含信号调制噪声和频率波动的时滞分数阶振子的随机共振

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摘　要：本文研究了含信号调制噪声和频率波动的小时滞线性分数阶振子的随机共振. 利用分数阶 Shapiro-Loginov 公式和 Laplace 变换技巧，本文首先推导了系统响应的一阶稳态矩和稳态响应振幅增益(Output Amplitude Gain，OAG)的解析表达式，然后讨论了分数阶、时滞和噪声参数对 OAG 的影响. 结果显示：各参数对 OAG 的影响均呈现非单调变化的特点，表明系统出现广义随机共振. 特别地，分数阶与时滞的协同作用可能诱导随机共振的多样化. 这就为在一定范围内调控随机共振提供了可能.

关键词：广义随机共振；线性分数阶振子；频率涨落；信号调制噪声；时滞
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Stochastic resonance of a time-delayed fractional oscillator with fluctuating frequency and signal-modulated noise

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Abstract: In this paper, stochastic resonance (SR) of time-delayed linear fractional oscillators subjected to both frequency-modulated and signal-modulated noises is investigated. By using the (fractional) Shapiro-Loginov formula and Laplace transform technique, the analytical expression of the output amplitude gain (OAG) is firstly derived, and the dependence of OAG on the system parameters such as fractional order, time delay and the parameters of noises is explored. It is shown that every dependence is non-monotonic, say, generalized stochastic resonance (GSR) happens. Particularly, cooperation of fractional order and time delay may result in diverse GSR behaviors of the system. In other word, the GSR behaviors can be controlled just by the system parameters.

Keywords: Generalized stochastic resonance; Linear fractional oscillator; Frequency fluctuation; Signal-modulated noise; Time delay

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1 Introduction

Since the term"stochastic resonance (SR)" was originally coined by Benzi, Sutera and Vulpiani [1] in 1981, SR has become a popular research topic due to its widespread applications in biology, physics, chemistry, engineering science, and even economics[2-5]. As a counter-intuitive concept, SR utilizes noise instead of eliminating it to improve signal-to-noise ratio (SNR) of nonlinear systems, which reflects the fact that noise can play a constructive rather than destructive role in stochastic systems. Early researches on noise-induced SR focused primarily on those nonlinear dynamical systems with periodic signal and additive noise[6, 7]. Subsequently, many studies have shown that SR can occur in linear systems with multiplicative noises[8-10]. Additionally, generalized SR (GSR), as the extension of SR, was introduced by Gitterman to depict the non-monotonic dependence of output amplitude of system steady response on the system parameters[10, 11].

In recent years, the discovery of anomalous diffusion in various fields has led to the study on numerous fractional stochastic models[12-14], in which the linear fractional models with noise have been of particular interest due to their ease of theoretical analysis compared to the nonlinear fractional stochastic models. Subsequently, the fractional oscillators (FOs) with mass fluctuation[15-18], damping fluctuation[19, 20] or frequency fluctuation[21-23] have attracted considerable attention, and their GSR phenomena have been extensively investigated. The majority of existing researches investigated the GSR behavior induced by additive and multiplicative noises. However, in the case of signal-modulated noise in optics and radio astronomy, the noise is often modulated by signal[24]. Recently, close attention is paid to GSR in the FOs subjected to signal-modulated noise[25-32]. For instance, He *et al.*[27] and Lin *et al.*[28] investigated the bias-signal-modulated dichotomous and trichotomous noise induced GSR phenomena in the FOs with random frequency respectively. Additionally, SR behaviors induced by the signal-modulated noises were also studied in the FOs with random mass or random damping[29-31], and the collective resonance behavior in coupled FOs subjected to noise-modulated external periodic force was studied in Ref. [32].

Moreover, time delays are inevitably present in nature due to the finite transmission speed of information and energy[33]. In some complex systems, time delay is an important factor that cannot be ignored in the study of dynamic problems, and even a small time delay can qualitatively change the dynamical characteristics. Therefore, stochastic time-delayed models have been employed to simulate and analyze the dynamic behaviors in scientific and engineering fields [34-37]. However, few literatures have considered the synergy in the time-delayed systems with the frequency fluctuation and signal-modulated noise. Hence, in this paper, we propose a time-delayed FO subjected to frequency fluctuation and signal-modulated dichotomous noise, here the noises are exponentially correlated. Then, we study the GSR behaviors and the nonlinear effect of time delay, and further reveal the cooperative mechanism of fractional order, time delay and noise parameters.

The remainder of this paper is structured as follows. We propose the system model and derive the analytical expression of the output amplitude gain(OAG) in Sections 2. The numerical results and discussion are presented in Section 3. A brief conclusion follows in Section 4.

2 The model

We consider a time-delayed fractional harmonic oscillator described by the following stochastic differential equation (SDE):

$$\ddot{x}(t)+\gamma\, {}_0^C D_t^\alpha x(t)+[\omega^2+\xi_1(t)]x(t-\tau)= \xi_2(t)[1+R\cos(\Omega t)]+\zeta(t) \quad (1)$$

where $x(t)$ is the displacement of Brownian particle at time t, $\gamma>0$ is the damping coefficient, ω

represents the system intrinsic frequency, $\tau>0$ is the time delay, and $\xi_2(t)[1+R\cos(\Omega t)]$ is a bias-signal-modulated noise. In various physical and biological environments, viscous media usually have power-law memory which represents the dependence of the viscous force on the velocity history of particle[38]. Therefore, the viscous force here is defined by fractional derivative $\gamma\,{}_0^{C}D_t^{\alpha}x(t)=\gamma\int_0^t\frac{1}{\Gamma(1-\alpha)}(t-s)^{-\alpha}\dot{x}(s)\mathrm{d}s$ with α-order Caputo form. The internal noise term $\zeta(t)$, defined by $\zeta(t)=\sqrt{2\kappa_B T}\frac{\mathrm{d}}{\mathrm{d}t}B_H(t)$, is a fractional Gaussian noise (fGn), here $B_H(t)$ is the fractional Brownian motion with Hurst parameter $H=1-\frac{\alpha}{2}$. In addition, the internal noise $\zeta(t)$ shares the same origin as the damping force of the system[39], and then $\zeta(t)$ is supposed to satisfy the generalized second fluctuation-dissipation theorem (FDT)[40]:

$$\langle\zeta(t)\rangle=0,\ \langle\zeta(t)\zeta(s)\rangle=\frac{\gamma\kappa_B T}{\Gamma(1-\alpha)}|t-s|^{-\alpha}\tag{2}$$

where the average $\langle\cdot\rangle$ is taken over an ensemble in thermal equilibrium, κ_B is the Boltzmann constant, and T is the absolute temperature. In addition, the frequency fluctuation $\xi_1(t)$ and signal-modulated noise $\xi_2(t)$ are modeled as symmetric dichotomous noises, taking two values $\xi_i(t)\in\{-\sigma_i,\sigma_i\}, i=1,2$, and have the following statistical properties:

$$\langle\xi_i(t)\rangle=0,\ \langle\xi_i(t)\xi_i(s)\rangle=\sigma_i^2\mathrm{e}^{-\lambda_i|t-s|},$$
$$\langle\xi_1(t)\xi_2(s)\rangle=\langle\xi_1(s)\xi_2(t)\rangle=\sigma_1\sigma_2\mathrm{e}^{-\lambda_3|t-s|}\tag{3}$$

where σ_i^2 and λ_i are the noise intensity and correlation rate of $\xi_i(t)$ $(i=1,2)$, λ_3 characterizes the correlation rate between $\xi_1(t)$ and $\xi_2(t)$, respectively. We further assume that they are uncorrelated with the additive fGn term

$$\langle\xi_1(t)\zeta(s)\rangle=\langle\xi_2(t)\zeta(s)\rangle=0.$$

Particularly, for $\tau=0$, Eq. (1) turns into the FO with signal-modulated noise, and it had been investigated in Ref. [27].

Performing an $O(\tau^2)$ Taylor expansion around $\tau=0$ on the function $x(t-\tau)$, Eq. (1) can be equivalently given by the following form without time delay:

$$\ddot{x}(t)+\gamma\,{}_0^{C}D_t^{\alpha}x(t)+[\omega^2+\xi_1(t)]\cdot[x(t)-\tau\dot{x}(t)]=\xi_2(t)[1+R\cos(\Omega t)]+\zeta(t)\tag{4}$$

It should be noted that Eq. (4) is an approximation to Eq. (1) only in the situation of small time delay. In order to analyze the system steady response, we average Eq. (4) and have

$$\langle\ddot{x}(t)\rangle+\gamma\,{}_0^{C}D_t^{\alpha}\langle x(t)\rangle+\omega^2\langle x(t)\rangle+\langle\xi_1(t)x(t)\rangle-\omega^2\tau\langle\dot{x}(t)\rangle-\tau\langle\xi_1(t)\dot{x}(t)\rangle=0\tag{5}$$

Meanwhile, we multiply both sides of Eq. (4) by $\xi_1(t)$ and then average all the terms, thus obtain

$$\langle\xi_1(t)\ddot{x}(t)\rangle+\gamma\langle\xi_1(t)\,{}_0^{C}D_t^{\alpha}x(t)\rangle+\omega^2\langle\xi_1(t)x(t)\rangle+\sigma_1^2\langle x(t)\rangle-\omega^2\tau\langle\xi_1(t)\dot{x}(t)\rangle-\sigma_1^2\tau\langle\dot{x}(t)\rangle=\sigma_1\sigma_2[1+R\cos(\Omega t)]\tag{6}$$

For splitting the correlations, we employ the Shapiro-Loginov (S-L) formula[41], which reads as

$$\langle\xi_i(t)\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\rangle=\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_i\right)^n\langle\xi_i(t)x(t)\rangle,\ i=1,2\tag{7}$$

At the same time, applying the fractional S-L formula[42] to the term $\langle\xi_i(t)\,{}_0^{C}D_t^{\alpha}x(t)\rangle$, we have

$$\langle\xi_i(t)\,{}_0^{C}D_t^{\alpha}x(t)\rangle=\mathrm{e}^{-\lambda_i t}\,{}_0^{C}D_t^{\alpha}(\langle\xi_i(t)x(t)\rangle\mathrm{e}^{\lambda_i t}),\quad i=1,2\tag{8}$$

Substituting S-L formula (7) and fractional S-L formula (8) into Eqs. (5)(6), we obtain the following fractional differential equations:

$$\begin{cases}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\omega^2\tau\frac{\mathrm{d}}{\mathrm{d}t}+\gamma\,{}_0^{C}D_t^{\alpha}+\omega^2\right)x_1(t)+\left[1-\tau\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_1\right)\right]x_2(t)=0,\\ \sigma_1^2\left(1-\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)x_1(t)+\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_1\right)^2-\omega^2\tau\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_1\right)+\omega^2\right]x_2(t)+\gamma\mathrm{e}^{-\lambda_1 t}\,{}_0^{C}D_t^{\alpha}(x_2(t)\mathrm{e}^{\lambda_1 t})=\sigma_1\sigma_2(1+R\cos(\Omega t))\end{cases}\tag{9}$$

with $x_1(t)\triangleq\langle x(t)\rangle$ and $x_2(t)\triangleq\langle\xi_1(t)x(t)\rangle$. Then we perform Laplace transform on Eq. (9) and obtain

$$\begin{cases}[s^2-\omega^2\tau s+\gamma s^\alpha+\omega^2]X_1(s)+\\ \quad(1-\tau s-\tau\lambda_1)X_2(s)=b_1,\\ \sigma_1^2(1-\tau s)X_1(s)+[(s+\lambda_1)^2-\omega^2\tau(s+\lambda_1)+\\ \quad\gamma(s+\lambda_1)^\alpha+\omega^2]X_2(s)=\\ \quad\sigma_1\sigma_2\left(\frac{1}{s}+\frac{Rs}{s^2+\Omega^2}\right)+b_2\end{cases} \tag{10}$$

where $X_i(s)=\mathscr{L}\{x_i(t)\}=\int_0^{+\infty}x_i(t)\mathrm{e}^{-st}\mathrm{d}t$ is the Laplace transform of $x_i(t)$, $i=1,2$, the coefficients b_1 and b_2 are respectively given by

$$b_1=(s+\gamma s^{\alpha-1}-\omega^2\tau)x_1(0)-\tau x_2(0)+x_3(0),$$
$$b_2=-\sigma_1^2\tau x_1(0)+[s+2\lambda_1+\gamma(s+\lambda_1)^{\alpha-1}-\omega^2\tau]x_2(0)+x_4(0),$$

with initial conditions $x_i(0)$, $x_{i+2}(0)=\dot{x}_i(0)$, $i=1,2$. Obviously, all the solutions $X_i(s)$ of Eq. (10) can be uniformly obtained. Specially, we mainly focus on the first-order moment of system steady response in present work, thus $X_1(s)$ can be written as

$$X_1(s)=H_{10}(s)\left(\frac{1}{s}+\frac{Rs}{s^2+\Omega^2}\right)+\sum_{k=1}^{4}H_{1k}(s)x_k(0) \tag{11}$$

with

$$H_{10}=\frac{-a_{12}\sigma_1\sigma_2}{a_{11}a_{22}-a_{12}a_{21}},$$
$$H_{11}=\frac{a_{22}(s-\omega^2\tau+\gamma s^{\alpha-1})+a_{12}\sigma_1^2\tau}{a_{11}a_{22}-a_{12}a_{21}},$$
$$H_{12}=\frac{-a_{22}\tau-a_{12}[s+2\lambda_1-\omega^2\tau+\gamma(s+\lambda_1)^{\alpha-1}]}{a_{11}a_{22}-a_{12}a_{21}},$$
$$H_{13}=\frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}},$$
$$H_{14}=\frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}},$$
$$a_{11}=s^2-\omega^2\tau s+\gamma s^\alpha+\omega^2,$$
$$a_{12}=1-\tau s-\tau\lambda_1,$$
$$a_{21}=\sigma_1^2(1-\tau s),$$
$$a_{22}=(s+\lambda_1)^2-\omega^2\tau(s+\lambda_1)+\gamma(s+\lambda_1)^\alpha+\omega^2.$$

Finally, we apply the inverse Laplace transform and obtain

$$\langle x(t)\rangle=x_1(t)=\int_0^t h_{10}(t-t')(1+R\cos(\Omega t'))\mathrm{d}t'+\sum_{k=1}^{4}h_{1k}(t)x_k(0) \tag{12}$$

where $H_{1k}(s)$ are the Laplace transforms of $h_{1k}(t)$, $k=1,\cdots,4$. In the long-time limit of $t\to\infty$, the influence of initial conditions will vanish, and the system steady response $\langle x(t)\rangle_{as}=\lim_{t\to\infty}\langle x(t)\rangle$ can be expressed as

$$\langle x(t)\rangle_{as}=A_{as}\cos(\Omega t+\varphi_{as})+K \tag{13}$$

with the output amplitude

$$A_{as}=R|H_{10}(\mathrm{j}\Omega)|=R\sqrt{\frac{u_1^2+u_2^2}{u_3^2+u_4^2}} \tag{14}$$

and phase shift

$$\varphi_{as}=\arg(H_{10}(\mathrm{j}\Omega))=\arctan\left(\frac{u_2u_3-u_1u_4}{u_1u_3+u_2u_4}\right) \tag{15}$$

where the constant

$$K=\frac{\sigma_1\sigma_2(\tau\lambda_1-1)}{\omega^2[\lambda_1^2+\gamma\lambda_1^\alpha+\omega^2(1-\tau\lambda_1)]-\sigma_1^2(1-\tau\lambda_1)} \tag{16}$$

The related coefficients in Eqs. (14)~(16) are listed as follows.

$$b=\sqrt{\Omega^2+\lambda_1^2},$$
$$\theta=\arctan\left(\frac{\Omega}{\lambda_1}\right),$$
$$f_1=\omega^2-\Omega^2+\gamma\Omega^\alpha\cos\left(\frac{\pi}{2}\alpha\right),$$
$$f_2=\gamma\Omega^\alpha\sin\left(\frac{\pi}{2}\alpha\right)-\omega^2\Omega\tau,$$
$$f_3=\lambda_1^2+\omega^2(1-\lambda_1\tau)-\Omega^2+\gamma b^\alpha\cos(\theta\alpha),$$
$$f_4=2\Omega\lambda_1-\omega^2\Omega\tau+\gamma b^\alpha\sin(\theta\alpha),$$
$$u_1=\sigma_1\sigma_2(\tau\lambda_1-1),$$
$$u_2=\sigma_1\sigma_2\Omega\tau,$$
$$u_3=f_1f_3-f_2f_4+\sigma_1^2(\tau\lambda_1-1)+\sigma_1^2\Omega^2\tau^2,$$
$$u_4=f_1f_4+f_2f_3+\sigma_1^2\Omega\tau(2-\tau\lambda_1).$$

Then the output amplitude gain (OAG) is given by

$$G=\frac{A_{as}}{R}=\sqrt{\frac{u_1^2+u_2^2}{u_3^2+u_4^2}} \tag{17}$$

3 Results and discussion

In this section, we explore the resonance behaviors of G based on the analytical expression of Eq. (17). Additionally, we investigate the dependence of G on various system parameters, including α, τ, Ω, σ_1 and λ_1.

3.1 GSR to the driving frequency

Firstly, we present the curves of G versus Ω

for different α and τ in Fig. 1. It is shown that $G(\Omega)$ can achieve a peak value no matter whether time delay τ exists or not, which indicates that GSR appears. From Fig. 1a, we can see that as α increases, the peak value decreases, and the peak position shifts to the left in the FO without time delay ($\tau=0$). Comparing with Fig. 1a, as α increases, the double-peak GSR disappears and the single-peak GSR occurs in the FO with small time delay ($\tau=0.03$) in Fig. 1b. Moreover, $G(\Omega)$ presents the same trend as α changes, but the maximum value goes up. Next, we choose the fixed fractional order $\alpha=0.4$ to investigate the effect of small time delay τ on the non-monotonic behaviors of $G(\Omega)$. It is observed that G becomes stronger and the GSR peak gets sharper as the time delay τ increases. The enhancement effect could be explained as follows. The time-delayed system is, in fact, a memory system, and bigger τ means stronger memory of the system. Thus, the energy can be accumulated by the memory effect, leading to an enhancement of the GSR intensity as τ increases. In addition, for small noise correlation rate $\lambda_1=0.1$ in Fig. 1c, there is a changing in the GSR from one peak to double peaks as τ increases. Specifically, two peaks are observed on $G(\Omega)$ for $\tau>0.07$. However, for $\lambda_1=0.5$ in Fig. 1d, the double-peak GSR phenomenon disappears, and the GSR intensity is weakened. All the above results indicate that properly increasing τ can enhance the GSR intensity, and induce more diverse GSR phenomena.

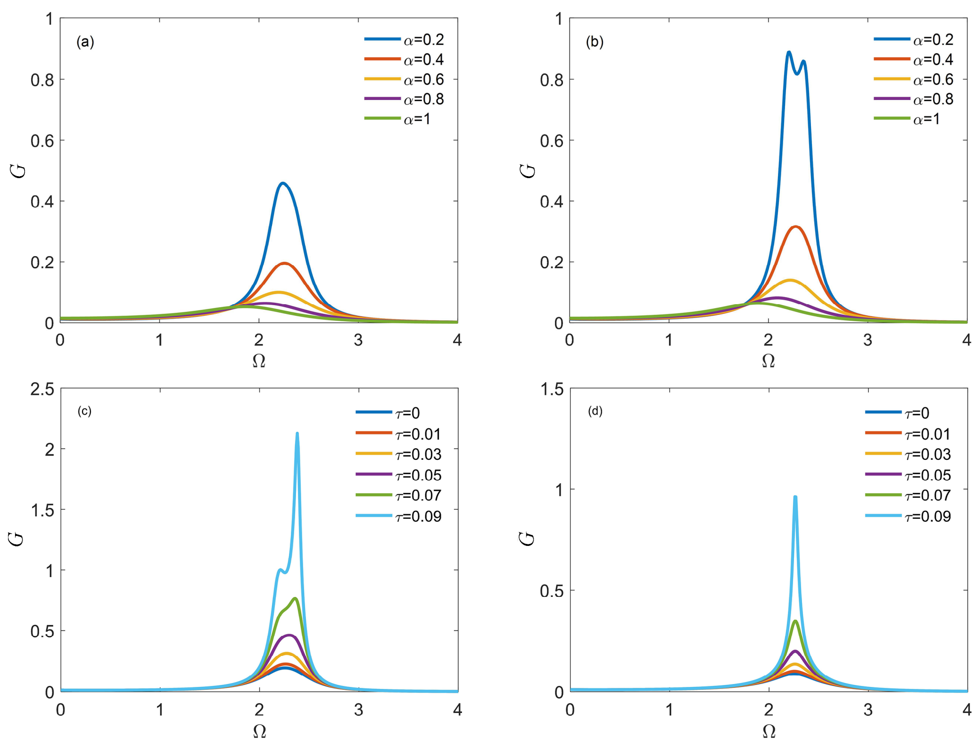

Fig. 1　GSR phenomena of $G(\Omega)$ for different α and τ: (a) $\tau=0$, $\sigma_1=0.5$, $\sigma_2=0.5$, $\lambda_1=0.1$; (b) $\tau=0.03$, $\sigma_1=0.5$, $\sigma_2=0.5$, $\lambda_1=0.1$; (c) $\alpha=0.4$, $\sigma_1=0.5$, $\sigma_2=0.5$, $\lambda_1=0.1$; (d) $\alpha=0.4$, $\sigma_1=0.5$, $\sigma_2=0.5$, $\lambda_1=0.5$. The other parameters are set as $\gamma=1$, $\omega=2$

3.2　GSR to the parameters of frequency fluctuation σ_1 and λ_1

In this subsection, our main focus is to investigate the effect of frequency fluctuation, particularly in terms of σ_1 and λ_1.

Ref. [20] has demonstrated that GSR of $G(\sigma)$ could still occur in some dynamic systems even when the intrinsic frequency ω is not equal to the driving frequency Ω. Without loss of generality, we take $\Omega=2$ and $\omega=1.5$ as an example. To further demonstrate the effect of time delay on $G(\sigma_1)$ in detail, we plot the three-dimensional and two-dimensional graphs of G versus σ_1 and τ in Fig. 2. As shown in Fig. 2b, we observe that the curves of $G(\sigma_1)$ present one peak, and the peak value increases sharply as τ increases, while the peak position moves to the smaller value of σ_1. These phenomena indicate that a small time delay τ can enhance the GSR intensity, mainly due to the enhancement of memory effect with increasing τ. In particular, a larger τ leads to a stronger system memory, which results in more efficient transformation of noise energy into signal energy. As a result, with increase of τ, G increases and attains maximum at smaller σ_1.

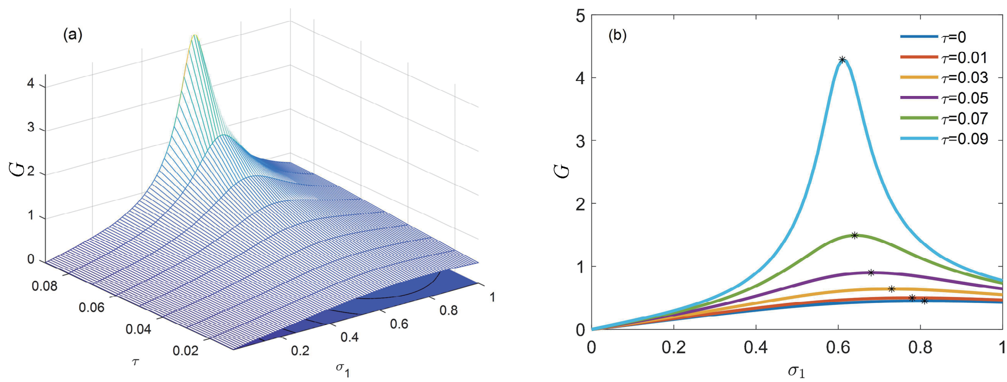

Fig. 2 (a) Three-dimensional graph of $G(\sigma_1, \tau)$; (b) $G(\sigma_1)$ curves with different τ. The other parameters are set as $\gamma=1$, $\omega=1.5$, $\Omega=2$, $\alpha=0.2$, $\lambda_1=0.1$, $\sigma_2=0.5$

In the following, we continue to study the effect of the noise correlation rate λ_1 on G for different α and τ. As seen in Fig. 3, the curves show that G attains a peak value with increasing λ_1, which means that GSR phenomenon takes place. Fig. 3(a) indicates that the bigger the fractional order α is, the bigger the peak value is, and the more the peak position shifts to the left. The GSR phenomenon disappears around $\alpha>0.3$. Fig. 3(b) shows that there is no resonance behavior for $\tau\leqslant 0.07$ and the single-peak resonance behavior occurs for $\tau>0.07$. From Fig. 3(b), we also observe that the bigger the time delay τ is, the smaller the GSR intensity is, and the more the peak position shifts to the right. Comparing Fig. 3(a) with Fig. 3(b), we can observe a similar trend, which can be explained as follows. In a time-delayed FO with noise, both α and τ reflect the system's memory property, and λ_1 reflects the noise's memory property. Specifically, smaller α and bigger τ indicate a stronger memory of the time-delayed FO. When the noise intensity is fixed, bigger λ_1 means that the correlation time of noise is shorter, *i. e.* the memory of noise is weaker. Consequently, bigger λ_1 can overcome the memory effect of the system caused by decreasing α or increasing τ.

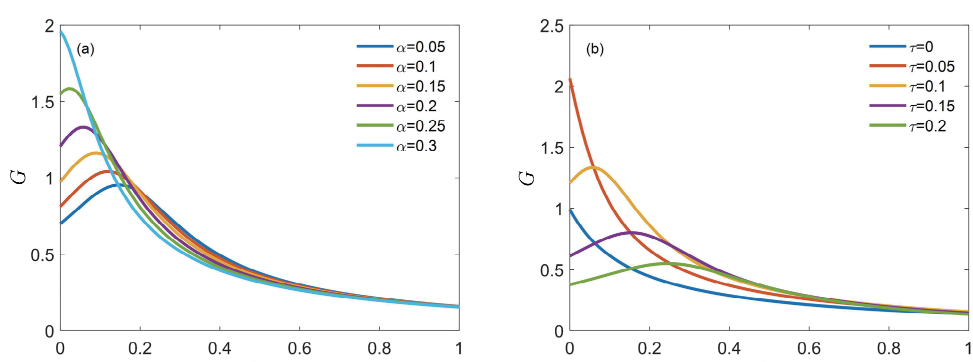

Fig. 3 GSR phenomena of $G(\lambda_1)$ for different α, τ, σ_1, σ_2: (a) $\tau=0.1$, $\sigma_1=0.9$, $\sigma_2=0.9$; (b) $\alpha=0.2$, $\sigma_1=0.9$, $\sigma_2=0.9$. The other parameters are set as $\gamma=1$, $\omega=2$, $\Omega=2$

3.3 GSR to the fractional order α

Finally, we investigate the effect of the fractional order α on G for different τ, and accordingly depict the three-dimensional and two-dimensional graphs of G versus α and τ in Fig. 4. As shown in Fig. 4(b), the curve of $G(\alpha)$ decreases monotonically when $\tau=0$, indicating that GSR does not occur in the absence of time delay. In addition, $G(\alpha)$ presents non-monotonic variation when $\tau>0$, *i. e.*, GSR appears in the FO with time delay. As τ increases, the peak value decreases first and then increases, and the peak position gradually shifts to the right.

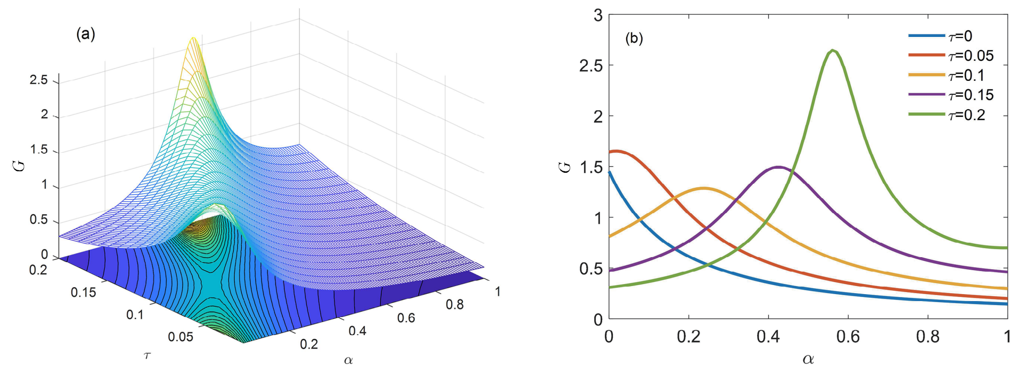

Fig. 4 (a) Three-dimensional graph of $G(\alpha,\tau)$; (b) $G(\alpha)$ curves with different τ. The other parameters are set as $\gamma=1$, $\omega=2$, $\Omega=2$, $\sigma_1=0.9$, $\lambda_1=0.1$, $\sigma_2=0.9$

4 Conclusions

The frequency fluctuation and signal-modulated noise exist widely in various complex physical and engineering systems. In this study, we propose a time-delayed FO with fluctuating frequency, and investigate the effect of small time delay on GSR phenomena driven by signal-modulated dichotomous noise. By using the (fractional) Shapiro-Loginov formula and Laplace transform, we obtain the analytical expression of the OAG. On that basis, we observe that diversified GSR phenomena take place in the system, and further explore the dependence on several parameters, including Ω, α, τ, σ_1 and λ_1. Specially, $G(\Omega)$ exhibits both single-peak and double-peak GSR phenomena, and the diversity of GSR is dependent on α and τ. At the same time, increasing τ can enhance the GSR intensity of $G(\Omega)$, while increasing α weakens the resonance intensity. In addition, GSR also occurs in response to variations in the frequency fluctuation parameter σ_1 and λ_1, and stronger system′s memory can enhance the GSR intensity of $G(\sigma_1)$, but weakens the GSR intensity of $G(\lambda_1)$. Finally, $G(\alpha)$ presents single-peak GSR phenomenon, and the peak value varies non-monotonously as τ increases. These results can provide theoretical support for further studies of weak signal detection in engineering applications.

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