$

即

$$\|x\|\leqslant\frac{c_6k_2+d_2c_5^{\gamma_1}k_2^2}{d_1k_1^2+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^3-k_1}. \tag{10}$$

再证y的范数亦有界. 由式(8)中左端不等式可得

$$x(t)\geqslant\sum_{s=0}^{b+2}G(t,s)(d_2(y(s+v-1))^{\gamma_1}-c_1).$$

两端乘以$\varphi(t)$,在$[v-1,b+v+1]$上求和,并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\geqslant\sum_{t=v-1}^{b+v+1}\varphi(t)\sum_{s=0}^{b+2}G(t,s)(d_2(y(s+v-1))^{\gamma_1}-c_1)\geqslant$$
$$k_1\sum_{s=0}^{b+2}\varphi(s+v-1)(d_2(y(s+v-1))^{\gamma_1}-c_1)=$$
$$k_1\sum_{t=v-1}^{b+v+1}\varphi(t)(d_2(y(t))^{\gamma_1}-c_1),$$

式中$y\in P_0$. 运用式(9)可得

$$d_2k_1\|y\|^{\gamma_1}\sum_{t=v-1}^{b+v+1}(q(t))^{\gamma_1}\varphi(t)\leqslant c_1k_1k_2+$$
$$\frac{c_6k_2+d_2c_5^{\gamma_1}k_2^2}{d_1k_1+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2-1},$$

即

$$\|y\|\leqslant\sqrt[\gamma_1]{\left(d_2k_1\sum_{t=v-1}^{b+v+1}(q(t))^{\gamma_1}\varphi(t)\right)^{-1}\left(c_1k_1k_2+\frac{c_6k_2+d_2c_5^{\gamma_1}k_2^2}{d_1k_1+d_2d_3^{\gamma_1}(\mathcal{M}(b+3))^{\gamma_1-1}k_1^2-1}\right)}. \tag{11}$$

至此我们估算了集合S中的先验界,并证明了它的有界性. 此时若取$R_1>\sup S, R_1>r_1$(r_1见 H2)),则有

$$(x,y)\neq A(x,y)+\lambda(\varphi,\varphi),x,y\in\partial B_{R_1}\cap P,\lambda\geqslant0. \tag{12}$$

由引理 7 可知

$$i(A,(B_{R_1}\times B_{R_1})\cap(P\times P),P\times P)=0. \tag{13}$$

另一方面需证明

$$(x,y)\neq\lambda A(x,y),\lambda\in[0,1],x,y\in\partial B_{r_1}\cap P, \tag{14}$$

式中r_1的意义见 H2). 事实上,若式(14)不成立,则存在x、$y\in\partial B_{r_1}\cap P,\lambda\in[0,1]$使得

$$(x,y)=\lambda A(x,y).$$

此即表明

$$x(t)=\lambda A_1(x,y)(t)\leqslant A_1(x,y)(t),y(t)\leqslant A_2(x,y)(t),t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.$$

根据 H2)和引理 9 可得

$$(y(t))^{\gamma_2}\leqslant d_6^{\gamma_2}\left(\sum_{s=0}^{b+2}G(t,s)(x(s+v-1))^{\Delta_2}\right)^{\gamma_2}\leqslant$$
$$d_6^{\gamma_2}(b+3)^{\gamma_2-1}\sum_{s=0}^{b+2}(G(t,s))^{\gamma_2}(x(s+v-1))^{\Delta_2\gamma_2}\leqslant$$
$$d_6^{\gamma_2}(\mathcal{M}(b+3))^{\gamma_2-1}\sum_{s=0}^{b+2}G(t,s)x(s+v-1). \tag{15}$$

将式(13)代入, 可得

$$x(t)\leqslant A_1(x,y)(t)\leqslant\sum_{s=0}^{b+2}G(t,s)(d_4x(s+v-1)+d_5(y(s+v-1))^{\gamma_2})\leqslant$$

$$d_4\sum_{s=0}^{b+2}G(t,s)x(s+v-1)+d_5d_6^{\gamma_2}(\mathcal{M}(b+3))^{\gamma_2-1}\sum_{s=0}^{b+2}G(t,s)\cdot$$

$$\sum_{\tau=0}^{b+2}G(s+v-1,\tau)x(\tau+v-1),$$

两端乘以$\varphi(t)$,在 $[v-1,\ b+v+1]$ 上求和, 并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\leqslant(d_4k_2+d_5d_6^{\gamma_2}(\mathcal{M}(b+3))^{\gamma_2-1}k_2^2)\sum_{t=v-1}^{b+v+1}x(t)\varphi(t),$$

由 H2)可得$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)=0$.

又因为φ在$[v-1,b+v+1]_{\mathbb{N}_{v-1}}$上不恒等于 0, 从而

$$x(t)\equiv 0,t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.\qquad(16)$$

根据式(15)、(16)可得

$$0\leqslant[y(t)]^{\gamma_2}\leqslant d_6^{\gamma_2}(\mathcal{M}(b+3))^{\gamma_2-1}\sum_{s=v-1}^{b+v+1}G(t,s-v+1)x(s)=0,$$

$$t\in[v-1,b+v+1]_{\mathbb{N}_{\sigma-1}}.$$

这表明

$$y(t)\equiv 0,t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.\qquad(17)$$

式(16)、(17)均与x、$y\in\partial B_{r_1}\cap P(r_1>0)$矛盾, 从而说明式(14)成立. 同时根据引理 8 可得

$$i(A,(B_{r_1}\times B_{r_1})\cap(P\times P),P\times P)=1.\qquad(18)$$

根据式(13)、(18), 有

$$i(A,((B_{R_1}\times B_{R_1})\backslash\overline{(B_{r_1}\times B_{r_1})})\cap(P\times P),P\times P)=$$
$$i(A,(B_{R_1}\times B_{R_1})\cap(P\times P),P\times P)-$$
$$i(A,(B_{r_1}\times B_{r_1})\cap(P\times P),P\times P)=0-1=-1.$$

从而可知算子A在$((B_{R_1}\times B_{R_1})\backslash\overline{(B_{r_1}\times B_{r_1})})\cap(P\times P)$中至少有 1 个不动点, 即式(1)在$((B_{R_1}\times B_{R_1})\backslash\overline{(B_{r_1}\times B_{r_1})})\cap(P\times P)$中至少存在 1 个解, 且是正解.

定理 2　若 H0)、H3)、H4)存在, 则式(1)至少存在 1 个正解.

证明　类似于定理 1 的证明, 某些计算和步骤可省略. 定义集合

$$W=\{(x,y)\in P\times P:(x,y)=\lambda A(x,y),\lambda\in[0,1]\}.$$

证明W是$P\times P$中的有界集. 事实上, 若存在$(x,y)\in W$, 则根据引理 6 可知x、$y\in P_0$且

$$x(t)\leqslant A_1(x,y)(t),y(t)\leqslant$$
$$A_2(x,y)(t),t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}.\qquad(19)$$

将 H4)中关于f_2的条件代入式(19)的右端不等式, 并运用引理 9 可得

$$(y(t))^{\gamma_4}\leqslant\left(\sum_{s=0}^{b+2}\frac{G(t,s)}{\mathcal{M}}\mathcal{M}(d_{12}(x(s+v-1))^{\Delta_4}+c_4)\right)^{\gamma_4}\leqslant$$

$$(b+3)^{\gamma_4-1}\sum_{s=0}^{b+2}\left(\frac{G(t,s)}{\mathcal{M}}\right)^{\gamma_4}\mathcal{M}^{\gamma_4}(d_{12}(x(s+v-1))^{\Delta_4}+c_4)^{\gamma_4}\leqslant$$

$$(2\mathcal{M}(b+3))^{\gamma_4-1}\sum_{s=0}^{b+2}G(t,s)(d_{12}^{\gamma_4}x(s+v-1)+c_4^{\gamma_4}).\qquad(20)$$

将此不等式代入式(19)的左端不等式, 有

$$x(t)\leqslant\sum_{s=0}^{b+2}G(t,s)(d_{10}x(s+v-1)+d_{11}(y(s+v-1))^{\gamma_4}+c_3)\leqslant d_{10}\cdot$$

$$\sum_{s=0}^{b+2}G(t,s)x(s+v-1)+d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}\sum_{s=0}^{b+2}G(t,s)\cdot$$

$$\sum_{\tau=0}^{b+2}G(s+v-1,s)(d_{12}^{\gamma_4}x(\tau+v-1)+c_4^{\gamma_4})+c_3k_2.$$

两端乘以$\varphi(t)$,在 $[v-1,\ b+v+1]$ 上求和, 并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\leqslant(d_{10}k_2+d_{11}d_{12}^{\gamma_4}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2)\cdot$$

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)+c_4^{\gamma_4}d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^3+c_3k_2^2,$$

解不等式可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\leqslant\frac{c_4^{\gamma_4}d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^3+c_3k_2^2}{1-(d_{10}k_2+d_{11}d_{12}^{\gamma_4}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2)},$$

式中$x\in P_0$. 从而可得

$$\|x\|\leqslant\frac{c_4^{\gamma_4}d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^3+c_3k_2^2}{k_1-k_1(d_{10}k_2+d_{11}d_{12}^{\gamma_4}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2)},\qquad(21)$$

在式(20)的两端乘以$\varphi(t)$, 在 $[v-1,\ b+v+1]$ 上求和, 并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}(y(t))^{\gamma_4}\varphi(t)\leqslant(2\mathcal{M}(b+3))^{\gamma_4-1}k_2\cdot$$

$$\sum_{s=0}^{b+2}\varphi(s+v-1)(d_{12}^{\gamma_4}x(s+v-1)+c_4^{\gamma_4}).$$

即

$$\sum_{t=v-1}^{b+v+1}(y(t))^{\gamma_4}\varphi(t)\leqslant(2\mathcal{M}(b+3))^{\gamma_4-1}k_2\left(c_4^{\gamma_4}k_2+\frac{c_4^{\gamma_4}d_{12}^{\gamma_4}d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^3+c_3k_2^2}{1-(d_{10}k_2+d_{11}d_{12}^{\gamma_4}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2)}\right).$$

式中$y\in P_0$. 从而可知

$$\|y\|\leqslant\sqrt[\gamma_4]{\frac{(2\mathcal{M}(b+3))^{\gamma_4-1}k_2\left(c_4^{\gamma_4}k_2+\frac{c_4^{\gamma_4}d_{12}^{\gamma_4}d_{11}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^3+c_3k_2^2}{1-(d_{10}k_2+d_{11}d_{12}^{\gamma_4}(2\mathcal{M}(b+3))^{\gamma_4-1}k_2^2)}\right)}{\sum_{t=v-1}^{b+v+1}(q(t))^{\gamma_4}\varphi(t)}}. \tag{22}$$

结合式(21)、(22)即证明了W的有界性. 若取$R_2>\sup W, R_2>r_2$(r_2的意义参见 H3)), 则

$$(x,y)\neq\lambda A(x,y), x,y\in\partial(B_{R_2}\times B_{R_2})\cap(P\times P),\lambda\in[0,1].$$

根据引理 8 可得

$$i(A,(B_{R_2}\times B_{R_2})\cap(P\times P),P\times P)=1. \tag{23}$$

另一方面需证明

$$(x,y)\neq A(x,y)+\lambda(\overline{\varphi},\overline{\varphi}), x,y\in\partial B_{r_2}\cap P,\lambda\geqslant 0, \tag{24}$$

式中: r_2的意义参见 H3); $\overline{\varphi}\in P$是一固定元素. 若式(24)不成立, 则存在x、$y\in\partial B_{r_2}\cap P,\lambda\geqslant 0$使得

$$(x,y)=A(x,y)+\lambda(\overline{\varphi},\overline{\varphi}).$$

按照已有的证明方法, 将 H3)中不等式的条件代入可得

$$(y(t))^{\gamma_3}\geqslant d_9^{\gamma_3}\left(\sum_{s=0}^{b+2}\frac{G(t,s)}{\mathcal{M}}\mathcal{M}(x(s+v-1))^{d_3}\right)^{\gamma_3}\geqslant$$
$$d_9^{\gamma_3}(\mathcal{M}(b+3))^{\gamma_3-1}\sum_{s=0}^{b+2}G(t,s)x(s+v+1).$$

将此式代入

$$x(t)\geqslant A_1(x,y)(t)\geqslant$$
$$\sum_{s=0}^{b+2}G(t,s)(d_7x(s+v-1)+$$
$$d_8(y(s+v-1))^{\gamma_3})\geqslant$$
$$d_7\sum_{s=0}^{b+2}G(t,s)x(s+v-1)+d_8d_9^{\gamma_3}(\mathcal{M}(b+3))^{\gamma_3-1}$$
$$\sum_{s=0}^{b+2}G(t,s)\sum_{\tau=0}^{b+2}G(s+v-1,\tau)x(\tau+v-1), \tag{25}$$

两端乘以$\varphi(t)$,在$[v-1,\ b+v+1]$上求和, 并运用式(3)可得

$$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)\geqslant(d_7k_1+d_8d_9^{\gamma_3}(\mathcal{M}(b+3))^{\gamma_3-1}k_1^2)\sum_{t=v-1}^{b+v+1}x(t)\varphi(t).$$

由 H3)可得$\sum_{t=v-1}^{b+v+1}x(t)\varphi(t)=0$, 从而

$$x(t)\equiv 0,\quad t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}. \tag{26}$$

再由式(25)的左端不等式可得

$$x(t)\geqslant d_8\sum_{s=0}^{b+2}G(t,s)(y(s+v-1))^{\gamma_3}=$$
$$d_8\sum_{s=v-1}^{b+v+1}G(t,s-v+1)(y(s))^{\gamma_3}, t\in(v-1,b+v+1)_{\mathbb{N}_{v-1}}.$$

从而

$$y(t)\equiv 0, t\in[v-1,b+v+1]_{\mathbb{N}_{v-1}}. \tag{27}$$

式(26)、(27)均与x、$y\in\partial B_{r_2}\cap P(r_2>0)$矛盾, 这表明式(24)成立. 从而根据引理 7 可得

$$i(A,(B_{r_2}\times B_{r_2})\cap(P\times P),P\times P)=0. \tag{28}$$

根据式(23)、(28), 有

$$i(A,((B_{R_2}\times B_{R_2})\backslash\overline{(B_{r_2}\times B_{r_2})})\cap(P\times P),P\times P)=$$
$$i(A,(B_{R_2}\times B_{R_2})\cap(P\times P),P\times P)-$$
$$i(A,(B_{r_2}\times B_{r_2})\cap(P\times P),P\times P)=1-0=1.$$

因此可知算子A在$((B_{R_2}\times B_{R_2})\backslash\overline{(B_{r_2}\times B_{r_2})})\cap(P\times P)$中至少存在 1 个不动点, 即式(1)在$((B_{R_2}\times B_{R_2})\backslash\overline{(B_{r_2}\times B_{r_2})})\cap(P\times P)$中至少存在 1 个解, 且是正解.

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Positive solutions for a class system of fractional difference equations boundary value problems

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Abstract The existence of positive solutions for a class system of fractional difference equations boundary value problems is studied by means of the theory of fixed point index. First, the problem is translated into its equivalent sum equation, and corresponding operator is established. Then the existence of positive fixed points is obtained under appropriate conditions from nonlinearities, positive solutions of the original problem are obtained.

Keywords system of fractional difference equations; boundary value problems; fixed point index; positive solutions

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