一类永久美式期权的定价问题 *

王 术[1)] 袁 芳[1,2)†]
(1)北京工业大学应用数理学院, 100124, 北京; 2)中国光大银行, 100054, 北京)

摘要 探索一类永久美式期权的定价问题, 在这类问题中, 允许波动率 σ 是一个间断函数, 使用微分方程理论等分析技巧, 克服波动率 σ 的间断性所带来的困难, 建立了一类期权定价公式.

关键词 Black-Scholes 方程; 永久美式期权; 间断波动率
中图分类号 O175.26 **DOI:** 10.12202/j.0476-0301.2020225

0 引言

探讨永久美式看涨期权的定价模型, 求$\{V(S),\omega\}$, 使得

$$\frac{\sigma^2}{2}S^2\frac{\mathrm{d}^2V}{\mathrm{d}S^2}+(r-q)S\frac{\mathrm{d}V}{\mathrm{d}S}-rV=0,\ 0<S<\omega, \tag{1}$$

$$V(\omega)=\omega-K, \tag{2}$$

$$V'(\omega)=1, \tag{3}$$

$$V(0)=0, \tag{4}$$

式中$V=V(S)$表示期权, S表示原生资产价格, σ为波动率, r为无风险利率, q为支付的红利率, K是期权的敲定价[1−4].

定解问题式(1)~(4), 在数学上称为自由边界问题(free boundary problem), 其中$S=\omega$称为自由边界[5−7]. 在金融学中, $S=\omega$称为最佳实施边界[1].

这类期权定价模型, 不仅具有重要的理论意义, 而且具有明确的金融意义, 因而受到广泛关注[8−11], 取得了丰富成果[12−15].

目前, 波动率σ是正常数的期权定价模型研究已经相当完善[1, 16]. 但是, 当波动率σ是间断函数时, 期权定价公式等问题还有待于进一步研究[3].

本文选取$\sigma=\sigma(S)$作为原生资产价格S的一个函数, 表达式为

$$\sigma=\sigma(S)=\begin{cases}\sigma_1, 0\leqslant S<\nu,\\ \sigma_2, \nu\leqslant S<+\infty,\end{cases} \tag{5}$$

式中: σ_1与σ_2是给定的正常数, 且$\sigma_1\neq\sigma_2$; ν是一个给定的正常数.

定理 1 假设式(5)成立. 定义

$$\alpha_1=\frac{-\left(r-q-\frac{\sigma_1^2}{2}\right)+\sqrt{\left(r-q-\frac{\sigma_1^2}{2}\right)+2r\sigma_1}}{\sigma_1^2}, \tag{6}$$

$$\alpha_2=\frac{-\left(r-q-\frac{\sigma_1^2}{2}\right)-\sqrt{\left(r-q-\frac{\sigma_1^2}{2}\right)+2r\sigma_1}}{\sigma_1^2}. \tag{7}$$

如果

$$\nu>\frac{\alpha_1K}{\alpha_1-1}, \alpha_1>1, \tag{8}$$

那么自由边界问题式(1)~(4)的解$(V(S),\omega)$具有下列表达式:

$$V(S)=\frac{S^{\alpha_1}}{\alpha_1\left(\frac{\alpha_1K}{\alpha_1-1}\right)^{\alpha_1-1}}, \tag{9}$$

$$\omega=\frac{\alpha_1K}{\alpha_1-1}. \tag{10}$$

为了方便讨论, 令

$$\beta_1=\frac{-\left(r-q-\frac{\sigma_2^2}{2}\right)+\sqrt{\left(r-q-\frac{\sigma_2^2}{2}\right)+2r\sigma_2}}{\sigma_2^2}, \tag{11}$$

$$\beta_2=\frac{-\left(r-q-\frac{\sigma_2^2}{2}\right)-\sqrt{\left(r-q-\frac{\sigma_2^2}{2}\right)+2r\sigma_2}}{\sigma_2^2}, \tag{12}$$

* 国家自然科学基金资助项目(11771031, 11531010)
† 通信作者: 袁芳(1990—)女, 博士生. 研究方向: 偏微分方程及其应用. E-mail: yuanfangrose@163.com
收稿日期: 2020-06-04

$$m=\beta_1-\beta_2>0, \tag{13}$$

$$a=\frac{K\beta_1}{\beta_2-2\beta_1}, \tag{14}$$

$$b=\frac{(\beta_2-1)(\beta_1-\alpha_1)\nu^{\beta_1-\beta_2}}{(\beta_2-2\beta_1)(\beta_2-\alpha_1)}, \tag{15}$$

$$c=\frac{-(\beta_1-\alpha_1)\beta_2K\nu^{\beta_1-\beta_2}}{(\beta_2-2\beta_1)(\beta_2-\alpha_1)}. \tag{16}$$

定理 2　设式（5）成立，再设函数

$$f(x)=x^{m+1}+ax^m+bx+c, \tag{17}$$

在区间$(\nu,+\infty)$内有$n(\geqslant 1)$个不同的正实数根

$$x=\omega_i\in(\nu,+\infty), i=1,2,\cdots,n,$$

式中 m、a、b、c 由式（13）~（16）定义．则，自由边界问题式(1)~(4)有n个解$(V^1(S),\omega_1),(V^2(S),\omega_2),\cdots,(V^n(S),\omega_n)$，式中$V^i(S)=\begin{cases}V_1^i(S), S\in[0,\nu),\\ V_2^i(S), S\in[\nu,\omega_i].\end{cases}$期权函数$V^i(S)$在间断点$S=\nu$处满足间断条件

$$V_1^i(\nu-0)=V_2^i(\nu+0), \tag{18}$$

$$(V_1^i)'(\nu-0)=(V_2^i)'(\nu+0), \tag{19}$$

式中 $i=1,2,\cdots,n$，$V_1^i(S)=A_1^iS^{\alpha_1}$，$V_2^i(S)=A_1^iS^{\beta_1}+A_2^iS^{\beta_2}$，$A_1^i=\frac{\nu^{1-\alpha_1}}{\alpha_1}(\beta_1A_2^i\nu^{\beta_1-1}+\beta_2B_2^i\nu^{\beta_2-1})$，　$A_2^i=\frac{\beta_2\omega_i^{-1}(\omega_i-K)-1}{(\beta_2-\beta_1)\omega_i^{\beta_1-1}}$，$B_2^i=\frac{1}{\beta_2-\beta_1}\left((\beta_2-2\beta_1)\omega_i^{1-\beta_2}+K\beta_1\omega_i^{-\beta_2}\right)$.

定理 3　设式（5）成立，再设

$$m=\beta_1-\beta_2=1, \tag{20}$$

$$c=\frac{-(\beta_1-\alpha_1)\beta_2K}{(\beta_2-\alpha_1)(\beta_2-2\beta_1)}\nu^{\beta_1-\beta_2}<0, \tag{21}$$

$$f(\nu)=\nu^2+(a+b)\nu+c<0. \tag{22}$$

则自由边界问题式（1）~（4）的一个解$(V(S),\omega)$具有下列性质：

1）自由边界$S=\omega$就是一元二次方程

$$f(\omega)=\omega^2+(a+b)\omega+c$$

在区间$(\nu,+\infty)$内唯一的正实数根，式中 a、b、c 由式（14）~（16）定义．

2）期权函数表达式为$V=\begin{cases}V_1(S), S\in[0,\nu),\\ V_2(S), S\in[\nu,\omega].\end{cases}$这个期权函数在间断点$S=\nu$处满足间断条件：

$$V_1(\nu-0)=V_2(\nu+0), \tag{23}$$

$$V_1'(\nu-0)=V_2'(\nu+0), \tag{24}$$

式中 $V_1(S)=A_1S^{\alpha_1}$，$V_2(S)=A_2S^{\beta_1}+B_2S^{\beta_2}$，$A_1=\frac{\nu^{1-\alpha_1}}{\alpha_1}\cdot(\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1})$，$A_2=\frac{\beta_2\omega^{-1}(\omega-K)-1}{(\beta_2-\beta_1)\omega^{\beta_1-1}}$，$B_2=\frac{1}{\beta_2-\beta_1}\cdot((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2})$.

美式期权是一张具有提前实施条款的合约．永久美式期权是一张没有终止期、生效后任何时间都可以实施的期权[1]. 美式期权由于可以提前实施，故持有人拥有比欧式期权更多的获利机会，通常比欧式期权更贵一些．在数学上，美式期权的定价问题是一个自由边界问题，是一条需要确定的交界线．这条交界线把区域$\{S>0\}$分成 2 个部分：一是继续持有区域，另一是终止持有区域．这个自由边界在金融上被称为最佳实施边界．就永久美式看涨期权而言，继续持有区域是$\{\omega>S>0\}$，另一部分是终止持有区域$\{S>\omega\}$，ω是最佳实施边界．定理 1~3 就是讨论一类永久美式看涨期权定价问题，即式（1）~（4）．通过计算给出了最佳实施边界ω的计算公式．

1　定理 1 的证明

令

$$\omega\in(0,\nu), \tag{25}$$

则对于所有的$S\in(0,\omega)\subset(0,\nu)$，都有$\sigma(S)=\sigma_1$.

在$(0,\omega)$上寻找定解问题式（1）~（4）的解．根据常微分方程理论，需要先找到 2 个线性无关的特解．

在式（1）中，选取$V=S^\alpha$，计算

$$0=\left(\frac{\sigma^2}{2}S^2\frac{d^2V}{dS^2}+(r-q)S\frac{dV}{dS}-rV\right)\Bigg|_{V=S^\alpha,\sigma=\sigma_1}=$$

$$\frac{\sigma_1^2}{2}S^2\frac{d^2(S^\alpha)}{dS^2}+(r-q)S\frac{d(S^\alpha)}{dS}-r(S^\alpha)=$$

$$\frac{\sigma_1^2}{2}S^2\cdot\alpha(\alpha-1)S^{\alpha-2}+(r-q)S\cdot\alpha S^{\alpha-1}-rS^\alpha=$$

$$\left(\frac{1}{2}\sigma_1^2\alpha(\alpha-1)+(r-q)\alpha-r\right)S^\alpha,$$

上式对于所有$S\in(0,\nu)$成立．于是，可得特征方程

$$\frac{1}{2}\sigma_1^2\alpha(\alpha-1)+(r-q)\alpha-r=0.$$

上述特征方程有 2 个特征根：

$$\alpha=\alpha_1, \tag{26}$$

$$\alpha=\alpha_2, \tag{27}$$

式中α_1与α_2分别由式（6）和（7）定义．于是可得通解

$$V(S)=AS^{\alpha_1}+BS^{\alpha_2},\tag{28}$$

式中 A、B 是待定的常数. 从式(26)和(27)容易看出

$$\alpha_1>0,\alpha_2<0.\tag{29}$$

应用自由边界条件式(4)计算

$$0=V(0)=(AS^{\alpha_1}+BS^{\alpha_2})|_{S=0},$$

可得

$$B=0.\tag{30}$$

应用自由边界条件式(2)得

$$\omega-K=V(\omega)=(AS^{\alpha_1}+BS^{\alpha_2})|_{S=\omega}=A\omega^{\alpha_1}+B\omega^{\alpha_2},$$

应用式(30)解得

$$A\omega^{\alpha_1}=\omega-K.\tag{31}$$

应用自由边界条件式(3)得

$$1=V'(\omega)=\left(\frac{\mathrm{d}V(S)}{\mathrm{d}S}\right)\bigg|_{S=\omega}=\left(\frac{\mathrm{d}(AS^{\alpha_1}+BS^{\alpha_2})}{\mathrm{d}S}\right)\bigg|_{S=\omega}=$$

$$(A\alpha_1S^{\alpha_1-1}+B\alpha_2S^{\alpha_2-1})\big|_{S=\omega}=A\alpha_1\omega^{\alpha_1-1}+B\alpha_2\omega^{\alpha_2-1}.$$

应用式(30)解得

$$A\alpha_1\omega^{\alpha_1-1}=1.\tag{32}$$

在式(31)的 2 边同乘以 $\alpha_1\omega^{-1}$, 得 $A\alpha_1\omega^{\alpha_1-1}=\alpha_1(\omega-K)\omega^{-1}$.

应用式(32)解得 $\alpha_1(\omega-K)\omega^{-1}=1$.

于是有

$$\omega=\frac{\alpha_1K}{\alpha_1-1}.\tag{33}$$

将式(33)代入(32)解得

$$A=\frac{1}{\alpha_1\left(\frac{\alpha_1K}{\alpha_1-1}\right)^{\alpha_1-1}}.\tag{34}$$

定理 1 得证.

2 定理 2 的证明

假设 $\nu\in(0,\omega)$, 考虑 2 种情况: 1) $S\in(0,\nu)$; 2) $S\in[\nu,\omega)$.

由情况 1)寻找式(1)的解.

根据式(5)可知 $\sigma=\sigma(S)=\sigma_1,S\in(0,\nu)$. 在 $(0,\nu)$ 上寻找定解问题式(1)~(4)的解. 根据常微分方程理论, 需要找到 2 个线性无关特解.

在式(1)中, 选取 $V=S^\alpha$, 计算

$$0=\left(\frac{\sigma^2}{2}S^2\frac{\mathrm{d}^2V}{\mathrm{d}S^2}+(r-q)S\frac{\mathrm{d}V}{\mathrm{d}S}-rV\right)\bigg|_{V=S^\alpha,\sigma=\sigma_1}=$$

$$\frac{\sigma_1^2}{2}S^2\frac{\mathrm{d}^2(S^\alpha)}{\mathrm{d}S^2}+(r-q)S\frac{\mathrm{d}(S^\alpha)}{\mathrm{d}S}-rS^\alpha=$$

$$\frac{\sigma_1^2}{2}S^2\cdot\alpha(\alpha-1)S^{\alpha-2}+(r-q)S\cdot\alpha S^{\alpha-1}-rS^\alpha=$$

$$\left(\frac{1}{2}\sigma_1^2\alpha(\alpha-1)+(r-q)\alpha-r\right)S^\alpha.$$

上式对于所有的 $S\in(0,\nu)$ 成立. 于是得到特征方程 $\frac{1}{2}\sigma_1^2\alpha(\alpha-1)+(r-q)\alpha-r=0$. 此特征方程有 2 个特征根:

$$\alpha=\alpha_1,\alpha=\alpha_2,\tag{35}$$

式中 α_1、α_2 分别由式(6)、(7)定义. 于是得到通解

$$V_1(S)=A_1S^{\alpha_1}+B_1S^{\alpha_2}.\tag{36}$$

应用式(29)和边界条件式(4)计算可得

$$0=V_1(0)=(A_1S^{\alpha_1}+B_1S^{\alpha_2})|_{S=0}.$$

上式表明

$$B_1=0.\tag{37}$$

由情况 2)寻找式(1)的解.

应用式(5)可知 $\sigma=\sigma(S)=\sigma_2,S\in[\nu,\omega)\subset[\nu,+\infty)$.

在 $[\nu,\omega)$ 上寻找定解问题式(1)~(4)的解. 根据常微分方程理论, 需要找出 2 个线性无关特解.

在式(1)中, 选取 $V=S^\alpha$, 有

$$0=\left(\frac{\sigma^2}{2}S^2\frac{\mathrm{d}^2V}{\mathrm{d}S^2}+(r-q)S\frac{\mathrm{d}V}{\mathrm{d}S}-rV\right)\bigg|_{V=S^\alpha,\sigma=\sigma_2}=$$

$$\frac{\sigma_2^2}{2}S^2\frac{\mathrm{d}^2(S^\alpha)}{\mathrm{d}S^2}+(r-q)S\frac{\mathrm{d}(S^\alpha)}{\mathrm{d}S}-rS^\alpha=$$

$$\frac{\sigma_2^2}{2}S^2\cdot\alpha(\alpha-1)S^{\alpha-2}+(r-q)S\cdot\alpha S^{\alpha-1}-rS^\alpha=$$

$$\left(\frac{1}{2}\sigma_2^2\alpha(\alpha-1)+(r-q)\alpha-r\right)S^\alpha,$$

上式对所有 $S\in[\nu,\omega)$ 成立, 得到特征方程

$$\frac{1}{2}\sigma_2^2\alpha(\alpha-1)+(r-q)\alpha-r=0,$$

这个特征方程有 2 个特征根 $\alpha=\beta_1$、$\alpha=\beta_2$, β_1、β_2 分别由式(11)与(12)定义. 于是得到通解

$$V_2(S)=A_2S^{\beta_1}+B_2S^{\beta_2},\tag{38}$$

式中 B_1、B_2 是待定常数. 应用与式(18)相应的间断条件, 再应用式(36)~(38), 有

$$0=V_1(\nu-0)-V_2(\nu+0)=\lim_{S\to\nu-0}V_1(S)-\lim_{S\to\nu+0}V_2(S)=$$
$$\lim_{S\to\nu-0}(A_1S^{\alpha_1}+B_1S^{\alpha_2})-\lim_{S\to\nu+0}(A_2S^{\alpha_1}+B_2S^{\alpha_2})=$$
$$A_1\nu^{\alpha_1}-(A_2\nu^{\beta_1}-B_2\nu^{\beta_2}).$$

上式表明

$$A_2\nu^{\beta_1}+B_2\nu^{\beta_2}=A_1\nu^{\alpha_1}. \tag{39}$$

应用与式(19)相应的间断条件，再应用式(36)~(38)，有

$$0=V_1'(\nu-0)-V_2'(\nu+0)=\lim_{S\to\nu-0}V_1'(S)-\lim_{S\to\nu+0}V_2'(S)=$$
$$\lim_{S\to\nu-0}\left(\frac{\mathrm{d}}{\mathrm{d}S}(A_1S^{\alpha_1}+B_1S^{\alpha_2})\right)-\lim_{S\to\nu+0}\left(\frac{\mathrm{d}}{\mathrm{d}S}(A_2S^{\beta_1}+B_2S^{\beta_2})\right)=$$
$$\lim_{S\to\nu-0}(\alpha_1A_1S^{\alpha_1-1})-\lim_{S\to\nu+0}(\beta_1A_2S^{\beta_1-1}+\beta_2B_2S^{\beta_2-1})=$$
$$\alpha_1A_1\nu^{\alpha_1-1}-(\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}).$$

上式表明

$$\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}=\alpha_1A_1\nu^{\alpha_1-1}. \tag{40}$$

应用自由边界条件式(2)，得

$$0=V_2(\omega)-(\omega-K)=(A_2S^{\beta_1}+B_2S^{\beta_2})\Big|_{S=\omega^{-(\omega-K)}}=$$
$$A_2\omega^{\beta_1}+B_2\omega^{\beta_2}-(\omega-K),$$

即

$$A_2\omega^{\beta_1}+B_2\omega^{\beta_2}=(\omega-K). \tag{41}$$

再应用自由边界条件式(3)得

$$0=V_2'(\omega)-1=\left(\frac{\mathrm{d}V_2(S)}{\mathrm{d}S}\right)\Bigg|_{S=\omega^{-1}}=$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}S}(A_2S^{\beta_1}+B_2S^{\beta_2})\right)\Bigg|_{S=\omega^{-1}}=$$
$$(\beta_1A_2S^{\beta_1-1}+\beta_2B_2S^{\beta_2-1})\Big|_{S=\omega^{-1}}=$$
$$(\beta_1A_2\omega^{\beta_1-1}+\beta_2B_2\omega^{\beta_2-1})-1,$$

即

$$\beta_1A_2\omega^{\beta_1-1}+\beta_2B_2\omega^{\beta_2-1}=1. \tag{42}$$

联立式(39)和(40)，有

$$0=\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}-\alpha_1A_1\nu^{\alpha_1-1}=$$
$$\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}-\alpha_1\nu^{-1}(A_1\nu^{\alpha_1})=$$
$$\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}-\alpha_1\nu^{-1}(A_2\nu^{\beta_1}+B_2\nu^{\beta_2})=$$
$$\beta_1A_2\nu^{\beta_1-1}+\beta_2B_2\nu^{\beta_2-1}-\alpha_1\nu^{\beta_1-1}A_2-\alpha_1\nu^{\beta_2-1}B_2=$$
$$(\beta_1-\alpha_1)A_2\nu^{\beta_1-1}+(\beta_2-\alpha_1)\nu^{\beta_2-1}B_2,$$

解之，得

$$(\beta_1-\alpha_1)A_2\nu^{\beta_1-1}+(\beta_2-\alpha_1)\nu^{\beta_2-1}B_2=0. \tag{43}$$

联立式(41)和(42)，计算得

$$0=\beta_1A_2\omega^{\beta_1-1}+\beta_2B_2\omega^{\beta_2-1}-1=$$
$$\beta_1A_2\omega^{\beta_1-1}+\beta_2\omega^{-1}\cdot B_2\omega^{\beta_2}-1=$$
$$\beta_1A_2\omega^{\beta_1-1}+\beta_2\omega^{-1}\cdot(\omega-K-A_2\omega^{\beta_1})-1=$$
$$\beta_1A_2\omega^{\beta_1-1}+(\omega-K)\beta_2\omega^{-1}-\beta_2\omega^{\beta_1-1}A_2-1=$$
$$(\beta_1-\beta_2)A_2\omega^{\beta_1-1}+(\omega-K)\beta_2\omega^{-1}-1=$$
$$(\beta_1-\beta_2)\omega^{\beta_1-1}\left(A_2+\frac{(\omega-K)\beta_2\omega^{-1}-1}{(\beta_1-\beta_2)\omega^{\beta_1-1}}\right),$$

式中，$\beta_1>\beta_2$且$\omega>0$，解之，得

$$A_2=\frac{(\omega-K)\beta_2\omega^{-1}-1}{(\beta_2-\beta_1)\omega^{\beta_1-1}}. \tag{44}$$

解式(42)，得

$$B_2=\frac{\omega^{1-\beta_2}}{\beta_2}-\frac{\beta_1\omega^{\beta_1-\beta_2}}{\beta_2}A_2. \tag{45}$$

将式(44)代入(45)得

$$B_2=\frac{1}{\beta_2}\omega^{1-\beta_2}-\frac{\beta_1\omega^{\beta_1-\beta_2}}{\beta_2}\cdot\frac{\beta_2\omega^{-1}(\omega-K)-1}{(\beta_2-\beta_1)\omega^{\beta_1-1}}=$$
$$\frac{\omega^{1-\beta_2}}{\beta_2}-\frac{\beta_1\omega^{1-\beta_2}}{\beta_2(\beta_2-\beta_1)}(\beta_2\omega^{-1}(\omega-K)-1)=$$
$$\frac{\omega^{1-\beta_2}}{\beta_2}-\frac{\beta_1\omega^{-\beta_2}(\omega-K)}{\beta_2-\beta_1}+\frac{\beta_1\omega^{1-\beta_2}}{\beta_2(\beta_2-\beta_1)}=$$
$$\frac{\omega^{1-\beta_2}}{\beta_2-\beta_1}-\frac{\beta_1\omega^{1-\beta_2}}{\beta_2-\beta_1}+\frac{K\beta_1\omega^{-\beta_2}}{\beta_2-\beta_1}=$$
$$\frac{(\beta_2-2\beta_1)\omega^{1-\beta_2}}{\beta_2-\beta_1}+\frac{K\beta_1\omega^{-\beta_2}}{\beta_2-\beta_1},$$

即

$$B_2=\frac{1}{\beta_2-\beta_1}((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2}). \tag{46}$$

将式(44)和(46)代入(43)得

$$0=(\beta_1-\alpha_1)\nu^{\beta_1-1}A_2+(\beta_2-\alpha_1)\nu^{\beta_1-1}B_2=$$
$$(\beta_1-\alpha_1)\nu^{\beta_1-1}\cdot\frac{\beta_2\omega^{-1}(\omega-K)-1}{(\beta_2-\beta_1)\omega^{\beta_1-1}}+$$
$$(\beta_2-\alpha_1)\nu^{\beta_2-1}\cdot\frac{1}{\beta_2-\beta_1}((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2})=$$
$$\frac{\beta_1-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_1-1}((\beta_2\omega^{-1}(\omega-K)-1)\omega^{1-\beta_1})+$$
$$\frac{\beta_2-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_2-1}((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2})=$$

$$\frac{\beta_1-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_1-1}((\beta_2-\beta_2K\omega^{-1}-1)\omega^{1-\beta_1})+$$
$$\frac{\beta_2-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_2-1}((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2})=$$
$$\frac{\beta_1-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_1-1}((\beta_2-1)\omega^{1-\beta_1}-\beta_2K\omega^{-\beta_1})+$$
$$\frac{\beta_2-\alpha_1}{\beta_2-\beta_1}\nu^{\beta_2-1}((\beta_2-2\beta_1)\omega^{1-\beta_2}+K\beta_1\omega^{-\beta_2})=$$
$$\frac{(\beta_1-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_1-1}}{\beta_2-\beta_1}\cdot\omega^{1-\beta_2}+\frac{(\beta_1-\alpha_1)K\beta_1\nu^{\beta_2-1}}{\beta_2-\beta_1}\omega^{-\beta_2}+$$
$$\frac{(\beta_1-\alpha_1)(\beta_2-1)\nu^{\beta_1-1}}{\beta_2-\beta_1}\omega^{1-\beta_1}-\frac{(\beta_1-\alpha_1)\beta_2K\nu^{\beta_1-1}}{\beta_2-\beta_1}\omega^{-\beta_1}=$$
$$\frac{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_2-1}\omega^{\beta_1}}{\beta_2-\beta_1}\Big(\omega^{1+\beta_1-\beta_2}+$$
$$\frac{(\beta_2-\alpha_1)K\beta_1\nu^{\beta_2-1}}{\beta_2-\beta_1}\cdot\frac{\beta_2-\beta_1}{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_2-1}}\cdot\omega^{\beta_1-\beta_2}+$$
$$\frac{(\beta_1-\alpha_1)(\beta_2-1)\nu^{\beta_1-1}}{\beta_2-\beta_1}\cdot\frac{\beta_2-\beta_1(\beta_2-1)}{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_2-1}}\cdot\omega-$$
$$\frac{(\beta_1-\alpha_1)\beta_2K\nu^{\beta_1-1}}{\beta_2-\beta_1}\cdot\frac{\beta_2-\beta_1}{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_2-1}}\Big)=$$
$$\frac{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_1-1}\omega^{-\beta_1}}{\beta_2-\beta_1}\cdot(\omega^{m+1}+a\omega^m+b\omega+c)=$$
$$\frac{(\beta_2-\alpha_1)(\beta_2-2\beta_1)\nu^{\beta_1-1}\omega^{-\beta_1}}{\beta_2-\beta_1}f(\omega).$$

应用上式,且有$\beta_2-\alpha_1<0$, $\beta_2-2\beta_1<0$, $\nu>0$, $\omega>0$, $\beta_2-\beta_1<0$, 则有

$$f(\omega)=0. \tag{47}$$

根据已知条件式(17),函数方程

$$f(\omega)=0$$

在$(\nu,+\infty)$内有$n\geqslant1$个不同的实数根

$$\omega=\omega_i(i=1,2,\cdots,n). \tag{48}$$

将式(48)代入(44)和(46),得

$$A_2^i=\frac{\beta_2\omega_i^{-1}(\omega_i-K)-1}{(\beta_2-\beta_1)\omega_i^{\beta_1-1}}, \tag{49}$$

$$B_2^i=\frac{1}{\beta_2-\beta_1}\Big((\beta_2-2\beta_1)\omega_i^{1-\beta_2}+K\beta_1\omega_i^{-\beta_2}\Big). \tag{50}$$

再将式(49)、(50)代入(40)得

$$A_1^i=\frac{1}{\alpha_1\nu^{\alpha_1-1}}(\beta_1A_2^i\nu^{\beta_1-1}+\beta_2\beta_2^i\nu^{\beta_2-1}). \tag{51}$$

应用式(49)~(51),并应用式(36)和(38),得出自由边界问题式(1)~(4)有n个解,即

$$(V^1(S),\omega_1),(V^2(S),\omega_2),\cdots,(V^n(S),\omega_n),$$

式中

$$V^i(S)=\begin{cases}V_1^i(S),S\in[0,\nu),\\ V_2^i(S),S\in[\nu,\omega_i],\end{cases}$$

这里 $V_1^i(S)=A_1^iS^{\alpha_1}$, $V_2^i(S)=A_2^iS^{\beta_1}+B_2^iS^{\beta_2}$, $A_1^i=\frac{\nu^{1-\alpha_1}}{\alpha_1}\cdot(\beta_1A_2^i\nu^{\beta_1-1}+\beta_2B_2^i\nu^{\beta_2-1})$, $A_2^i=\frac{\beta_2\omega_i^{-1}(\omega_i-K)-1}{(\beta_2-\beta_1)\omega_i^{\beta_1-1}}$, $B_2^i=\frac{1}{\beta_2-\beta_1}\cdot\Big((\beta_2-2\beta_1)\omega_i^{1-\beta_2}+K\beta_1\omega_i^{-\beta_2}\Big)$.

定理 2 得证.

3 定理 3 的证明

应用式(14)~(16),一元二次方程

$$f(X)=X^2+(a+b)X+c=0 \tag{52}$$

有如下 2 个实数根:

$$X_1=\frac{-(a+b)+\sqrt{(a+b)^2-4c}}{2}, \tag{53}$$

$$X_2=\frac{-(a+b)-\sqrt{(a+b)^2-4c}}{2}. \tag{54}$$

应用式(21),得

$$c<0. \tag{55}$$

由式(53)~(55)可知,式(52)在$(-\infty,+\infty)$内只有 1 个正实数根

$$X_1=\frac{-(a+b)+\sqrt{(a+b)^2-4c}}{2},$$

这表明

$$f(x)>0,\ x\in(X_1,+\infty). \tag{56}$$

另外,应用式(22),得

$$f(\nu)<0.$$

应用上式及式(56),可得

$$\nu\in(0,X_1).$$

特别地,选取自由边界$\omega=X_1$,则式(52)在$(\nu,+\infty)$内只有 1 个正实数根.

应用定理 2,即可得到定理 3. 定理 3 得证.

4 结论

当波动率σ是间断函数时,找到了一个永久美式看涨期权的定价公式. 这个公式丰富了期权定价理论,它不仅给投资者提供了规避金融风险的理论,也为管理者制定金融政策提供了重要参考.

5 参考文献

[1] JIANG L. Mathematical modeling and methods of option

pricing[M]. River Edge, NJ: World Scientific Publishing Co Inc, 2005
[2] JIANG L, DAI M. Convergence of explicit difference scheme and the binomial tree method for American options[J]. Journal of Computational Mathematics, 2004(22): 371
[3] JIANG L, DAI M. Convergence of binomial tree methods for European/American path-dependent options[J]. SIAM Journal of Numerical Analysis, 2004(42): 1094
[4] JIANG L, CHEN Q, WANG L, et al. A new well-posed algorithm to recover implied local volatility[J]. Quantitative Finance, 2003(3): 451
[5] FRIEDMAN A. Variational principles and free-boundary problems[M]. New York: John Wiley & Sons Inc, 1982
[6] HULL J. Options, futures, and other derivatives[M]. 4th ed. Upper Saddle River: Prentice-Hall, 2006
[7] JIANG L. Analysis of pricing American options on the maximum (minimum) of two risk assets[J]. Interface & Free Boundary, 2002(4): 27
[8] KARATZAS I, Shreve S. Methods of mathematical finance[M]. New York: Springer-Verlag,1998
[9] MERTON R. Continuous-time finance[M]. Greenwich: Wiley, 1992
[10] MUSIELA M, RUTKOWSKI M. Martingale methods in financial modelling[M]. Berlin: Springer-Verlag, 1998
[11] WOLMOTT P, DEWYNNE J, HOWISON S. Option pricing: mathematical models and computation[M]. Oxford: Oxford Financial Press, 1993
[12] JIANG L, BIAN B. The regularized implied local volatility equations: a new model to recover the volatility of underlying asset from option pricing, working paper[C]. Shanghai: Tongji University, 2005: 1
[13] JIANG L, DAI M. On path-dependent options[C]//Mathematical Finance-Theory & Applications. Beijing: Higher Education Press, 2000: 290
[14] JIANG L, YANG D, ZHANG S. On pricing model of reset option with N predetermined levels[J]. Journal of Systems Science & Complexity, 2004(17): 137
[15] WILMOTT P. Derivatives: The theory and practice of financial engineering[M]. New York: John Wiley & Sons Inc, 1998
[16] WANG S, YUAN F. The asymptotic behavior of the solutions of the Black-Scholes equation as volatility $\sigma \to 0^+$[J]. Computers and Mathematics with Applications, 2019(78): 1037

The pricing problem for a class of permanent American option

WANG Shu[1)]　YUAN Fang[1,2)†]
(1)College of Applied Sciences, Beijing University of Technology, 100124, Beijing, China;
2)China Everbright Bank, 100054, Beijing, China)

Abstract　For the pricing problem of a class of permanent American option, we allow volatility to be a discontinuous function. Analytical techniques such as differential equation theory were used to overcome difficulty due to discontinuity in volatility and to establish a type of option pricing formula.

Keywords　Black-Scholes equation; permanent American option; discontinuous volatility

【责任编辑：陆有忠】