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磁粘弹性演化模型解的局部存在唯一性

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摘　要：文章证明了在$\mathbb{R}^2$、$\mathbb{R}^3$中磁粘弹性演化模型解的局部存在唯一性。该模型由不可压缩的 Navier-Stokes 系统、变形梯度演化的正则化系统和磁化动力学的 Landau-Lifshitz-Gilber 系统组成。该方法依赖于用一系列扰动系统来逼近该系统。

关键词：磁粘弹性演化模型；局部解；唯一性；扰动系统
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Local existence and uniqueness of solutions to the evolutionary model for magnetoviscoelasticity

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Abstract: In this paper, we prove the local existence and uniqueness of solutions to the evolutionary model for magnetoviscoelasticity in $\mathbb{R}^2$, $\mathbb{R}^3$. This model consists of an incompressible Navier-Stokes, a regularized system for the evolution of the deformation gradient and the Landau-Lifshitz-Gilbert system for the dynamics of the magnetization. Our approach depends on approximating the system with a sequence of perturbed systems.

Key words: the evolutionary model for magnetoviscoelasticity; local solution; uniqueness; perturbed systems

1　Introduction

In this paper, the following evolutionary model for magnetoviscoelasticity with an initial value problem is considered[1]:

$$\begin{cases}\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla P = \mu\Delta\boldsymbol{v} - \nabla\cdot \\ \quad (2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M} - \mathbb{W}'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}) + \mu_0\nabla H_{ext}^{\mathrm{T}}\boldsymbol{M}, \\ \nabla\cdot\boldsymbol{v} = 0, \\ \partial_t\boldsymbol{F} + \boldsymbol{v}\cdot\nabla\boldsymbol{F} - \nabla\boldsymbol{v}\boldsymbol{F} = \kappa\Delta\boldsymbol{F}, \\ \partial_t\boldsymbol{M} + \boldsymbol{v}\cdot\nabla\boldsymbol{M} = -\gamma\boldsymbol{M}\times(2A\Delta\boldsymbol{M} + \mu_0 H_{ext}) - \\ \quad \lambda\boldsymbol{M}\times[\boldsymbol{M}\times(2A\Delta\boldsymbol{M} + \mu_0 H_{ext})], \\ (\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})(x,0) = (\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0), |\boldsymbol{M}_0| = 1.\end{cases} \tag{1}$$

The dimension here is $d = 2,3$. The first equation of system (1) describes the balance of momentum in Eulerian coordinates with the velocity $\boldsymbol{v}:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ and the pressure $P:\mathbb{R}^d\times[0,T]\to\mathbb{R}$. The second equation of system (1) describes an incompressible condition. Analogously, the third equation of system (1) describes the evolution of the deformation gradient in Eulerian coordinates with deformation gradient $\boldsymbol{F} = (\boldsymbol{F}^{ij})_{1\leqslant i,j\leqslant n}\in\mathbb{R}^{d\times d}$ and the fourth equation of system (1) is a variant of the Landau-Lifschit-Gilbert(LLG) evolution equation for the magnetization $\boldsymbol{M}$: $\mathbb{R}^d\times[0,T]\to\mathbb{S}^2$. In this system (1), H_{ext} stands for the external magnetic field, $\mathbb{W}(\boldsymbol{F})$ represents elastic energy, and $\mu > 0$ as the viscosity of the fluid. γ is the electron gyromagnetic ratio, $\lambda > 0$ as a phenomenological damping pa-

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rameter. The notation × represents the cross product for vectors in $\mathbb{R}^3$, and the term $\nabla \boldsymbol{M} \odot \nabla \boldsymbol{M}$ denotes the $\boldsymbol{M} \times \boldsymbol{M}$ matrix whose (i,j)-th entry is given by $\partial_i \boldsymbol{M} \cdot \partial_j \boldsymbol{M} (1 \leqslant i,j \leqslant 3)$.

For the sake of simplicity in this article, we assume $W(\boldsymbol{F}) = |\boldsymbol{F}|^2$, $H_{ext} = 0$ and $\alpha = 2\gamma A, \beta = 2\lambda A$, $A \geqslant \frac{1}{2}$. The constants α, β and μ, κ are positive constants. The constant $\beta > 0$ is called the Gilbert damping coefficient. Subsequently, we will explore the following simplified system:

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla P = \mu \Delta \boldsymbol{v} - \nabla \cdot \\ \quad (2A \nabla \boldsymbol{M} \odot \nabla \boldsymbol{M} - W'(\boldsymbol{F}) \boldsymbol{F}^{\mathrm{T}}), \\ \nabla \cdot \boldsymbol{v} = 0, \\ \partial_t \boldsymbol{F} + \boldsymbol{v} \cdot \nabla \boldsymbol{F} - \nabla \boldsymbol{v} \boldsymbol{F} = \kappa \Delta \boldsymbol{F}, \\ \partial_t \boldsymbol{M} + \boldsymbol{v} \cdot \nabla \boldsymbol{M} = -\alpha \boldsymbol{M} \times \Delta \boldsymbol{M} - \beta \boldsymbol{M} \times (\boldsymbol{M} \times \Delta \boldsymbol{M}), \\ (\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{M})(x,0) = (\boldsymbol{v}_0, \boldsymbol{F}_0, \boldsymbol{M}_0), |\boldsymbol{M}_0| = 1. \end{cases} \tag{2}$$

Firstly, the phenomenon of elastic deformation of magnetoelastic materials is caused by a change in their magnetization state. Due to their significant changes under external stress, magnetoelastic materials are considered to be intelligent materials. In the following section, we will present some research on magnetoelastic models. The modeling of magnetoelastic materials originated from Refs. [2-3], and there have been numerous studies on this topic since then. DeSimone, et al.[4-5], explored magnetoelasticity under static circumstances by considering minimum energy, and made significant progress in this area. By employing a nonlinear hyperbolic parabola system, Chipot, et al.[6] characterized the evolution of magnetoelastic materials in 2009. Recently, Kalousek, et al.[7] applied the Galerkin method to explore the local time existence of strong solutions for this two-dimensional model. However, in this paper, we obtain the existence of local strong solutions for magnetoelastic models in $\mathbb{R}^d$ directly from the approximating system and energy estimates.

Secondly, system (2) evolves into an incompressible Navier-Stokes-Landau-Lifshitz (NSLL) equation, while the deformation gradient F remains constant:

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla P = \mu \Delta \boldsymbol{v} - \nabla \cdot 2A \nabla \boldsymbol{M} \odot \nabla \boldsymbol{M}, \\ \nabla \cdot \boldsymbol{v} = 0, \\ \partial_t \boldsymbol{M} + \boldsymbol{v} \cdot \nabla \boldsymbol{M} = -\alpha \boldsymbol{M} \times \Delta \boldsymbol{M} - \beta \boldsymbol{M} \times \boldsymbol{M} \times \Delta \boldsymbol{M}. \end{cases} \tag{3}$$

There are numerous associated works on the study of Navier-Stokes-Landau-Lifshitz (NSLL) equation, and we only briefly introduce some of them in the following. The regularity criterion of NSLL smooth solutions in Besov space and multiplication subspace was discovered by Fan, et al.[8] in 2010. In 2017, Wang, et al.[9] proved the existence of weak solutions to the NSLL equation with finite energy and an incompressibility condition in $\mathbb{R}^2$. In 2019, through additional research, they further demonstrated the global existence of weak solutions to the quantum NSLL equation in Ref. [10]. Recently, A blowup criterion for the incompressible NSLL equation with finite positive initial density was discovered by Qiu, et al.[11].

Finally, let us discuss the research work surrounding the Landau-Lifshitz (LL) equation. As early as 1985, Visintin[12] obtained the existence of weak solutions to the LL equation with magnetostrictive effect. In 1986, Sulem, et al.[13] drew a conclusion by applying the difference method: In the case that there is no dissipative term in $\mathbb{R}^d$. It is particularly worth mentioning that this conclusion was subsequently improved by Ding, et al.[14-15]. Indeed, in 2001, Ding, et al.[14] proved that the Cauchy problem of Schrodinger flows mapping from compact Riemannian manifolds to complete Kähler manifolds has a locally smooth unique solution. This provided a new idea and theoretical basis for research in related fields. In the same year, Carbou, et al.[16-17] demonstrated the global existence of regular solutions for LL equations in $\mathbb{R}^3$, which further promoted the research in related directions. Recently, Fratta, et al.[18] explored the weak-strong uniqueness problem of LLG equation in micromagnetics and provided a proof.

Based on the approximation of a perturbed parabolic system and inspired by the methods of Refs. [14] and [19], this paper strictly proves the local existence uniqueness of system (1) with finite data.

Next, we will look at some of the symbols mentioned in this article. Let $\mathbb{Z}_+ = 0, 1, 2, \cdots$, $[q]$ as the integer part of the positive real number q. For $k \in \mathbb{Z}_+$, $p \in [1, \infty]$, set $H^k(\mathbb{R}^d)$, $W^{k,p}(\mathbb{R}^d)$ represents the usual Sobolev spaces of functions on $\mathbb{R}^d$. We can view $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ as a submanifold of $\mathbb{R}^3$, then the mapping $\boldsymbol{M}$ can be expressed as $\boldsymbol{M} = (\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3)$ with $\boldsymbol{M}_i$ being globally defined functions on $\mathbb{R}^d$. Denote ∇, D are expressed as the general derivative for functions on $\mathbb{R}^d$. Then for $Q \in \mathbb{S}^2$, we define the extrinsic Sobolev spaces:

$$W_Q^{k,p} = \{f: \mathbb{R}^d \to \mathbb{R}^3 : |f(x)| = 1 \text{ a. e. and } f - Q \in W^{k,p}\}$$

with the induced distance $d_Q^{k,p}(f,g) = \| f - g \|_{W_Q^{k,p}}$ and Q is an arbitrarily point in $\mathbb{S}^2$. For simplicity of notation, let $\| f \|_{W_Q^{k,p}} = d_Q^{k,p}(f, Q)$ and further denote $H_Q^k := W_Q^{k,2}$. The main results of this paper are as follows.

Theorem 1 If Cauchy problem (2) with $(\boldsymbol{v}_0, \boldsymbol{F}_0, \boldsymbol{M}_0) \in H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d) \times H_Q^{k+1}(\mathbb{R}^d)$ for any integer $k \geqslant [\frac{d}{2}] + 1$, admits a unique local solution $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{M})$ satisfying

$$\begin{aligned} &\| \boldsymbol{v} \|_{H^k}^2 + \| \boldsymbol{F} \|_{H^k}^2 + \| \nabla \boldsymbol{M} \|_{H_Q^k}^2 + \\ &\quad \mu \int_0^t \| \nabla \boldsymbol{v} \|_{H^k}^2 \mathrm{d}s + \kappa \int_0^t \| \nabla \boldsymbol{F} \|_{H^k}^2 \mathrm{d}s + \\ &\quad \beta \int_0^t \| \nabla^2 \boldsymbol{M} \|_{H_Q^k}^2 \mathrm{d}s \leqslant \\ &\quad C(k, \| \boldsymbol{v}_0 \|_{H^k}^2, \| \boldsymbol{F}_0 \|_{H^k}^2, \| \nabla \boldsymbol{M}_0 \|_{H_Q^k}^2) \end{aligned}$$

for any $t \in [0, T]$, where $T = T(\| \boldsymbol{v}_0 \|_{H^2(\mathbb{R}^d)}, \| \boldsymbol{F}_0 \|_{H^2(\mathbb{R}^d)}, \| \nabla \boldsymbol{M}_0 \|_{H_Q^2(\mathbb{R}^d)})$.

We can first establish the approximation system of (1) with the initial value $(\boldsymbol{v}_0, \boldsymbol{F}_0, \boldsymbol{M}_0) \in C^\infty$ to obtain the uniform boundedness of the approximation solution on $\mathbb{T}^d$, and then use the weak $*$ column compactness of Sobolev space to get the local existence of $\mathbb{T}^d$. Finally, the existence and uniqueness of local solutions on the whole space $\mathbb{R}^d$ are obtained by using the continuation theorem. The proof method of Theorem 1 depends on Refs. [14] and [19].

The remaining part of this paper will be developed around the following structure. In Section 2, relevant background knowledge will be introduced, with a focus on reviewing the fundamental properties of Sobolev spaces. In Section 3, a uniform energy estimate will be derived via an approximation system, accompanied by a proof of local existence of the problem on the entire space $\mathbb{R}^d$. In Section 4, we will demonstrate the uniqueness by employing a Gronwall inequality.

2 Preliminaries

In order to further prove Theorem 1, we will introduce some necessary lemmas by providing background knowledge in this section.

Lemma 1 (Gagliardo-Nirenberg inequality[20])

Set Ω be $\mathbb{R}^d$ or a bounded Lipshitz domain in $\mathbb{R}^d$ with $\partial\Omega$, and let u be any function in $W^{m,r}(\Omega) \cap W^{k,q}(\Omega)$. $1 \leqslant r, q \leqslant +\infty$. For any integer $0 \leqslant k \leqslant j < m$, and for any number α in the interval $0 < \alpha \leqslant 1$, set

$$\frac{1}{p} - \frac{j}{d} = \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + (1 - \alpha)\left(\frac{1}{q} - \frac{k}{d}\right).$$

Then

$$\| D^j u \|_{L^p(\mathbb{R}^d)} \leqslant C(\| u \|_{W^{m,r}(\mathbb{R}^d)})^\alpha (\| u \|_{W^{k,q}(\mathbb{R}^d)})^{1-\alpha}.$$

Lemma 2 (Gagliardo-Nirenberg-Moser inequality (Proposition 3.7[21]))

$$\| f \cdot g \|_{H^l} \leqslant C(\| f \|_{L^\infty} \| g \|_{H^l} + \| f \|_{H^l} \| g \|_{L^\infty}).$$

Lemma 3 (Lemma II. 4. 12[22]) Set $f: \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function which is non-decreasing such that $f > 0$ on $(0, \infty)$ and $\int_1^\infty \frac{1}{f} dx < \infty$. Let y be a non-negative continuous function on $\mathbb{R}^+$ and let g be a function which is non-negative in $L_{loc}^1(\mathbb{R}^+)$. We assume that there exists a positive number $y_0 > 0$ such that for all $t \geqslant 0$, we have the inequality

$$y(t) \leqslant y_0 + \int_0^t g(s) \mathrm{d}s + \int_0^t f(y(s)) \mathrm{d}s.$$

Then, there exists a positive number T^* depending only on y_0 and f, such that for all $T < T^*$, there holds true

$$\sup_{0 \leqslant t \leqslant T} y(t) \leqslant C(T, y_0)$$

for some constant $C(T, y_0)$.

Lemma 4 (Gronwall inequality).

(i) Let $\eta(\cdot)$ be a non-negative, absolutely continuous function on $[0, T]$, which satisfies for a. e. t the differential inequality

$$\eta'(t) \leqslant \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are non-negative, summable functions on $[0, T]$. Then

$$\eta(t) \leqslant \exp\left\{\int_0^t \phi(s) \mathrm{d}s\right\} \left[\eta(0) + \int_0^t \psi(s) \mathrm{d}s\right]$$

for all $0 \leqslant t \leqslant T$.

(ii) In particular, if

$$\eta' \leqslant \phi\eta, \text{ on } [0, T] \text{ and } \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0, T].$$

Lemma 5 (Lemma 3. 4[19]) Let $k > d/2$ and $\boldsymbol{M} \in$

$H_Q^k(\mathbb{R}^d,\mathbb{S}^2)$. Then, there exists a sequence of map $\boldsymbol{M}_i - Q \in H^k(\mathbb{R}^d,\mathbb{S}^2) \cap C_0^\infty(\mathbb{R}^d,\mathbb{R}^3)$ such that $\boldsymbol{M}_i \to \boldsymbol{M}$ in $H_Q^k(\mathbb{R}^d,\mathbb{S}^2)$.

3 Local existence of solutions

3.1 Local existence of solutions in $\mathbb{T}^d$

In this section, we prove the local existence of smooth solutions for system (2) with the initial value $(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0) \in C^\infty(\Omega\times\Omega\times\Omega,\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2)$. (Let Ω be a flat torus $\mathbb{T}^d$).

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla P = \mu\Delta\boldsymbol{v} - \nabla\cdot \\ \quad (2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M} - W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}), \\ \nabla\cdot\boldsymbol{v} = 0, \\ \partial_t\boldsymbol{F} + \boldsymbol{v}\cdot\nabla\boldsymbol{F} - \nabla\boldsymbol{v}\boldsymbol{F} = k\Delta\boldsymbol{F}, \\ \partial_t\boldsymbol{M} + \boldsymbol{v}\cdot\nabla\boldsymbol{M} = -\alpha\boldsymbol{M}\times\Delta\boldsymbol{M} - \beta\boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}), \\ (\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})(x,0) = (\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0) \in \\ \quad C^\infty(\Omega\times\Omega\times\Omega,\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2), \\ |\boldsymbol{M}_0| = 1. \end{cases} \tag{4}$$

We may employ an approximate procedure and solve first the following perturbed problem:

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla P = \mu\Delta\boldsymbol{v} - \nabla\cdot \\ \quad (2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M} - W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}), \\ \nabla\cdot\boldsymbol{v} = 0, \\ \partial_t\boldsymbol{F} + \boldsymbol{v}\cdot\nabla\boldsymbol{F} - \nabla\boldsymbol{v}\boldsymbol{F} = k\Delta\boldsymbol{F}, \\ \partial_t\boldsymbol{M} + \boldsymbol{v}\cdot\nabla\boldsymbol{M} = -\alpha\boldsymbol{M}\times\Delta\boldsymbol{M} - \\ \quad \beta\boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}) + \varepsilon\Delta\boldsymbol{M}, \\ (\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})(x,0) = (\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0) \in \\ \quad C^\infty(\Omega\times\Omega\times\Omega,\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2), \\ |\boldsymbol{M}_0| = 1, \end{cases} \tag{5}$$

where $\varepsilon \in (0,1)$ is the perturbation constant.

Inspired by Ding, et al.[14], this paper proves the existence of the smooth solutions of system (4) by constructing the perturbation system (5). First, the perturbed system (5) is uniformly parabolic at $\varepsilon \in (0,1)$ for which system (5) has a unique smooth solution $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\boldsymbol{M}_\varepsilon)$ for T_ε. In other words, as long as we can obtain uniform boundedness estimates in T_ε, and uniform bounds for various norms of $\boldsymbol{v}_\varepsilon$, $\boldsymbol{F}_\varepsilon$, $\boldsymbol{M}_\varepsilon$ for t in $[0,T]$, so that a subsequence of $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\boldsymbol{M}_\varepsilon)$ converges to the smooth solution of (4) when $\varepsilon\to 0$. Therefore, for the initial-value problem (4), we have

Lemma 6 Let Ω be a flat torus $\mathbb{T}^d$. Set $m_0 = [d/2]+1=2$ and let $(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0) \in C^\infty(\Omega\times\Omega\times\Omega,\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2)$. There exists a constant $T = T(\|\boldsymbol{v}_0\|_{H^2(\Omega)}, \|\boldsymbol{F}_0\|_{H^2(\Omega)}, \|\nabla\boldsymbol{M}_0\|_{H^2(\Omega)}) > 0$, independent of $\varepsilon \in (0,1]$, such that if $(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M}) \in C^\infty(\Omega\times[0,T_\varepsilon])$ is a solution of (4) with $\varepsilon\in(0,1]$, then

$$T_\varepsilon \geqslant T(\|v_0\|_{H^2(\Omega)}, \|\boldsymbol{F}_0\|_{H^2(\Omega)}, \|\nabla\boldsymbol{M}_0\|_{H^2(\Omega)}),$$

and

$$\begin{aligned} &\|\boldsymbol{v}\|_{H^k(\Omega)}^2 + \|\boldsymbol{F}\|_{H^k(\Omega)}^2 + \|\nabla\boldsymbol{M}\|_{H^k(\Omega)}^2 + \\ &\quad \mu\|\nabla\boldsymbol{v}\|_{L^2([0,t];H^k(\Omega))}^2 + \kappa\|\nabla\boldsymbol{F}\|_{L^2([0,t];H^k(\Omega))}^2 + \\ &\quad \beta\|\nabla^2\boldsymbol{M}\|_{L^2([0,t];H^k(\Omega)}^2 \leqslant C(k, \|\boldsymbol{v}_0\|_{H^k(\Omega)}^2, \\ &\quad \|\boldsymbol{F}_0\|_{H^k(\Omega)}^2, \|\nabla\boldsymbol{M}_0\|_{H^k(\Omega)}^2), t\in[0,T] \end{aligned}$$

for all $k \geqslant 2$.

Proof For simplicity, denote $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\boldsymbol{M}_\varepsilon) := (\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})$ be a solution of (5), and denote $H^l = H^l(\Omega)$ for any integer $l \geqslant 0$.

Multiplying the first equation of system (5) by $\boldsymbol{v}$ and then integrate x over Ω with respect to x using $\nabla\cdot\boldsymbol{v}=0$, we have

$$\begin{aligned} &\langle\partial_t\boldsymbol{v},\boldsymbol{v}\rangle + \langle\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}\rangle + \langle\nabla P,\boldsymbol{v}\rangle = \\ &\quad \mu\langle\Delta\boldsymbol{v},\boldsymbol{v}\rangle - \langle\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M}),\boldsymbol{v}\rangle + \\ &\quad \langle\nabla\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),\boldsymbol{v}\rangle, \end{aligned}$$

where

$$\langle\partial_t\boldsymbol{v},\boldsymbol{v}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{v}|^2\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|_{L^2}^2,$$

$$\langle\mu\Delta\boldsymbol{v},\boldsymbol{v}\rangle = -\mu\langle\nabla\boldsymbol{v},\nabla\boldsymbol{v}\rangle = -\mu\|\nabla\boldsymbol{v}\|_{L^2}^2,$$

$$\begin{aligned} &\langle\nabla\cdot(W'(\boldsymbol{F})\cdot\boldsymbol{F}^{\mathrm{T}}),\boldsymbol{v}\rangle = -\langle(|\boldsymbol{F}|^2)'\cdot\boldsymbol{F},\nabla\boldsymbol{v}\rangle = \\ &\quad -\langle 2\nabla\boldsymbol{v}\boldsymbol{F},\boldsymbol{F}\rangle. \end{aligned}$$

Using $\nabla\cdot\boldsymbol{v}=0$, we have

$$\begin{aligned} &-\langle(\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M})),\boldsymbol{v}\rangle = \\ &\quad -2A\langle\partial_j(\partial_j\boldsymbol{M}_k\cdot\partial_i\boldsymbol{M}_k),\boldsymbol{v}_i\rangle = \\ &\quad -2A\langle\partial_j\partial_j\boldsymbol{M}_k\cdot\partial_i\boldsymbol{M}_k + \partial_j\boldsymbol{M}_k\cdot\partial_j\partial_i\boldsymbol{M}_k,\boldsymbol{v}_i\rangle = \\ &\quad -2A\langle\partial_j\partial_j\boldsymbol{M}_k\cdot\partial_i\boldsymbol{M}_k,\boldsymbol{v}_i\rangle = -2A\langle\Delta\boldsymbol{M},\boldsymbol{v}\cdot\nabla\boldsymbol{M}\rangle, \end{aligned}$$

$$\langle\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}\rangle = -\langle\boldsymbol{v}_i,\frac{1}{2}\nabla_i|\boldsymbol{v}_j|^2\rangle = 0,$$

$$\langle\nabla P,\boldsymbol{v}\rangle = -\langle P,\nabla\cdot\boldsymbol{v}\rangle = 0.$$

Multiplying the third equation of system (5) by $2\boldsymbol{F}$ and then integrate over Ω with respect to x.

$$\begin{aligned} &\langle\partial_t\boldsymbol{F},2\boldsymbol{F}\rangle = -\langle\boldsymbol{v}\cdot\nabla\boldsymbol{F},2\boldsymbol{F}\rangle + \langle\nabla\boldsymbol{v}\boldsymbol{F},2\boldsymbol{F}\rangle + \\ &\quad \kappa\langle\Delta\boldsymbol{F},2\boldsymbol{F}\rangle, \end{aligned}$$

where

$$\langle \partial_t \boldsymbol{F}, 2\boldsymbol{F}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{F}|^2\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{F}\|_{L^2}^2,$$

$$\kappa\langle \Delta\boldsymbol{F}, 2\boldsymbol{F}\rangle = -2\kappa\langle \nabla\boldsymbol{F}, \nabla\boldsymbol{F}\rangle = -2\kappa\|\nabla\boldsymbol{F}\|_{L^2}^2.$$

Using $\nabla\cdot\boldsymbol{v}=0$, we have

$$\langle \boldsymbol{v}\cdot\nabla\boldsymbol{F}, 2\boldsymbol{F}\rangle = -\langle \boldsymbol{v}, \nabla|\boldsymbol{F}|^2\rangle = 0.$$

Multiplying the fourth equation of system (5) by $-2A\Delta\boldsymbol{M}$ and then integrate over Ω with respect to x.

$$\begin{aligned}\langle \partial_t\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle = {} & -\langle \boldsymbol{v}\cdot\nabla\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle - \\ & \langle \alpha\boldsymbol{M}\times\Delta\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle - \langle \beta\boldsymbol{M}\times\boldsymbol{M}\times\Delta\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle + \\ & \langle \varepsilon\Delta\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle,\end{aligned}$$

where

$$\begin{aligned}\langle \partial_t\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle = {} & A\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\boldsymbol{M}|^2\mathrm{d}x = \\ & A\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{M}\|_{L^2}^2,\end{aligned}$$

$$-\langle \alpha\boldsymbol{M}\times\Delta\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle = 0,$$

$$\langle \varepsilon\Delta\boldsymbol{M}, -2A\Delta\boldsymbol{M}\rangle = -2A\varepsilon\|\Delta\boldsymbol{M}\|_{L^2}^2,$$

$$\begin{aligned}& -\langle \beta\boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}), -2A\Delta\boldsymbol{M}\rangle = \\ & \int_\Omega 2A\beta\boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M})\cdot 2A\Delta\boldsymbol{M}\mathrm{d}x = \\ & -\int_\Omega (2A\beta\boldsymbol{M}\times\Delta\boldsymbol{M})\cdot(2A\boldsymbol{M}\times\Delta\boldsymbol{M})\mathrm{d}x = \\ & -2A^2\beta\|\boldsymbol{M}\times\Delta\boldsymbol{M}\|_{L^2}^2.\end{aligned}$$

Then, we can get

$$\begin{aligned}& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\boldsymbol{v}\|_{L^2}^2 + \|\boldsymbol{F}\|_{L^2}^2 + A\|\nabla\boldsymbol{M}\|_{L^2}^2\right) + \\ & \mu\|\nabla\boldsymbol{v}\|_{L^2}^2 + 2\kappa\|\nabla\boldsymbol{F}\|_{L^2}^2 + \\ & 2A^2\beta\|\boldsymbol{M}\times\Delta\boldsymbol{M}\|_{L^2}^2 + 2A\varepsilon\|\Delta\boldsymbol{M}\|_{L^2}^2 = 0. \qquad (6)\end{aligned}$$

From (6), we obtain that the energy

$$\begin{aligned}E(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M}) := {} & \frac{1}{2}\|\boldsymbol{v}\|_{L^2}^2 + \|\boldsymbol{F}\|_{L^2}^2 + A\|\nabla\boldsymbol{M}\|_{L^2}^2 + \\ & \mu\int_0^t(\|\nabla\boldsymbol{v}\|_{L^2}^2 + 2\kappa\|\nabla\boldsymbol{F}\|_{L^2}^2 + 2A^2\beta\|\boldsymbol{M}\times \\ & \Delta\boldsymbol{M}\|_{L^2}^2 + 2A\varepsilon\|\Delta\boldsymbol{M}\|_{L^2}^2)\mathrm{d}s. \qquad (7)\end{aligned}$$

Then, using Eq. (6) and a Gronwall inequality, we obtain $E(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})$ is uniformly bounded for $t\in[0,T_\varepsilon)$.

In order to obtain $\nabla^2\boldsymbol{M}$ of energy estimate in H^n, we need to use the fact: If we take the inner product of the fourth equation of system (5) with $\boldsymbol{M}$ and using $\nabla\cdot\boldsymbol{v}=0$, $|\boldsymbol{M}_0|=1$, we can get

$$|\boldsymbol{M}| = 1 \text{ almost everywhere in } \mathbb{R}^d\times[0,T].$$

Now, rewrite the fourth equation of system (5).

$$\begin{aligned}\partial_t\boldsymbol{M} + \boldsymbol{v}\cdot\nabla\boldsymbol{M} = {} & -\alpha\boldsymbol{M}\times\Delta\boldsymbol{M} + \beta|\nabla\boldsymbol{M}|^2\boldsymbol{M} + \\ & (\beta+\varepsilon)\Delta\boldsymbol{M}. \qquad (8)\end{aligned}$$

Here we use the following facts:

$$|\boldsymbol{M}| = 1,$$

$$\begin{aligned}\beta\boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}) = {} & \beta((\boldsymbol{M}\cdot\Delta\boldsymbol{M})\boldsymbol{M} - (\boldsymbol{M}\cdot\boldsymbol{M})\Delta\boldsymbol{M}) = \\ & \beta\left(\frac{1}{2}\Delta|\boldsymbol{M}|^2 - |\nabla\boldsymbol{M}|^2\right)\boldsymbol{M} - \beta|\boldsymbol{M}|^2\Delta\boldsymbol{M} = \\ & -\beta|\nabla\boldsymbol{M}|^2\boldsymbol{M} - \beta\Delta\boldsymbol{M}.\end{aligned}$$

As $\boldsymbol{M}$ belongs to $L^\infty((0,T);H^2(\Omega))$ and $|\boldsymbol{M}|=1$ in $\mathbb{R}^d\times[0,T]$, system (5) is equivalent to (8).

Next, we will discuss the higher order derivative term about $\boldsymbol{v}$, $\boldsymbol{F}$ and $\boldsymbol{M}$. We shall firstly define the multiple index operator $D^\xi = \prod_{i=1}^n \partial_{x_i}^{\xi_i}$, $|\xi| = \sum_{i=1}^n \xi_i$, where ξ is a multi-index of length n, i. e., $\xi=(\xi_1,\cdots,\xi_n)$ and n are nonnegative integers. For simplicity, let D^n denote any kind of D^ξ, where $|\xi|=n$, D^1 denote as D.

Then, differentiating the first equation of system (5) with D^n, multiplying by $D^n v$ and then integrate over Ω with respect to x.

$$\begin{aligned}\langle D^n(\partial_t\boldsymbol{v}), D^n\boldsymbol{v}\rangle = {} & -\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{v}), D^n\boldsymbol{v}\rangle - \\ & \langle D^n\nabla P, D^n\boldsymbol{v}\rangle + \mu\langle D^n\Delta\boldsymbol{v}, D^n\boldsymbol{v}\rangle - \\ & \langle D^n(\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M})), D^n\boldsymbol{v}\rangle + \\ & \langle D^n\nabla\cdot(\mathbb{W}'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}), D^n\boldsymbol{v}\rangle,\end{aligned}$$

where

$$\langle D^n(\partial_t\boldsymbol{v}), D^n\boldsymbol{v}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |D^n\boldsymbol{v}|^2\mathrm{d}x,$$

$$\begin{aligned}\langle D^n\nabla P, D^n\boldsymbol{v}\rangle = {} & \langle D^n\nabla_i P, D^n\boldsymbol{v}_i\rangle = \\ & -\langle D^n P, D^n\nabla_i\cdot\boldsymbol{v}_i\rangle = 0,\end{aligned}$$

$$\begin{aligned}\mu\langle D^n\Delta\boldsymbol{v}, D^n\boldsymbol{v}\rangle = {} & -\mu\langle D^n\nabla\boldsymbol{v}, D^n\nabla\boldsymbol{v}\rangle = \\ & -\mu\int_\Omega |D^n\nabla\boldsymbol{v}|^2\mathrm{d}x.\end{aligned}$$

Differentiating the third equation of system (5) with D^n, multiplying by $D^n\boldsymbol{F}$ and then integrate over Ω with respect to x.

$$\begin{aligned}\langle D^n(\partial_t\boldsymbol{F}), D^n\boldsymbol{F}\rangle = {} & -\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{F}), D^n\boldsymbol{F}\rangle + \\ & \langle D^n(\nabla\boldsymbol{v}\boldsymbol{F}), D^n\boldsymbol{F}\rangle + \kappa\langle D^n\Delta\boldsymbol{F}, D^n\boldsymbol{F}\rangle,\end{aligned}$$

where

$$\langle D^n(\partial_t\boldsymbol{F}), D^n\boldsymbol{F}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |D^n\boldsymbol{F}|^2\mathrm{d}x,$$

$$\int_\Omega \kappa D^n\Delta\boldsymbol{F}\cdot D^n\boldsymbol{F}\mathrm{d}x = -\kappa\int_\Omega |D^n\nabla\boldsymbol{F}|^2\mathrm{d}x.$$

Differentiating Eq. (8) with D^{n+1}, multiplying by $D^{n+1}\boldsymbol{M}$ and then integrate over Ω with respect to x.

$$\begin{aligned}\langle D^{n+1}(\partial_t\boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle = {} & -\langle D^{n+1}(\boldsymbol{v}\cdot\nabla\boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle - \\ & \alpha\langle D^{n+1}(\boldsymbol{M}\times\Delta\boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle + \beta\langle D^{n+1}(|\nabla\boldsymbol{M}|^2\boldsymbol{M}),\end{aligned}$$

$$D^{n+1}\boldsymbol{M}\rangle+(\beta+\varepsilon)\langle D^{n+1}\Delta\boldsymbol{M},D^{n+1}\boldsymbol{M}\rangle,$$

where

$$\langle D^{n+1}(\partial_t\boldsymbol{M}),D^{n+1}\boldsymbol{M}\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|D^n\nabla\boldsymbol{M}|^2\mathrm{d}x,$$

$$\langle(\beta+\varepsilon)D^{n+1}\Delta\boldsymbol{M},D^{n+1}\boldsymbol{M}\rangle=-(\beta+\varepsilon)\int_{\Omega}|D^{n+1}\nabla\boldsymbol{M}|^2\mathrm{d}x.$$

Then, we can get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|D^n\boldsymbol{v}\|_L^2+\|D^n\boldsymbol{F}\|_{L^2}^2+\|D^n\nabla\boldsymbol{M}\|_{L^2}^2)+\\&\quad(\mu\|D^n\nabla\boldsymbol{v}\|_{L^2}^2+\kappa\|D^n\nabla\boldsymbol{v}\|_{L^2}^2+\\&\quad(\beta+\varepsilon)\|D^{n+1}\nabla\boldsymbol{M}\|_{L^2}^2)=-\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{v}),D^n\boldsymbol{v}\rangle-\\&\quad\langle D^n\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M}),D^n\boldsymbol{v}\rangle+\\&\quad\langle D^n\nabla\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),D^n\boldsymbol{v}\rangle-\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),\nabla^n\boldsymbol{F}\rangle+\\&\quad\langle D^n(\nabla\boldsymbol{v}\boldsymbol{F}),D^n\boldsymbol{F}\rangle-\langle D^{n+1}(\boldsymbol{v}\cdot\nabla\boldsymbol{M}),D^{n+1}\boldsymbol{M}\rangle-\\&\quad\alpha\langle D^{n+1}(\boldsymbol{M}\times\Delta\boldsymbol{M}),D^{n+1}\boldsymbol{M}\rangle+\\&\quad\beta\langle D^{n+1}(|\nabla\boldsymbol{M}|^2\boldsymbol{M}),D^{n+1}\boldsymbol{M}\rangle=\\&\quad I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.\end{aligned}\tag{9}$$

Then, we can define the energy function by

$$\begin{aligned}&E_n(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M}):=\frac{1}{2}(\|D^n\boldsymbol{v}\|_{L^2}^2+\|D^n\boldsymbol{F}\|_{L^2}^2+\\&\quad\|D^n\nabla\boldsymbol{M}\|_{L^2}^2)+(\mu\int_0^t\|D^n\nabla\boldsymbol{v}\|_{L^2}^2+\kappa\int_0^t\|D^n\nabla\boldsymbol{F}\|_{L^2}^2+\\&\quad(\beta+\varepsilon)\int_0^t\|D^{n+1}\nabla\boldsymbol{M}\|_{L^2}^2).\end{aligned}\tag{10}$$

Next we will estimate every term at the right end of (9). Indeed, from Lemma 1, we obtain

when $d=2$(2 - dimensional),

$$\|\boldsymbol{v}\|_{L^\infty}\leqslant C\|\boldsymbol{v}\|_{H^2}^{\frac{1}{2}}\|\boldsymbol{v}\|_{L^2}^{\frac{1}{2}}\leqslant C\|\boldsymbol{v}\|_{H^2},$$

$$\|D\boldsymbol{M}\|_{L^\infty}\leqslant C\|D\boldsymbol{M}\|_{H^2}^{\frac{1}{2}}\|D\boldsymbol{M}\|_{L^2}^{\frac{1}{2}}\leqslant C\|D\boldsymbol{M}\|_{H^2},$$

$$\|D\boldsymbol{v}\|_{L^4}\leqslant C\|\boldsymbol{v}\|_{H^2}^{\frac{3}{4}}\|\boldsymbol{v}\|_{L^2}^{\frac{1}{4}}\leqslant C\|\boldsymbol{v}\|_{H^2},$$

$$\|D^2\boldsymbol{M}\|_{L^4}\leqslant C\|D\boldsymbol{M}\|_{H^2}^{\frac{3}{4}}\|D\boldsymbol{M}\|_{L^2}^{\frac{1}{4}}\leqslant C\|D\boldsymbol{M}\|_{H^2}.$$

When $d=3$(3 - dimensional),

$$\|\boldsymbol{v}\|_{L^\infty}\leqslant C\|\boldsymbol{v}\|_{H^2}^{\frac{3}{4}}\|\boldsymbol{v}\|_{L^2}^{\frac{1}{4}}\leqslant C\|\boldsymbol{v}\|_{H^2},$$

$$\|D\boldsymbol{M}\|_{L^\infty}\leqslant C\|D\boldsymbol{M}\|_{H^2}^{\frac{3}{4}}\|D\boldsymbol{M}\|_{L^2}^{\frac{1}{4}}\leqslant C\|D\boldsymbol{M}\|_{H^2},$$

$$\|D\boldsymbol{v}\|_{L^4}\leqslant C\|\boldsymbol{v}\|_{H^2}^{\frac{7}{8}}\|\boldsymbol{v}\|_{L^2}^{\frac{1}{8}}\leqslant C\|\boldsymbol{v}\|_{H^2},$$

$$\|D^2\boldsymbol{M}\|_{L^4}\leqslant C\|D\boldsymbol{M}\|_{H^2}^{\frac{7}{8}}\|D\boldsymbol{M}\|_{L^2}^{\frac{1}{8}}\leqslant C\|D\boldsymbol{M}\|_{H^2}.$$

Now, using the above estimate, we can obtain every term estimate at the right end of (9).

We can prove the bound of I_1,

$$I_1\leqslant\begin{cases}C\|\boldsymbol{v}\|_{H^2}^2\cdot\|D\boldsymbol{v}\|_{H^2}, & \text{if}\quad 1\leqslant n\leqslant 2,\\ C\|\boldsymbol{v}\|_{H^n}^2\cdot\|D\boldsymbol{v}\|_{H^n}, & \text{if}\quad n\geqslant 3,\end{cases}\tag{11}$$

where the constant C depends on n.

For $1\leqslant n\leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n=1$,

$$\begin{aligned}I_1&=-\langle D(\boldsymbol{v}\cdot\nabla\boldsymbol{v}),D\boldsymbol{v}\rangle=-\langle D\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\boldsymbol{v}\cdot\nabla D\boldsymbol{v},D\boldsymbol{v}\rangle=\\&\quad-\langle D\boldsymbol{v}\cdot\nabla\boldsymbol{v},D\boldsymbol{v}\rangle=\langle D(\nabla\cdot\boldsymbol{v})\boldsymbol{v},D\boldsymbol{v}\rangle+\\&\quad\langle D\boldsymbol{v}\cdot\boldsymbol{v},\nabla D\boldsymbol{v}\rangle\leqslant C\|D\boldsymbol{v}\|_{L^2}\|D^2\boldsymbol{v}\|_{L^2}\|\boldsymbol{v}\|_{L^\infty}\leqslant\\&\quad C\|\boldsymbol{v}\|_{H^2}^2\|D\boldsymbol{v}\|_{H^2}.\end{aligned}$$

Secondly, for $n=2$,

$$\begin{aligned}I_1&=-\langle D^2(\boldsymbol{v}\cdot\nabla\boldsymbol{v}),D^2\boldsymbol{v}\rangle=\\&\quad-\langle D^2\boldsymbol{v}\cdot\nabla\boldsymbol{v}+2D\boldsymbol{v}\cdot D\nabla\boldsymbol{v}+\boldsymbol{v}D^2\nabla\boldsymbol{v},D^2\boldsymbol{v}\rangle=\\&\quad-\langle D^2\boldsymbol{v}\cdot\nabla\boldsymbol{v}+2D\boldsymbol{v}\cdot D\nabla\boldsymbol{v},D^2\boldsymbol{v}\rangle\leqslant\\&\quad 3\|D^2\boldsymbol{v}\|_{L^2}^2\|\nabla\boldsymbol{v}\|_{L^\infty}\leqslant C\|\boldsymbol{v}\|_{H^2}^2\|\nabla\boldsymbol{v}\|_{H^2}.\end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}I_1&=\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{v}),D^n\boldsymbol{v}\rangle\leqslant\\&\quad\|D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_{L^2}\|D^n\boldsymbol{v}\|_{L^2}\leqslant\\&\quad\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{H^n}\|\boldsymbol{v}\|_{H^n}\leqslant\\&\quad C(\|\boldsymbol{v}\|_{L^\infty}\|\nabla\boldsymbol{v}\|_{H^n}+\|\boldsymbol{v}\|_{H^n}\|\nabla\boldsymbol{v}\|_{L^\infty})\|\boldsymbol{v}\|_{H^n}\leqslant\\&\quad C(\|\boldsymbol{v}\|_{H^2}\|\nabla\boldsymbol{v}\|_{H^n}+\|\boldsymbol{v}\|_{H^n}\|\nabla\boldsymbol{v}\|_{H^2})\|\boldsymbol{v}\|_{H^n}\leqslant\\&\quad C\|\boldsymbol{v}\|_{H^n}^2\|\nabla\boldsymbol{v}\|_{H^n}.\end{aligned}$$

We can prove the bound of I_2

$$I_2\leqslant\begin{cases}C\|D\boldsymbol{M}\|_{H^2}^2\cdot\|D\boldsymbol{v}\|_{H^2}, & \text{if}\quad 1\leqslant n\leqslant 2,\\ C\|D\boldsymbol{M}\|_{H^n}^2\cdot\|D\boldsymbol{v}\|_{H^n}, & \text{if}\quad n\geqslant 3,\end{cases}\tag{12}$$

where the constant C depends on n.

For $1\leqslant n\leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n=1$,

$$\begin{aligned}I_2&=-\langle D(\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M})),D\boldsymbol{v}\rangle=\\&\quad-2A\langle D\partial_i(\partial_i\boldsymbol{M}\cdot\partial_j\boldsymbol{M}),D\boldsymbol{v}_j\rangle=\\&\quad 2A\langle D\partial_i\boldsymbol{M}_k\cdot\partial_j\boldsymbol{M}_k+\partial_i\boldsymbol{M}_k\cdot D\partial_j\boldsymbol{M}_k,\partial_iD\boldsymbol{v}_j\rangle\leqslant\\&\quad 3A\|D^2\boldsymbol{M}\|_{L^2}\|D\boldsymbol{M}\|_{L^\infty}\|D^2\boldsymbol{v}\|_{L^2}\leqslant\\&\quad C\|D\boldsymbol{M}\|_{H^1}\|D\boldsymbol{M}\|_{H^2}\|D\boldsymbol{v}\|_{H^1}\leqslant\\&\quad C\|D\boldsymbol{M}\|_{H^2}^2\|D\boldsymbol{v}\|_{H^2}.\end{aligned}$$

Secondly, for $n=2$,

$$\begin{aligned}I_2&=-\langle D^2(\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M})),D^2\boldsymbol{v}\rangle=\\&\quad-2A\langle D^2\partial_i(\partial_i\boldsymbol{M}_k\cdot\partial_j\boldsymbol{M}_k),D^2\boldsymbol{v}_j\rangle=\\&\quad 2A\langle D^2(\partial_i\boldsymbol{M}\cdot\partial_j\boldsymbol{M}),\partial_iD^2\boldsymbol{v}_j\rangle=\\&\quad 2A\langle D^2\partial_i\boldsymbol{M}_k\cdot\partial_j\boldsymbol{M}_k+2D\partial_i\boldsymbol{M}_k\cdot D\partial_j\boldsymbol{M}_k+\\&\quad\partial_i\boldsymbol{M}_k\cdot D^2\partial_j\boldsymbol{M}_k,\partial_iD^2\boldsymbol{v}_j\rangle\leqslant\\&\quad 2A(2\|D^3\boldsymbol{M}\|_{L^2}\|D\boldsymbol{M}\|_{L^\infty}+2\|D^2\boldsymbol{M}\|_{L^4}^2)\|D^3\boldsymbol{v}\|_{L^2}\leqslant\\&\quad C(\|D\boldsymbol{M}\|_{H^2}\|D\boldsymbol{M}\|_{H^2}+\|D\boldsymbol{M}\|_{H^2}^2)\|D\boldsymbol{v}\|_{H^1}\leqslant\\&\quad C\|D\boldsymbol{M}\|_{H^2}^2\|D\boldsymbol{v}\|_{H^2}.\end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}I_2 = &-\langle D^n(\nabla\cdot(2A\nabla\boldsymbol{M}\odot\nabla\boldsymbol{M})),D^n\boldsymbol{v}\rangle =\\ &-2A\langle D^n\partial_i(\partial_i\boldsymbol{M}_k\cdot\partial_j\boldsymbol{M}_k),D^n\boldsymbol{v}\rangle =\\ &2A\langle D^n(\partial_i\boldsymbol{M}_k\cdot\partial_j\boldsymbol{M}_k),\partial_i D^n\boldsymbol{v}\rangle\leqslant\\ &2A\|D^n(\nabla\boldsymbol{M}\cdot\nabla\boldsymbol{M})\|_{L^2}\|D^n\nabla\boldsymbol{v}\|_{L^2}\leqslant\\ &2A\|\nabla\boldsymbol{M}\cdot\nabla\boldsymbol{M}\|_{H^n}\|\nabla\boldsymbol{v}\|_{H^n}\leqslant\\ &C\|\nabla\boldsymbol{M}\|_{L^\infty}\|\nabla\boldsymbol{M}\|_{H^n}\|\nabla\boldsymbol{v}\|_{H^n}\leqslant\\ &C\|\nabla\boldsymbol{M}\|_{H^n}^2\|\nabla\boldsymbol{v}\|_{H^n}.\end{aligned}$$

We can prove the bound I_3,

$$I_3\leqslant\begin{cases}C\|\boldsymbol{F}\|_{H^2}^2\cdot\|D\boldsymbol{v}\|_{H^2}, & \text{if}\quad 1\leqslant n\leqslant 2,\\ C\|\boldsymbol{F}\|_{H^n}^2\cdot\|D\boldsymbol{v}\|_{H^n}, & \text{if}\quad n\geqslant 3,\end{cases}\tag{13}$$

where the constant C depends on n.

For $1\leqslant n\leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n=1$,

$$\begin{aligned}I_3 = &-\langle D\nabla\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),D\boldsymbol{v}\rangle =\\ &-\langle D(2\boldsymbol{F}^2),D\nabla\boldsymbol{v}\rangle = -\langle 4\boldsymbol{F}D\boldsymbol{F},D^2\boldsymbol{v}\rangle\leqslant\\ &4\|\boldsymbol{F}\|_{L^\infty}\|D\boldsymbol{F}\|_{L^2}\|D^2\boldsymbol{v}\|_{L^2}\leqslant\\ &C\|\boldsymbol{F}\|_{H^2}\|\boldsymbol{F}\|_{H^1}\|D\boldsymbol{v}\|_{H^1}\leqslant\\ &C\|\boldsymbol{F}\|_{H^2}^2\|D\boldsymbol{v}\|_{H^2}.\end{aligned}$$

Secondly, for $n=2$,

$$\begin{aligned}I_3 = &-\langle D^2\nabla\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),D^2\boldsymbol{v}\rangle =\\ &-\langle D^2(2\boldsymbol{F}^2),D^2\nabla\boldsymbol{v}\rangle = -\langle D(4\boldsymbol{F}\nabla\boldsymbol{F}),D^2\nabla\boldsymbol{v}\rangle =\\ &-4\langle|\nabla\boldsymbol{F}|^2+\boldsymbol{F}\cdot D\nabla\boldsymbol{F},D^2\nabla\boldsymbol{v}\rangle\leqslant\\ &4(\|D\boldsymbol{F}\|_{L^4}^2+\|\boldsymbol{F}\|_{L^\infty}\|D^2\boldsymbol{F}\|_{L^2})\|D^3\boldsymbol{v}\|_{L^2}\leqslant\\ &C(\|\boldsymbol{F}\|_{H^2}^2+\|\boldsymbol{F}\|_{H^2}^2)\|D^3\boldsymbol{v}\|_{H^2}\leqslant\\ &C\|\boldsymbol{F}\|_{H^2}^2\|D\boldsymbol{v}\|_{H^2}.\end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}I_3 = &\langle D^n\nabla\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),D^n\boldsymbol{v}\rangle =\\ &-\langle D^n\cdot(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}),D^n\nabla\boldsymbol{v}\rangle\leqslant\\ &(\|D^n(W'(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}})\|_{L^2}\|D^n\nabla\boldsymbol{v}\|_{L^2})\leqslant\\ &2\|\boldsymbol{F}\cdot\boldsymbol{F}\|_{H^n}\|\nabla\boldsymbol{v}\|_{H^n}\leqslant\\ &C(\|\boldsymbol{F}\|_{L^\infty}\|\boldsymbol{F}\|_{H^n}+\|\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{L^\infty})\|\nabla\boldsymbol{v}\|_{H^n}\leqslant\\ &C\|\boldsymbol{F}\|_{H^n}^2\|\nabla\boldsymbol{v}\|_{H^n}.\end{aligned}$$

We can prove the bound I_4,

$$I_4\leqslant\begin{cases}C\|\boldsymbol{v}\|_{H^2}\cdot\|\nabla\boldsymbol{F}\|_{H^2}\|\boldsymbol{F}\|_{H^2}, & \text{if}\quad 1\leqslant n\leqslant 2.\\ C\|\boldsymbol{v}\|_{H^n}\cdot\|\nabla\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{H^n}, & \text{if}\quad n\geqslant 3,\end{cases}\tag{14}$$

where the constant C depends on n.

For $1\leqslant n\leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n=1$

$$\begin{aligned}I_4 = &-\langle D(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),D\boldsymbol{F}\rangle = \langle(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),D^2\boldsymbol{F}\rangle\leqslant\\ &\|\boldsymbol{v}\|_{L^\infty}\|\nabla\boldsymbol{F}\|_{L^2}\|D^2\boldsymbol{F}\|_{L^2}\leqslant\\ &C\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^1}\|D\boldsymbol{F}\|_{H^1}\leqslant\\ &C\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^2}.\end{aligned}$$

Secondly, for $n=2$,

$$\begin{aligned}I_4 = &-\langle D^2(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),D^2\boldsymbol{F}\rangle = \langle D(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),D^3\boldsymbol{F}\rangle =\\ &\langle D\boldsymbol{v}\cdot\nabla\boldsymbol{F}+\boldsymbol{v}\cdot D\nabla\boldsymbol{F},D^3\boldsymbol{F}\rangle\leqslant\\ &C(\|D\boldsymbol{v}\|_{L^4}\|\nabla\boldsymbol{F}\|_{L^4}+\|\boldsymbol{v}\|_{L^\infty}\|D^2\boldsymbol{F}\|_{L^2})\|D^3\boldsymbol{F}\|_{L^2}\leqslant\\ &(\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}+\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2})\|D\boldsymbol{F}\|_{H^2}\leqslant\\ &C\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^2}.\end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}I_4 = &\langle D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{F}),D^n\boldsymbol{F}\rangle\leqslant\\ &\|D^n(\boldsymbol{v}\cdot\nabla\boldsymbol{F})\|_{L^2}\|D^n\boldsymbol{F}\|_{L^2}\leqslant\\ &\|\boldsymbol{v}\cdot\nabla\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{H^n}\leqslant\\ &C(\|\boldsymbol{v}\|_{L^\infty}\|\nabla\boldsymbol{F}\|_{H^n}+\|\boldsymbol{v}\|_{H^n}\|\nabla\boldsymbol{F}\|_{L^\infty})\|\boldsymbol{F}\|_{H^n}\leqslant\\ &C(\|\boldsymbol{v}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^n}+\|\boldsymbol{v}\|_{H^n}\|\nabla\boldsymbol{F}\|_{H^2})\|\boldsymbol{F}\|_{H^n}\leqslant\\ &C\|\boldsymbol{v}\|_{H^n}\|\boldsymbol{F}\|_{H^n}\|\nabla\boldsymbol{F}\|_{H^n}.\end{aligned}$$

We can prove the bound I_5,

$$I_5\leqslant\begin{cases}C\|\boldsymbol{v}\|_{H^2}\cdot\|\nabla\boldsymbol{F}\|_{H^2}\|\boldsymbol{F}\|_{H^2}, & \text{if}\quad 1\leqslant n\leqslant 2,\\ C\|\boldsymbol{v}\|_{H^n}\cdot\|\nabla\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{H^n}, & \text{if}\quad n\geqslant 3,\end{cases}\tag{15}$$

where the constant C depends on n.

For $1\leqslant n\leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n=1$,

$$\begin{aligned}I_5 = &-\langle D(\nabla\boldsymbol{v}\boldsymbol{F}),D\boldsymbol{F}\rangle = \langle(\nabla\boldsymbol{v}\cdot\boldsymbol{F}),D^2\boldsymbol{F}\rangle\leqslant\\ &\|\nabla\boldsymbol{v}\|_{L^2}\|\boldsymbol{F}\|_{L^\infty}\|D^2\boldsymbol{F}\|_{L^2}\leqslant\\ &C\|\boldsymbol{v}\|_{H^1}\|\boldsymbol{F}\|_{H^2}\|D\boldsymbol{F}\|_{H^1}\leqslant\\ &C\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^2}.\end{aligned}$$

Secondly, for $n=2$,

$$\begin{aligned}I_5 = &-\langle D^2(\nabla\boldsymbol{v}\boldsymbol{F}),D^2\boldsymbol{F}\rangle = \langle D(\nabla\boldsymbol{v}\cdot\boldsymbol{F}),D^3\boldsymbol{F}\rangle =\\ &\langle D\nabla\boldsymbol{v}\cdot\boldsymbol{F}+\nabla\boldsymbol{v}\cdot D\boldsymbol{F},D^3\boldsymbol{F}\rangle\leqslant\\ &(\|D^2\boldsymbol{v}\|_{L^2}\|\boldsymbol{F}\|_{L^\infty}+\|D\boldsymbol{v}\|_{L^4}\|\nabla\boldsymbol{F}\|_{L^4})\|D^3\boldsymbol{F}\|_{L^2}\leqslant\\ &C(\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}+\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2})\|D\boldsymbol{F}\|_{H^2}\leqslant\\ &C\|\boldsymbol{v}\|_{H^2}\|\boldsymbol{F}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^2}.\end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}I_5 = &\langle D^n(\nabla\boldsymbol{v}\boldsymbol{F}),D^n\boldsymbol{F}\rangle\leqslant\|D^n(\nabla\boldsymbol{v}\boldsymbol{F})\|_{L^2}\|D^n\boldsymbol{F}\|_{L^2}\leqslant\\ &\|\nabla\boldsymbol{v}\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{H^n}\leqslant C(\|\nabla\boldsymbol{v}\|_{L^\infty}\|\boldsymbol{F}\|_{H^n}+\\ &\|\nabla\boldsymbol{v}\|_{H^n}\|\boldsymbol{F}\|_{L^\infty})\|\boldsymbol{F}\|_{H^n}\leqslant C\|\boldsymbol{F}\|_{H^n}^2\|\nabla\boldsymbol{v}\|_{H^n}.\end{aligned}$$

We can prove the bound I_6,

$$I_6 \leqslant \begin{cases} C\|\boldsymbol{v}\|_{H^2} \cdot \|\nabla \boldsymbol{M}\|_{H^2} \|D^2\boldsymbol{M}\|_{H^2}, \text{ if } 1 \leqslant n \leqslant 2, \\ C\|\boldsymbol{v}\|_{H^n} \cdot \|\nabla \boldsymbol{M}\|_{H^n} \|D^2\boldsymbol{M}\|_{H^n}, \text{ if } n \geqslant 3, \end{cases} \tag{16}$$

where the constant C depends on n.

For $1 \leqslant n \leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n = 1$,

$$\begin{aligned} I_6 &= -\langle D^2(\boldsymbol{v} \cdot \nabla \boldsymbol{M}), D^2\boldsymbol{M}\rangle = \langle D(\boldsymbol{v} \cdot \nabla \boldsymbol{M}), D^3\boldsymbol{M}\rangle = \\ &\langle (D\boldsymbol{v} \cdot \nabla \boldsymbol{M} + \boldsymbol{v} \cdot D\nabla \boldsymbol{M}), D^3\boldsymbol{M}\rangle \leqslant \\ &C\|\boldsymbol{v}\|_{H^2} \|\nabla \boldsymbol{M}\|_{H^2} \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

Secondly, for $n = 2$,

$$\begin{aligned} I_6 &= -\langle D^3(\boldsymbol{v} \cdot \nabla \boldsymbol{M}), D^3\boldsymbol{M}\rangle = \langle D^2(\boldsymbol{v} \cdot \nabla \boldsymbol{M}), D^4\boldsymbol{M}\rangle = \\ &\langle D^2\boldsymbol{v} \cdot \nabla \boldsymbol{M} + 2D\boldsymbol{v} \cdot D\nabla \boldsymbol{M} + \boldsymbol{v}D^2\nabla \boldsymbol{M}, D^4\boldsymbol{M}\rangle \leqslant \\ &C(\|\boldsymbol{v}\|_{H^2} \|D\boldsymbol{M}\|_{H^2} + \\ &\|\boldsymbol{v}\|_{H^2} \|D\boldsymbol{M}\|_{H^2}) \|D^2\boldsymbol{M}\|_{H^2} \leqslant \\ &C\|\boldsymbol{v}\|_{H^2} \|D\boldsymbol{M}\|_{H^2} \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

For $n \geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned} I_6 &= -\langle D^{n+1}(\boldsymbol{v} \cdot \nabla \boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle \leqslant \\ &\|D^n(\boldsymbol{v} \cdot \nabla \boldsymbol{M})\|_{L^2} \|D^{n+1}\nabla \boldsymbol{M}\|_{L^2} \leqslant \\ &\|\boldsymbol{v} \cdot \nabla \boldsymbol{M}\|_{H^n} \|D^2\boldsymbol{M}\|_{H^n} \leqslant \\ &C(\|\boldsymbol{v}\|_{L^\infty} \|\nabla \boldsymbol{M}\|_{H^n} + \\ &\|\boldsymbol{v}\|_{H^n} \|\nabla \boldsymbol{M}\|_{L^\infty}) \|D^2\boldsymbol{M}\|_{H^n} \leqslant \\ &C(\|\boldsymbol{v}\|_{H^2} \|\nabla \boldsymbol{M}\|_{H^n} + \\ &\|\boldsymbol{v}\|_{H^n} \|\nabla \boldsymbol{M}\|_{H^2}) \|D^2\boldsymbol{M}\|_{H^n} \leqslant \\ &C\|\boldsymbol{v}\|_{H^n} \|\nabla \boldsymbol{M}\|_{H^n} \|D^2\boldsymbol{M}\|_{H^n}. \end{aligned}$$

We can prove the bound I_7,

$$I_7 \leqslant \begin{cases} C\|\nabla \boldsymbol{M}\|_{H^2}^2 \|D^2\boldsymbol{M}\|_{H^2}, \quad \text{if} \quad 1 \leqslant n \leqslant 2, \\ C\|\nabla \boldsymbol{M}\|_{H^n}^2 \|D^2\boldsymbol{M}\|_{H^n}, \quad \text{if} \quad n \geqslant 3, \end{cases} \tag{17}$$

where the constant C depends on n.

For $1 \leqslant n \leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n = 1$,

$$\begin{aligned} I_7 &\leqslant |-\alpha\langle D^2(\boldsymbol{M} \times \Delta\boldsymbol{M}), D^2\boldsymbol{M}\rangle| = \\ &|-\alpha\langle D^2(\boldsymbol{M} \times \nabla \boldsymbol{M}), \nabla D^2\boldsymbol{M}\rangle| = \\ &|-\alpha\langle D^2\boldsymbol{M} \times \nabla \boldsymbol{M} + 2D\boldsymbol{M} \times \\ &D\nabla \boldsymbol{M} + \boldsymbol{M} \times D^2\nabla \boldsymbol{M}, \nabla D^2\boldsymbol{M}\rangle| \leqslant \\ &3\alpha\|D^2\boldsymbol{M}\|_{L^2} \|\nabla \boldsymbol{M}\|_{L^\infty} \|D^3\boldsymbol{M}\|_{L^2} \leqslant \\ &C\|D\boldsymbol{M}\|_{H^1} \|\nabla \boldsymbol{M}\|_{H^2} \|D^2\boldsymbol{M}\|_{H^1} \leqslant \\ &C\|D\boldsymbol{M}\|_{H^2}^2 \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

Secondly, for $n = 2$,

$$\begin{aligned} I_7 &\leqslant \left| -\alpha\langle D^3(\boldsymbol{M} \times \Delta\boldsymbol{M}), D^3\boldsymbol{M}\rangle \right| = \\ &\left| \alpha\langle D^3(\boldsymbol{M} \times \nabla \boldsymbol{M}), \nabla D^3\boldsymbol{M}\rangle \right| = \\ &\left| \alpha\langle D^3\boldsymbol{M} \times \nabla \boldsymbol{M} + 3D^2\boldsymbol{M} \times D\nabla \boldsymbol{M} + \right. \\ &\left. 3D\boldsymbol{M} \times D^2\nabla \boldsymbol{M} + \boldsymbol{M} \times D^3\nabla \boldsymbol{M}, \nabla D^3\boldsymbol{M}\rangle \right| \leqslant \\ &\alpha(4\|D^3\boldsymbol{M}\|_{L^2} \|\nabla \boldsymbol{M}\|_{L^\infty} + 3\|D^2\boldsymbol{M}\|_{L^4}^2) \|D^4\boldsymbol{M}\|_{L^2} \leqslant \\ &C(\|D\boldsymbol{M}\|_{H^2}^2 + \|D\boldsymbol{M}\|_{H^2}^2) \|D^2\boldsymbol{M}\|_{H^2} \leqslant \\ &C\|D\boldsymbol{M}\|_{H^2}^2 \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

For $n \geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned} I_7 &= \left| -\alpha\langle D^{n+1}(\boldsymbol{M} \times \Delta\boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle \right| = \\ &\left| \alpha\langle D^{n+1}(\boldsymbol{M} \times \nabla \boldsymbol{M}), D^{n+1}\nabla \boldsymbol{M}\rangle \right| = \\ &\left| \alpha\langle (D^{n+1}\boldsymbol{M} \times \nabla \boldsymbol{M} + \boldsymbol{M} \times D^{n+1}\nabla \boldsymbol{M} + \right. \\ &\left. \sum_{i=1}^{n} C_i D^i\boldsymbol{M} \times D^{n+1-i}\nabla \boldsymbol{M}), D^{n+1}\nabla \boldsymbol{M}\rangle \right| \leqslant \\ &C\alpha(2^{n+1} - 1) \|D\boldsymbol{M}\|_{L^\infty} \|\nabla \boldsymbol{M}\|_{H^n} \|D^{n+1}\nabla \boldsymbol{M}\|_{L^2} \leqslant \\ &C\|D\boldsymbol{M}\|_{H^2} \|\nabla \boldsymbol{M}\|_{H^n} \|D^2\boldsymbol{M}\|_{H^n} \leqslant \\ &C\|D\boldsymbol{M}\|_{H^n}^2 \|D^2\boldsymbol{M}\|_{H^n}. \end{aligned}$$

We can prove the bound I_8,

$$I_8 \leqslant \begin{cases} C(\nabla \boldsymbol{M}\|_{H^2}^2 + \nabla \boldsymbol{M}\|_{H^2}^3) \|D^2\boldsymbol{M}\|_{H^2}, \\ \quad \text{if} \quad 1 \leqslant n \leqslant 2, \\ C(\nabla \boldsymbol{M}\|_{H^n}^2 \|\boldsymbol{M}\|_{H^n} + \|\nabla \boldsymbol{M}\|_{H^n}^2) \|D^2\boldsymbol{M}\|_{H^n}, \\ \quad \text{if} \quad n \geqslant 3, \end{cases} \tag{18}$$

where the constant C depends on n.

For $1 \leqslant n \leqslant 2$, using Hölder's inequality and Lemma 1, we have firstly, for $n = 1$,

$$\begin{aligned} I_8 &\leqslant |\beta\langle D^2(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^2\boldsymbol{M}\rangle| = \\ &|-\beta\langle D(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^3\boldsymbol{M}\rangle| = \\ &|-\beta\langle 2\nabla \boldsymbol{M}D\nabla \boldsymbol{M} \cdot \boldsymbol{M} + |\nabla \boldsymbol{M}|^2 D\boldsymbol{M}, D^3\boldsymbol{M}\rangle| \leqslant \\ &\beta(2\|\nabla \boldsymbol{M}\|_{L^\infty} \|D\nabla \boldsymbol{M}\|_{L^2} + \\ &\|D\boldsymbol{M}\|_{L^2} \|\nabla \boldsymbol{M}\|_{L^\infty}^2) \|D^3\boldsymbol{M}\|_{L^2} \leqslant \\ &C(\|\nabla \boldsymbol{M}\|_{H^2} \|D\boldsymbol{M}\|_{H^1} + \\ &\|D\boldsymbol{M}\|_{L^2} \|\nabla \boldsymbol{M}\|_{H^2}^2) \cdot \|D^2\boldsymbol{M}\|_{H^1} \leqslant \\ &C(\|\nabla \boldsymbol{M}\|_{H^2}^2 + \|\nabla \boldsymbol{M}\|_{H^2}^3) \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

Secondly, for $n = 2$,

$$\begin{aligned} I_8 &= \beta\langle D^3(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^3\boldsymbol{M}\rangle = \\ &-\beta\langle D^2(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^4\boldsymbol{M}\rangle = \\ &-\beta\langle 2|D^2\boldsymbol{M}|^2 \cdot \boldsymbol{M} + 2|\nabla \boldsymbol{M}|D^3\boldsymbol{M} \cdot \\ &\boldsymbol{M} + 5|\nabla \boldsymbol{M}|^2 \cdot D\nabla \boldsymbol{M}, D^4\boldsymbol{M}\rangle \leqslant \\ &\beta(2\|D^2\boldsymbol{M}\|_{L^4}^2 \|\boldsymbol{M}\|_{L^\infty} + \\ &2\|\nabla \boldsymbol{M}\|_{L^\infty} \|\boldsymbol{M}\|_{L^\infty} \|D^3\boldsymbol{M}\|_{L^2} + 5\|\nabla \boldsymbol{M}\|_{L^\infty}^2 \cdot \\ &\|D^2\boldsymbol{M}\|_{L^2}) \|D^4\boldsymbol{M}\|_{L^2} \leqslant \\ &C(2\|D\boldsymbol{M}\|_{H^2}^2 + 2\|\nabla \boldsymbol{M}\|_{H^2}^2 + \\ &5\|\nabla \boldsymbol{M}\|_{H^2}^2 \|D\boldsymbol{M}\|_{H^2}) \|D^2\boldsymbol{M}\|_{H^2} \leqslant \\ &C(\|D\boldsymbol{M}\|_{H^2}^2 + \|\nabla \boldsymbol{M}\|_{H^2}^3) \|D^2\boldsymbol{M}\|_{H^2}. \end{aligned}$$

For $n\geq 3$, using Hölder's inequality and Lemma 2, we have

$$\begin{aligned}
I_8 = {} & \beta\langle D^{n+1}(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^{n+1}\boldsymbol{M}\rangle = \\
& \beta\langle D^{n}(|\nabla \boldsymbol{M}|^2\boldsymbol{M}), D^{n+1}\nabla\boldsymbol{M}\rangle \leqslant \\
& \beta\| D^{n}(|\nabla \boldsymbol{M}|^2\boldsymbol{M})\|_{L^2}\cdot\| D^{n+1}\nabla\boldsymbol{M}\|_{L^2} \leqslant \\
& \beta\| |\nabla \boldsymbol{M}|^2\boldsymbol{M}\|_{H^n}\cdot\| D^{2}\boldsymbol{M}\|_{H^n} \leqslant \\
& C(\|\nabla\boldsymbol{M}\|_{L^\infty}^2\|\boldsymbol{M}\|_{H^n} + \\
& \| |\nabla\boldsymbol{M}|^2\|_{H^n}\|\boldsymbol{M}\|_{L^\infty})\|D^2\boldsymbol{M}\|_{H^n} \leqslant \\
& C(\|\nabla\boldsymbol{M}\|_{H^2}^2\|\boldsymbol{M}\|_{H^n} + \\
& \|D\boldsymbol{M}\|_{H^n}\|\nabla\boldsymbol{M}\|_{L^\infty})\|D^2\boldsymbol{M}\|_{H^n} \leqslant \\
& C(\|\nabla\boldsymbol{M}\|_{H^2}^2\|\boldsymbol{M}\|_{H^n} + \\
& \|D\boldsymbol{M}\|_{H^n}\|\nabla\boldsymbol{M}\|_{H^2})\|D^2\boldsymbol{M}\|_{H^n} \leqslant \\
& C(\|\nabla\boldsymbol{M}\|_{H^n}^2\|\boldsymbol{M}\|_{H^n} + \|\nabla\boldsymbol{M}\|_{H^n}^2)\|D^2\boldsymbol{M}\|_{H^n}.
\end{aligned}$$

Next, we consider the bound the energy of $\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{M}$. we think about the first case $1\leqslant n\leqslant 2$, according to Eqs. (6), (9) and (11) ~ (18), we can obtain

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2) + \\
& \mu\|\nabla\boldsymbol{v}\|_{H^2}^2 + \kappa\|\nabla\boldsymbol{F}\|_{H^2}^2 + \\
& (\beta+\varepsilon)\|D^2\boldsymbol{M}\|_{H^2}^2 \leqslant (\beta+\varepsilon)\|D^2\boldsymbol{M}\|_{L^2}^2 + \\
& C\|\nabla\boldsymbol{v}\|_{H^2}(\|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2) + \\
& C\|\boldsymbol{v}\|_{H^2}\|\nabla\boldsymbol{F}\|_{H^2}\|\boldsymbol{F}\|_{H^2} + \\
& C\|\boldsymbol{v}\|_{H^2}\|\nabla\boldsymbol{M}\|_{H^2}\|D^2\boldsymbol{M}\|_{H^2} + \\
& C(\|\nabla\boldsymbol{M}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^3)\|D^2\boldsymbol{M}\|_{H^2} \leqslant \\
& \frac{\mu}{2}\|\nabla\boldsymbol{v}\|_{H^2}^2 + C(1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2)^2 + \\
& C\|\boldsymbol{v}\|_{H^2}^2\|\boldsymbol{F}\|_{H^2}^2 + \frac{\kappa}{2}\|\nabla\boldsymbol{F}\|_{H^2}^2 + \\
& C\|\boldsymbol{v}\|_{H^2}^2\|\nabla\boldsymbol{M}\|_{H^2}^2 + \frac{\beta+\varepsilon}{4}\|D^2\boldsymbol{M}\|_{H^2}^2 + \\
& C(\|\nabla\boldsymbol{M}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^3)^2 + \frac{\beta+\varepsilon}{4}\|D^2\boldsymbol{M}\|_{H^2}^2 \leqslant \\
& \frac{\mu}{2}\|\nabla\boldsymbol{v}\|_{H^2}^2 + \frac{\kappa}{2}\|\nabla\boldsymbol{F}\|_{H^2}^2 + \frac{\beta+\varepsilon}{2}\|D^2\boldsymbol{M}\|_{H^2} + \\
& C(1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2)^3.
\end{aligned}$$

From above calculating, we can obtain

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2) + \mu\|\nabla\boldsymbol{v}\|_{H^2}^2 + \\
& \kappa\|\nabla\boldsymbol{F}\|_{H^2}^2 + (\beta+\varepsilon)\|D^2\boldsymbol{M}\|_{H^2}^2 \leqslant \\
& C(1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2)^3.
\end{aligned} \tag{19}$$

If we set

$$y(t) = 1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2,$$

$$f(y(t)) = (1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2)^3,$$

then, we have

$$\begin{gathered}
y(t)' \leqslant C(f(y(t))) = C(y(t))^3, \\
y(0) = 1 + \|\boldsymbol{v}_0\|_{H^2}^2 + \|\boldsymbol{F}_0\|_{H^2}^2 + \|\nabla\boldsymbol{M}_0\|_{H^2}^2,
\end{gathered} \tag{20}$$

where C depends on Ω.

Using Lemma 3, we obtain that there exists $T(\Omega, \|\boldsymbol{v}_0\|_{H^2}, \|\boldsymbol{F}_0\|_{H^2}, \|\nabla\boldsymbol{M}_0\|_{H^2}) > 0$ such that

$$\sup_{0\leqslant t\leqslant T}\|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2 \leqslant \tilde{K}_2,\ t\in[0,t].$$

Hence, by this and Eqs. (20), there exist $K_2 \geqslant 0$ such that

$$\begin{aligned}
& \sup_{0\leqslant t\leqslant T}(\|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2) + \\
& \mu\int_0^t\|\nabla\boldsymbol{v}\|_{H^2}^2\mathrm{d}s + k\int_0^t\|\nabla\boldsymbol{F}\|_{H^2}^2\mathrm{d}s + \\
& (\beta+\varepsilon)\int_0^t\|\nabla\boldsymbol{M}\|_{H^3}^2\mathrm{d}s \leqslant K_2.
\end{aligned} \tag{21}$$

We continue to bound the higher-order energy of $\boldsymbol{v}$, $\boldsymbol{F}$, $\boldsymbol{M}$. We think about the second case $n\geqslant 3$.

According to Eqs. (9), (11) ~ (18) and (19), we can obtain

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|_{H^n}^2 + \|\boldsymbol{F}\|_{H^n}^2 + \|\nabla\boldsymbol{M}\|_{H^n}^2) + \\
& \mu\|\nabla\boldsymbol{v}\|_{H^n}^2 + k\|\nabla\boldsymbol{F}\|_{H^n}^2 + (\beta+\varepsilon)\|D^2\boldsymbol{M}\|_{H^n}^2 \leqslant \\
& C(1 + \|\boldsymbol{v}\|_{H^2}^2 + \|\boldsymbol{F}\|_{H^2}^2 + \|\nabla\boldsymbol{M}\|_{H^2}^2)^3 + \\
& C\|\nabla\boldsymbol{v}\|_{H^n}(\|v\|_{H^n}^2 + \|\boldsymbol{F}\|_{H^n}^2 + \|\nabla\boldsymbol{M}\|_{H^n}^2) + \\
& \|\boldsymbol{v}\|_{H^n}\|\nabla\boldsymbol{F}\|_{H^n}\|\boldsymbol{F}\|_{H^n} + C\|\boldsymbol{v}\|_{H^n}\cdot \\
& \|\nabla\boldsymbol{M}\|_{H^n}\|D^2\boldsymbol{M}\|_{H^n} + \\
& C\|\nabla\boldsymbol{M}\|_{H^n}^2(\|\boldsymbol{M}\|_{H^n} + 1)\|D^2\boldsymbol{M}\|_{H^n} \leqslant \\
& \frac{\mu}{4}\|\nabla\boldsymbol{v}\|_{H^n}^2 + C(1 + \|\boldsymbol{v}\|_{H^n}^2 + \|\boldsymbol{F}\|_{H^n}^2 + \|\nabla\boldsymbol{M}\|_{H^n}^2)^3 + \\
& C\|\boldsymbol{v}\|_{H^n}^2\|\boldsymbol{F}\|_{H^n}^2 + \frac{\kappa}{2}\|\nabla\boldsymbol{F}\|_{H^n}^2 + \\
& C\|\boldsymbol{v}\|_{H^n}^2\|\nabla\boldsymbol{M}\|_{H^n}^2 + \frac{\beta+\varepsilon}{4}\|D^2\boldsymbol{M}\|_{H^n}^2 + \\
& C\|\nabla\boldsymbol{M}\|_{H^n}^4(\|\boldsymbol{M}\|_{H^n} + 1)^2 + \frac{\beta+\varepsilon}{4}\|D^2\boldsymbol{M}\|_{H^n}^2 \leqslant \\
& \frac{\mu}{2}\|\nabla\boldsymbol{v}\|_{H^n}^2 + \frac{\kappa}{2}\|\nabla\boldsymbol{F}\|_{H^n}^2 + \frac{\beta+\varepsilon}{4}\|D^2\boldsymbol{M}\|_{H^n}^2 + \\
& C(1 + \|\boldsymbol{v}\|_{H^n}^2 + \|\boldsymbol{F}\|_{H^n}^2 + \\
& \|\nabla\boldsymbol{M}\|_{H^n}^2)^3(\|\boldsymbol{M}\|_{H^n} + 1)^2.
\end{aligned} \tag{22}$$

Here $\|\boldsymbol{M}\|_{H^n(\Omega)} = \|D\boldsymbol{M}\|_{H^{n-1}(\Omega)} + |\Omega|$.

From Eq. (21), using mathematical induction, we may assume that for any $3\leqslant l\leqslant n-1$, there exists $K_l \geqslant 0$ such that

$$\begin{aligned}
& \|\boldsymbol{v}\|_{H^l}^2 + \|\boldsymbol{F}\|_{H^l}^2 + \|\nabla\boldsymbol{M}\|_{H^l}^2 + \\
& \mu\int_0^t\|\nabla\boldsymbol{v}\|_{H^l}^2\mathrm{d}s + k\int_0^t\|\nabla\boldsymbol{F}\|_{H^l}^2\mathrm{d}s +
\end{aligned}$$

$$(\beta+\varepsilon)\int_0^t \|\nabla^2 \boldsymbol{M}\|_{H^l}^2 \mathrm{d}s \leqslant K_l,\ t\in[0,T]. \tag{23}$$

From Eqs. (22) and (23), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2)+\mu\|\nabla \boldsymbol{v}\|_{H^n}^2+ \\ \kappa\|\nabla \boldsymbol{F}\|_{H^n}^2+(\beta+\varepsilon)\|\nabla^2 \boldsymbol{M}\|_{H^n}^2\leqslant \\ C(1+\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2)^3(K_{n-1}^{\frac{1}{2}}+1)^2. \tag{24}$$

If we set

$$y(t)=1+\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2,$$

$$f(y(t))=(1+\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2)^3,$$

then, using Eqs. (22) and (23), we have

$$\begin{aligned} &y(t)'\leqslant C(K_{n-1}^{\frac{1}{2}}+1)^2 f_n=C(K_{n-1}^{\frac{1}{2}}+1)^2 y_n^3(t),\\ &y(0)=1+\|\boldsymbol{v}_0\|_{H^n}^2+\|\boldsymbol{F}_0\|_{H^n}^2+\|\nabla \boldsymbol{M}_0\|_{H^n}^2, \end{aligned} \tag{25}$$

where C depend on Ω.

Using Lemma 3, we obtain that there exists $\tilde{K}_n$ such that

$$\sup_{0\leqslant t\leqslant T}\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2\leqslant \tilde{K}_n,\\ t\in[0,T].$$

Hence, by this and Eq. (22), we obtain that there exists $K_n\geqslant 0$ such

$$\sup_{0\leqslant t\leqslant T}(\|\boldsymbol{v}\|_{H^n}^2+\|\boldsymbol{F}\|_{H^n}^2+\|\nabla \boldsymbol{M}\|_{H^n}^2)+ \\ \mu\int_0^t\|\nabla \boldsymbol{v}\|_{H^n}^2\mathrm{d}s+k\int_0^t\|\nabla \boldsymbol{F}\|_{H^n}^2\mathrm{d}s+ \\ (\beta+\varepsilon)\int_0^t\|\nabla^2 \boldsymbol{M}\|_{H^n}^2\mathrm{d}s\leqslant K_n,\ t\in[0,T]. \tag{26}$$

System (5) must have a solution that exists on $[0,T]$. If not, we may always expand the temporal period of existence to cover $[0,T]$, i. e., which means that we always have $T_\varepsilon\geqslant T$.

Finally, from Eq. (26), we obtain $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\boldsymbol{M}_\varepsilon)$ subconverge to a smooth solution of (4) as $\varepsilon\to 0$. That concludes the proof of Lemma 6.

3.2 Local existence of solutions in $\mathbb{R}^d$

Proof of Theorem 1

The proof of Theorem 1 below depends on [19].

From $(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0)\in H^k\times H^k\times H_Q^{k+1}$ for $k\geqslant 2$, by the density theorem of Sobolev spaces and Lemma 5, searching a sequence

$$(\boldsymbol{v}_{i0},\boldsymbol{F}_{i0})\in C_0^\infty(\mathbb{R}^d),$$

$$\boldsymbol{M}_{i0}-Q\in H^k(\mathbb{R}^d,\mathbb{S}^2)\cap C_0^\infty(\mathbb{R}^d,\mathbb{R}^3)$$

such that,

$$(\boldsymbol{v}_{i0},\boldsymbol{F}_{i0},\boldsymbol{M}_{i0})\to(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0)\in H^k(\mathbb{R}^d)\times \\ H^k(\mathbb{R}^d)\times H_Q^{k+1}(\mathbb{R}^d),\ \text{as } i\to\infty.$$

As a result, there exists a smooth solution $(\boldsymbol{v}_i,\boldsymbol{F}_i,\boldsymbol{M}_i)$, defined on the time interval $[0,T_i]$, system (2) with $(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0)$ substituted by $(\boldsymbol{v}_{i0},\boldsymbol{F}_{i0},\boldsymbol{M}_{i0}-Q)$. Furthermore, Lemma 6 states that if i is high enough, then there exists a uniform positive lower bound of T_i. In other words, using Lemma 6, we can obtain

$$\sup_{0\leqslant t\leqslant T}(\|\boldsymbol{v}_i\|_{H^k(\mathbb{T}_i^d)}^2+\|\boldsymbol{F}_i\|_{H^k(\mathbb{T}_i^d)}^2+\|\nabla \boldsymbol{M}_i\|_{H_Q^k(\mathbb{T}_i^d)}^2+ \\ \mu\int_0^t\|\nabla \boldsymbol{v}_i\|_{H^k(\mathbb{T}_i^d)}^2\mathrm{d}s+k\int_0^t\|\nabla \boldsymbol{F}_i\|_{H^k(\mathbb{T}_i^d)}^2\mathrm{d}s+ \\ \beta\int_0^t\|\nabla \boldsymbol{M}_i\|_{H_Q^k(\mathbb{T}_i^d)}^2\mathrm{d}s\leqslant C(T,\|\boldsymbol{v}_0\|_{H^k(\mathbb{T}_i^d)}^2, \\ \|\boldsymbol{F}_0\|_{H^k(\mathbb{T}_i^d)}^2,\|\nabla \boldsymbol{M}_0\|_{H_Q^k(\mathbb{T}_i^d)}^2),\ t\in[0,T].$$

Next, using extension (see Theorem 4.9[23]), let R_i be the support of $(\boldsymbol{v}_i,\boldsymbol{F}_i,\boldsymbol{M}_i)$, and satisfy $\mathbb{T}_i^d\subset\subset[-R_i,R_i]^{2d}$. If we regard each $(\boldsymbol{v}_i,\boldsymbol{F}_i,\boldsymbol{M}_i)$ as a function from $[-R_i,R_i]^d\times[-R_i,R_i]^d\times[-R_i,R_i]^d$ into $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^2$, then there exists a $(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})\in L^\infty([0,T];H^n(\mathbb{R}^d)\times H^n(\mathbb{R}^d)\times H_Q^{n+1}(\mathbb{R}^d))$ and a subsequence which is still denoted by $(\boldsymbol{v}_i,\boldsymbol{F}_i,\boldsymbol{M}_i)$ such that for any compact domain $E_1,E_2,E_3\subset\mathbb{R}^d$

$$(\boldsymbol{v}_i,\boldsymbol{F}_i,\boldsymbol{M}_i)\to(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})\ \text{weakly}^*\ \text{in} \\ L^\infty([0,T];H^n(E_1))\times L^\infty([0,T];H^n(E_2))\times \\ L^\infty([0,T];H_Q^{n+1}(E_3)).$$

And then, we easily obtain that $(\boldsymbol{v},\boldsymbol{F},\boldsymbol{M})$ is a strong solution of system (2).

3.3 Invariance of modules

Multiply Eq. (8) by $\boldsymbol{M}$ and using $\Delta|\boldsymbol{M}|^2=2(\boldsymbol{M}\cdot\Delta\boldsymbol{M})+2|\nabla\boldsymbol{M}|^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{M}|^2+\frac{1}{2}\boldsymbol{v}\cdot\nabla|\boldsymbol{M}|^2-\frac{1}{2}(\beta+\varepsilon)\Delta|\boldsymbol{M}|^2- \\ \beta|\nabla\boldsymbol{M}|^2(|\boldsymbol{M}|^2-1)+\varepsilon|\nabla\boldsymbol{M}|^2=0. \tag{27}$$

Denote $b=|\boldsymbol{M}|^2-1$ and then taking the scalar product of Eq. (27) with b

$$\begin{cases}\langle\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}b,b\rangle+\langle\frac{1}{2}\boldsymbol{v}\cdot\nabla b,b\rangle-\langle\frac{1}{2}(\beta+\varepsilon)\Delta b,b\rangle- \\ \langle\beta|\nabla\boldsymbol{M}|^2 b,b\rangle+\langle\varepsilon|\nabla\boldsymbol{M}|^2,b\rangle=0, \\ b(0)=|\boldsymbol{M}_0|-1=0.\end{cases} \tag{28}$$

Since $\boldsymbol{M}\in L^\infty([0,T],H^2(\Omega))$, $H^2\subset L^\infty(\Omega)$, we obtain that $|\nabla\boldsymbol{M}|^2\in L^1(0,T;L^\infty)$.

Therefore, we can obtain that the energy estimate of (28)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^2}^2+\frac{1}{2}(\beta+\varepsilon)\|\nabla b\|_{L^2}^2+\langle\varepsilon|\nabla\boldsymbol{M}|^2,b\rangle\leqslant$$
$$\beta\|\nabla\boldsymbol{M}\|_{L^\infty}^2\|b\|_{L^2}. \tag{29}$$

Using Gronwall's inequality from Eq. (29), we obtain $\|b\|_{L^2}^2=0$. So we have proved that $|\boldsymbol{M}|=1$ in $[0,T]\times\Omega$, as soon as $|\boldsymbol{M}_0|=1$ in Ω. Thus, we obtain that Eq. (8) is equivalent to the fourth equation of the symstem (5).

4 Uniqueness of local solution

In this section, we will research the uniqueness of the local solution.

Suppose that $(\boldsymbol{v}_1,\boldsymbol{F}_1,\boldsymbol{M}_1),(\boldsymbol{v}_2,\boldsymbol{F}_2,\boldsymbol{M}_2)\in H^3\times H^3\times H_Q^5$ are two solutions to system (2) with the same initial $(\boldsymbol{v}_0,\boldsymbol{F}_0,\boldsymbol{M}_0)\in H^3\times H^3\times H_Q^5$ on an interval $[0,T]$. For simplicity, denote $\boldsymbol{v}_1-\boldsymbol{v}_2:=\tilde{v}$, $\boldsymbol{F}_1-\boldsymbol{F}_2:=\tilde{F}$, $\boldsymbol{M}_1-\boldsymbol{M}_2:=\widetilde{\boldsymbol{M}}$.

Using $(\boldsymbol{v}_1,\boldsymbol{F}_1,\boldsymbol{M}_1)$ and $(\boldsymbol{v}_2,\boldsymbol{F}_2,\boldsymbol{M}_2)$ as two solution to the first equation of system (2), we have

$$\partial_t\tilde{\boldsymbol{v}}+\boldsymbol{v}_1\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_2=\mu\Delta\tilde{v}-\nabla\cdot(2A\nabla\boldsymbol{M}_1\odot\nabla\boldsymbol{M}_1-$$
$$2A\nabla\boldsymbol{M}_2\odot\nabla\boldsymbol{M}_2)+\nabla\cdot(W'(\boldsymbol{F}_1)\boldsymbol{F}_1^{\mathrm{T}}-W'(\boldsymbol{F}_2)\boldsymbol{F}_2^{\mathrm{T}}). \tag{30}$$

Multiplying Eq. (30) by $\tilde{v}$ and then integrate over $\mathbb{R}^d$ with respect to x.

$$\langle\partial_t\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\rangle+\langle\boldsymbol{v}_1\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_2,\tilde{\boldsymbol{v}}\rangle=\mu\langle\Delta\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\rangle-$$
$$\langle\nabla\cdot(2A\nabla\boldsymbol{M}_1\odot\nabla\boldsymbol{M}_1-2A\nabla\boldsymbol{M}_2\odot\nabla\boldsymbol{M}_2),\tilde{\boldsymbol{v}}\rangle+$$
$$\langle\nabla\cdot(W'(\boldsymbol{F}_1)\boldsymbol{F}_1^{\mathrm{T}}-W'(\boldsymbol{F}_2)\boldsymbol{F}_2^{\mathrm{T}}),\tilde{\boldsymbol{v}}\rangle.$$

We can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}\|_{L_2}^2+\mu\|\nabla(\tilde{\boldsymbol{v}})\|_{L_2}^2=-\langle\boldsymbol{v}_1\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_2,\tilde{\boldsymbol{v}}\rangle-$$
$$\langle\nabla\cdot(2A\nabla\boldsymbol{M}_1\odot\nabla\boldsymbol{M}_1-2A\nabla\boldsymbol{M}_2\odot\nabla\boldsymbol{M}_2),\tilde{\boldsymbol{v}}\rangle+$$
$$\langle\nabla\cdot(W'(\boldsymbol{F}_1)\boldsymbol{F}_1^{\mathrm{T}}-W'(\boldsymbol{F}_2)\boldsymbol{F}_2^{\mathrm{T}}),\tilde{\boldsymbol{v}}\rangle=$$
$$II_1+II_2+II_3. \tag{31}$$

Next, we will estimate each term on the right-hand side of the above equation. By Hölder's inequality and Lemma 1, we obtain

$$II_1=-\langle\boldsymbol{v}_1\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_2,\tilde{\boldsymbol{v}}\rangle=$$
$$-\langle\boldsymbol{v}_1\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_1+\boldsymbol{v}_2\nabla\boldsymbol{v}_1-\boldsymbol{v}_2\nabla\boldsymbol{v}_2,\tilde{\boldsymbol{v}}\rangle=$$
$$-\langle\tilde{\boldsymbol{v}}\nabla\boldsymbol{v}_1+\boldsymbol{v}_2\nabla\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\rangle=\langle\nabla\cdot\tilde{\boldsymbol{v}}\boldsymbol{v}_1,\tilde{\boldsymbol{v}}\rangle+$$
$$\langle\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}_1,\nabla\tilde{\boldsymbol{v}}\rangle+\langle\boldsymbol{v}_2,\frac{1}{2}\nabla|\tilde{\boldsymbol{v}}|^2\rangle\leqslant$$
$$C\|\boldsymbol{v}_1\|_{L^\infty}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\tilde{\boldsymbol{v}}\|_{L^2}\leqslant$$
$$C\|\boldsymbol{v}_1\|_{H^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\tilde{\boldsymbol{v}}\|_{L^2},$$

$$II_2=-\langle\nabla\cdot(2A\nabla\boldsymbol{M}_1\odot\nabla\boldsymbol{M}_1-2A\nabla\boldsymbol{M}_2\odot\nabla\boldsymbol{M}_2),\tilde{\boldsymbol{v}}\rangle=$$
$$-\langle\partial_j(2A(\partial_j\boldsymbol{M}_1\cdot\partial_i\boldsymbol{M}_1-\partial_j\boldsymbol{M}_1\cdot\partial_i\boldsymbol{M}_2+$$
$$\partial_j\boldsymbol{M}_1\cdot\partial_i\boldsymbol{M}_2-\partial_j\boldsymbol{M}_2\cdot\partial_i\boldsymbol{M}_2)),\tilde{\boldsymbol{v}}\rangle=$$
$$2A\langle\partial_j\boldsymbol{M}_1\partial_i\widetilde{\boldsymbol{M}}+\partial_j\widetilde{\boldsymbol{M}}\partial_i\boldsymbol{M}_2,\partial_j\tilde{\boldsymbol{v}}\rangle\leqslant$$
$$2A(\|\nabla\boldsymbol{M}_1\|_{L^\infty}+\|\nabla\boldsymbol{M}_2\|_{L^\infty})\cdot$$
$$\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\leqslant C(\|\nabla\boldsymbol{M}_1\|_{H^2}+$$
$$\|\nabla\boldsymbol{M}_2\|_{H^2})\cdot\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2},$$

$$II_3=\langle\nabla\cdot(W'(\boldsymbol{F}_1)\boldsymbol{F}_1^{\mathrm{T}}-W'(\boldsymbol{F}_2)\boldsymbol{F}_2^{\mathrm{T}}),\tilde{\boldsymbol{v}}\rangle=$$
$$-2\langle\nabla\cdot(\boldsymbol{F}_1^2-\boldsymbol{F}_2^2),\tilde{\boldsymbol{v}}\rangle=2\langle\widetilde{\boldsymbol{F}}(\boldsymbol{F}_1+\boldsymbol{F}_2),\nabla\tilde{\boldsymbol{v}}\rangle\leqslant$$
$$2(\|\boldsymbol{F}_1\|_{L^\infty}+\|\boldsymbol{F}_2\|_{L^\infty})\|\widetilde{\boldsymbol{F}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\leqslant$$
$$C(\|\boldsymbol{F}_1\|_{H^2}+\|\boldsymbol{F}_2\|_{H^2})\|\widetilde{\boldsymbol{F}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}.$$

Using above estimates and Eq. (31), we can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\boldsymbol{v}}\|_{L_2}^2+\mu\|\nabla\tilde{\boldsymbol{v}}\|_{L_2}^2\leqslant$$
$$C(\|\boldsymbol{v}_2\|_{H^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\tilde{\boldsymbol{v}}\|_{L^2}+(\|\nabla\boldsymbol{M}_1\|_{H^2}+$$
$$\|\nabla\boldsymbol{M}_2\|_{H^2})\cdot\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}+$$
$$(\|\boldsymbol{F}_1\|_{H^2}+\|\boldsymbol{F}_2\|_{H^2})\|\widetilde{\boldsymbol{F}}\|_{L^2}\|\nabla\tilde{\boldsymbol{v}}\|_{H^2})\leqslant$$
$$C\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}(\|\tilde{\boldsymbol{v}}\|_{L^2}+\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}+\|\widetilde{\boldsymbol{F}}\|_{L^2})\leqslant$$
$$\frac{\mu}{6}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}^2+C(\|\tilde{\boldsymbol{v}}\|_{L^2}^2+\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}^2+\|\widetilde{\boldsymbol{F}}\|_{L^2}^2). \tag{32}$$

Using $(\boldsymbol{v}_1,\boldsymbol{F}_1,\boldsymbol{M}_1)$ and $(\boldsymbol{v}_2,\boldsymbol{F}_2,\boldsymbol{M}_2)$ as two solutions to the third equation of system (2), we have

$$\partial_t\widetilde{\boldsymbol{F}}+(\boldsymbol{v}_1\nabla\boldsymbol{F}_1-\boldsymbol{v}_2\nabla\boldsymbol{F}_2)=\kappa\Delta\widetilde{\boldsymbol{F}}+(\nabla\boldsymbol{v}_1\boldsymbol{F}_1-\nabla\boldsymbol{v}_2\nabla\boldsymbol{F}_2). \tag{33}$$

Multiplying Eq. (33) by $\widetilde{\boldsymbol{F}}$ and then integrates over $\mathbb{R}^d$ with respect to x.

$$\langle\partial_t\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{F}}\rangle+\langle(\boldsymbol{v}_1\nabla\boldsymbol{F}_1-\boldsymbol{v}_2\nabla\boldsymbol{F}_2),\widetilde{\boldsymbol{F}}\rangle=$$
$$\kappa\langle\Delta\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{F}}\rangle+\langle(\nabla\boldsymbol{v}_1\boldsymbol{F}_1-\nabla\boldsymbol{v}_2\boldsymbol{F}_2),\widetilde{\boldsymbol{F}}\rangle.$$

We can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\boldsymbol{F}}\|_{L_2}^2+\kappa\|\nabla\widetilde{\boldsymbol{F}}\|_{L_2}^2=-\langle\boldsymbol{v}_1\nabla\boldsymbol{F}_1-\boldsymbol{v}_2\nabla\boldsymbol{F}_2,\widetilde{\boldsymbol{F}}\rangle+$$
$$\langle\nabla\boldsymbol{v}_1\boldsymbol{F}_1-\nabla\boldsymbol{v}_2\boldsymbol{F}_2,\widetilde{\boldsymbol{F}}\rangle=II_4+II_5. \tag{34}$$

Next, we will estimate each term on the right-hand side of the above. By Hölder's inequality and Lemma 1, we obtain

$$II_4=-\langle\boldsymbol{v}_1\nabla\boldsymbol{F}_1-\boldsymbol{v}_2\nabla\boldsymbol{F}_2,\widetilde{\boldsymbol{F}}\rangle=-\langle\tilde{\boldsymbol{v}}\nabla\boldsymbol{F}_1+\boldsymbol{v}_2\nabla\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{F}}\rangle=$$
$$\langle\nabla\cdot\tilde{\boldsymbol{v}}\boldsymbol{F}_1,\widetilde{\boldsymbol{F}}\rangle+\langle\tilde{\boldsymbol{v}}\cdot\boldsymbol{F}_1,\nabla\widetilde{\boldsymbol{F}}\rangle+\boldsymbol{v}_2,\frac{1}{2}\nabla|\widetilde{\boldsymbol{F}}|^2\rangle\leqslant$$
$$\|\boldsymbol{F}_1\|_{L^\infty}\|\tilde{\boldsymbol{v}}\|_{L^2}\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2}\leqslant$$
$$C\|\boldsymbol{F}_1\|_{H^2}\|\tilde{\boldsymbol{v}}\|_{L^2}\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2}.$$

Next, we will estimate each term on the right-hand side of the above

$$\begin{aligned}II_5 =& \langle(\nabla \boldsymbol{v}_1\boldsymbol{F}_1 - \nabla \boldsymbol{v}_2\boldsymbol{F}_2),\widetilde{\boldsymbol{F}}\rangle = \\ & \langle \nabla \tilde{\boldsymbol{v}}\boldsymbol{F}_1 + \nabla \boldsymbol{v}_2\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{F}}\rangle = \\ & \langle \nabla \tilde{\boldsymbol{v}}\boldsymbol{F}_1,\widetilde{\boldsymbol{F}}\rangle - 2\langle \boldsymbol{v}_2\nabla\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{F}}\rangle \leqslant \\ & (\|\boldsymbol{F}_1\|_{L^\infty}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2} + \\ & 2\|\boldsymbol{v}_2\|_{L^\infty}\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2})\|\widetilde{\boldsymbol{F}}\|_{L^2} \leqslant \\ & C(\|\boldsymbol{F}_1\|_{H^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2} + \\ & \|\boldsymbol{v}_2\|_{H^2}\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2})\|\widetilde{\boldsymbol{F}}\|_{L^2}.\end{aligned}$$

Using above estimates and Eq. (34), we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\boldsymbol{F}}\|_{L_2}^2 + \kappa\|\nabla\widetilde{\boldsymbol{F}}\|_{L_2}^2 \leqslant \\ &\quad C(\|\boldsymbol{F}_1\|_{H^2}\|\tilde{\boldsymbol{v}}\|_{L^2} + \|\boldsymbol{v}_2\|_{H^2}\|\widetilde{\boldsymbol{F}}\|_{L^2})\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2} + \\ &\quad C\|\boldsymbol{F}_1\|_{H^2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\widetilde{\boldsymbol{F}}\|_{L^2} \leqslant \\ &\quad C(\|\tilde{\boldsymbol{v}}\|_{L^2} + \|\widetilde{\boldsymbol{F}}\|_{L^2})\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2} + \\ &\quad C\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\widetilde{\boldsymbol{F}}\|_{L^2} \leqslant C(\|\tilde{\boldsymbol{v}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{F}}\|_{L^2}^2) + \\ &\quad \frac{\kappa}{2}\|\nabla\widetilde{\boldsymbol{F}}\|_{L^2} + \frac{\mu}{6}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2}. \qquad (35)\end{aligned}$$

Similarly, we can use the same conditions to show that the fourth equation of system (2) is equivalent to the case of $\varepsilon \to 0$ in system (8). Next, we use $(\boldsymbol{v}_1, \boldsymbol{F}_1, \boldsymbol{M}_1)$ and $(\boldsymbol{v}_2, \boldsymbol{F}_2, \boldsymbol{M}_2)$ as two solutions of system (8), we have

$$\begin{aligned}\partial_t\widetilde{\boldsymbol{M}} =& -(\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2) - \alpha(\boldsymbol{M}_1\times\Delta\boldsymbol{M}_1 - \\ & \Delta\boldsymbol{M}_2\times\boldsymbol{M}_2) + \beta(\Delta\boldsymbol{M}_1 - \Delta\boldsymbol{M}_2). \qquad (36)\end{aligned}$$

Multiplying Eq. (36) by $\widetilde{M}$ and then integrate over $\mathbb{R}^d$ with respect to x.

$$\begin{aligned}&\langle\partial_t\widetilde{\boldsymbol{M}},\widetilde{\boldsymbol{M}}\rangle + \langle\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ &\quad\langle\alpha(\boldsymbol{M}_1\times\Delta\boldsymbol{M}_1 - \boldsymbol{M}_2\times\Delta\boldsymbol{M}_2),\widetilde{\boldsymbol{M}}\rangle + \\ &\quad\beta\langle\Delta\boldsymbol{M}_1 - \Delta\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle + \\ &\quad\beta\langle|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - |\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle.\end{aligned}$$

We can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\boldsymbol{M}}\|_{L_2}^2 + \beta\|\Delta\widetilde{\boldsymbol{M}}\|_{L_2}^2 = \\ &\quad -\langle\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle - \\ &\quad \langle\alpha(\boldsymbol{M}_1\times\Delta\boldsymbol{M}_1 - \boldsymbol{M}_2\times\Delta\boldsymbol{M}_2),\widetilde{\boldsymbol{M}}\rangle + \\ &\quad \beta\langle|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - |\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ &\quad II_6 + II_7 + II_8. \qquad (37)\end{aligned}$$

Next, we will estimate each term on the right-hand side of the above equation. By Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}II_6 =& -\langle\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ & -\langle\boldsymbol{v}_1\nabla\widetilde{\boldsymbol{M}} + \tilde{\boldsymbol{v}}\nabla\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle \leqslant (\|\boldsymbol{v}_1\|_{L^\infty}\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2} + \\ & \|\tilde{\boldsymbol{v}}\|_{L^2}\|\nabla\boldsymbol{M}_2\|_{L^\infty})\|\widetilde{\boldsymbol{M}}\|_{L^2} \leqslant \\ & C(\|\boldsymbol{v}_1\|_{H^2}\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2} + \\ & \|\tilde{\boldsymbol{v}}\|_{L^2}\|\nabla\boldsymbol{M}_2\|_{H^2})\|\widetilde{\boldsymbol{M}}\|_{L^2}.\end{aligned}$$

Next, we will estimate each term on the right-hand side of the above equation

$$\begin{aligned}II_7 =& -\langle\alpha(\boldsymbol{M}_1\times\Delta\boldsymbol{M}_1 - \boldsymbol{M}_2\times\Delta\boldsymbol{M}_2),\widetilde{\boldsymbol{M}}\rangle = \\ & -\alpha\langle\boldsymbol{M}_1\times\Delta\boldsymbol{M}_1 - \boldsymbol{M}_1\times\Delta\boldsymbol{M}_2 + \boldsymbol{M}_1\times \\ & \Delta\boldsymbol{M}_2 - \boldsymbol{M}_2\times\Delta\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ & -\alpha\langle\boldsymbol{M}_1\times\Delta\widetilde{\boldsymbol{M}} + \widetilde{\boldsymbol{M}}\times\Delta\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ & \alpha\langle\nabla\boldsymbol{M}_1\times\nabla\widetilde{\boldsymbol{M}},\widetilde{\boldsymbol{M}}\rangle + \alpha\langle\boldsymbol{M}_1\times\nabla\widetilde{\boldsymbol{M}},\nabla\widetilde{\boldsymbol{M}}\rangle \leqslant \\ & C\|\nabla\boldsymbol{M}_1\|_{L^\infty}\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}\|\widetilde{\boldsymbol{M}}\|_{L^2},\end{aligned}$$

$$\begin{aligned}II_8 =& \beta\langle|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - |\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ & -\beta\langle(|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - |\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_2 + \\ & |\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_2 - |\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2),\widetilde{\boldsymbol{M}}\rangle = \\ & \beta\langle|\nabla\boldsymbol{M}_1|^2\widetilde{\boldsymbol{M}} + (|\nabla\boldsymbol{M}_1|^2 - |\nabla\boldsymbol{M}_2|^2)\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle = \\ & \beta\langle|\nabla\boldsymbol{M}_1|^2\widetilde{\boldsymbol{M}} + (|\nabla\boldsymbol{M}_1| - |\nabla\boldsymbol{M}_2|)(|\nabla\boldsymbol{M}_1| + \\ & |\nabla\boldsymbol{M}_2|)\boldsymbol{M}_2,\widetilde{\boldsymbol{M}}\rangle \leqslant \beta\|\nabla\boldsymbol{M}_1\|_{L^\infty}^2\|\widetilde{\boldsymbol{M}}\|_{L^2}^2 + \\ & \beta\|\nabla\boldsymbol{M}_1 - \nabla\boldsymbol{M}_2\|_{L^2}(\|\nabla\boldsymbol{M}_1\|_{L^\infty} + \|\nabla\boldsymbol{M}_2\|_{L^\infty})\cdot \\ & \|\boldsymbol{M}_2\|_{L^\infty}\|\widetilde{\boldsymbol{M}}\|_{L^2} \leqslant C\|\nabla\boldsymbol{M}_1\|_{H^2}^2\|\widetilde{\boldsymbol{M}}\|_{L^2}^2 + \\ & C\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2}(\|\nabla\boldsymbol{M}_1\|_{H^2} + \|\nabla\boldsymbol{M}_2\|_{H^2})\|\widetilde{\boldsymbol{M}}\|_{L^2}.\end{aligned}$$

Differentiating Eq. (36) with ∇, multiplying by $\nabla\widetilde{\boldsymbol{M}}$ and then integrate over $\mathbb{R}^d$ with respect to x.

$$\begin{aligned}&\langle\nabla\partial_t\widetilde{\boldsymbol{M}},\nabla\widetilde{\boldsymbol{M}}\rangle = -\langle\nabla(\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle - \\ &\quad\langle\alpha\nabla(\boldsymbol{M}_1\times\boldsymbol{M}_1 - \boldsymbol{M}_2\times\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle + \\ &\quad\beta\langle\nabla\Delta\widetilde{\boldsymbol{M}},\nabla\widetilde{\boldsymbol{M}}\rangle + \beta\langle\nabla(|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - \\ &\quad|\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle,\end{aligned}$$

We can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\boldsymbol{M}}\|_{L_2}^2 + \beta\|\Delta\widetilde{\boldsymbol{M}}\|_{L_2}^2 = \\ &\quad -\langle\nabla(\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle - \\ &\quad \langle\alpha\nabla(\boldsymbol{M}_1\times\boldsymbol{M}_1 - \boldsymbol{M}_2\times\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle + \\ &\quad \beta\langle\nabla(|\nabla\boldsymbol{M}_1|^2\boldsymbol{M}_1 - |\nabla\boldsymbol{M}_2|^2\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle = \\ &\quad III_1 + III_2 + III_3. \qquad (38)\end{aligned}$$

Next, we will estimate each term on the right-hand side of the above equation. By Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}III_1 =& -\langle\nabla(\boldsymbol{v}_1\nabla\boldsymbol{M}_1 - \boldsymbol{v}_2\nabla\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle = \\ & -\langle\nabla(\boldsymbol{v}_1\nabla\widetilde{\boldsymbol{M}} + \tilde{\boldsymbol{v}}\nabla\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle = \\ & -\langle\nabla\boldsymbol{v}_1\cdot\nabla\widetilde{\boldsymbol{M}} + \boldsymbol{v}_1\cdot\nabla\nabla\widetilde{\boldsymbol{M}} + \\ & \nabla\tilde{\boldsymbol{v}}\nabla\boldsymbol{M}_2 + \tilde{\boldsymbol{v}}\nabla\nabla\boldsymbol{M}_2),\nabla\widetilde{\boldsymbol{M}}\rangle \leqslant \\ & (\|\nabla\boldsymbol{v}_1\|_{L^\infty}\|\nabla\widetilde{\boldsymbol{M}}\|_{L^2} + \|\nabla\tilde{\boldsymbol{v}}\|_{L^2}\|\nabla\boldsymbol{M}_2\|_{L^\infty} + \\ & \|\tilde{v}\|_{L^2}\|\nabla^2\boldsymbol{M}_2\|_{L^\infty})\cdot\end{aligned}$$

$$
\begin{aligned}
&\| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} + \langle \boldsymbol{v}_1 , \frac{1}{2} \nabla | \nabla \widetilde{\boldsymbol{M}} |^2 \rangle \leqslant \\
&(\| \nabla \boldsymbol{v}_1 \|_{H^2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} + \| \nabla \tilde{\boldsymbol{v}} \|_{L^2} \| \nabla \boldsymbol{M}_2 \|_{H^2} + \\
&\| \tilde{\boldsymbol{v}} \|_{L^2} \| \nabla^2 \boldsymbol{M}_2 \|_{H^2}) \cdot \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}.
\end{aligned}
$$

Next, we will estimate each term on the right-hand side of the above equation

$$
\begin{aligned}
III_2 = &-\langle \alpha \nabla (\boldsymbol{M}_1 \times \Delta \boldsymbol{M}_1 - \boldsymbol{M}_2 \times \Delta \boldsymbol{M}_2) , \nabla \widetilde{\boldsymbol{M}} \rangle = \\
&-\alpha \langle \nabla (\boldsymbol{M}_1 \times \Delta \boldsymbol{M}_1 - \boldsymbol{M}_1 \times \Delta \boldsymbol{M}_2 + \boldsymbol{M}_1 \times \Delta \boldsymbol{M}_2 - \\
&\boldsymbol{M}_2 \times \Delta \boldsymbol{M}_2) , \nabla \widetilde{\boldsymbol{M}} \rangle = \\
&-\alpha \langle \nabla (\boldsymbol{M}_1 \times \Delta \widetilde{\boldsymbol{M}} + \widetilde{\boldsymbol{M}} \times \Delta \boldsymbol{M}_2) , \nabla \widetilde{\boldsymbol{M}} \rangle = \\
&\alpha \langle (\boldsymbol{M}_1 \times \Delta \widetilde{\boldsymbol{M}} , \Delta \widetilde{\boldsymbol{M}} \rangle + \\
&\alpha \langle \nabla \widetilde{\boldsymbol{M}} \times \Delta \boldsymbol{M}_2 + \widetilde{\boldsymbol{M}} \times \nabla \Delta \boldsymbol{M}_2 , \nabla \widetilde{\boldsymbol{M}} \rangle \leqslant \\
&\alpha \| \widetilde{\boldsymbol{M}} \|_{L^2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} \| D^3 \boldsymbol{M}_2 \|_{L^\infty} \leqslant \\
&C \| \widetilde{\boldsymbol{M}} \|_{L^2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} \| D^3 \boldsymbol{M}_2 \|_{H^2} ,
\end{aligned}
$$

$$
\begin{aligned}
III_3 = &\beta \langle \nabla (| \nabla \boldsymbol{M}_1 |^2 \boldsymbol{M}_1 - | \nabla \boldsymbol{M}_2 |^2 \boldsymbol{M}_2) , \nabla \widetilde{\boldsymbol{M}} \rangle = \\
&\beta \langle \nabla (| \nabla \boldsymbol{M}_1 |^2 \widetilde{\boldsymbol{M}} + \\
&(| \nabla \boldsymbol{M}_1 |^2 - | \nabla \boldsymbol{M}_2 |^2)) \boldsymbol{M}_2 , \nabla \widetilde{\boldsymbol{M}} \rangle = \\
&\beta \langle 2 \nabla \boldsymbol{M}_1 D^2 \boldsymbol{M}_1 \widetilde{\boldsymbol{M}} + | \nabla \boldsymbol{M}_1 |^2 \nabla \widetilde{\boldsymbol{M}} , \nabla \widetilde{\boldsymbol{M}} \rangle + \\
&(| \nabla \boldsymbol{M}_1 | - | \nabla \boldsymbol{M}_2 |) \cdot \\
&(| \nabla \boldsymbol{M}_1 | + | \nabla \boldsymbol{M}_2 |) \boldsymbol{M}_2) , \Delta \widetilde{\boldsymbol{M}} \rangle \leqslant \\
&2\beta \| \nabla \boldsymbol{M}_1 \|_{L^\infty} \| D^2 \boldsymbol{M}_1 \|_{L^\infty} \| \widetilde{\boldsymbol{M}} \|_{L^2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} + \\
&\beta \| \nabla \boldsymbol{M}_1 \|_{L^\infty}^2 \cdot \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}^2 + \\
&\beta \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} (\| \nabla \boldsymbol{M}_1 \|_{L^\infty} + \\
&\| \nabla \boldsymbol{M}_2 \|_{L^\infty}) \| \boldsymbol{M}_2 \|_{L^\infty} \| \Delta \widetilde{\boldsymbol{M}} \|_{L^2} \leqslant \\
&C (\| \nabla \boldsymbol{M}_1 \|_{H^2} \| D^2 \boldsymbol{M}_1 \|_{H^2} \| \widetilde{\boldsymbol{M}} \|_{L^2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} + \\
&\| \nabla \boldsymbol{M}_1 \|_{H^2}^2 \cdot \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}^2 + \\
&\| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} (\| \nabla \boldsymbol{M}_1 \|_{H^2} + \| \nabla \boldsymbol{M}_2 \|_{H^2}) \| \Delta \widetilde{\boldsymbol{M}} \|_{L^2}).
\end{aligned}
$$

Using above estimates and Eqs. (37) and (38), we can obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\| \widetilde{\boldsymbol{M}} \|_{L_2}^2 + \| \nabla \widetilde{\boldsymbol{M}} \|_{L_2}^2) + \\
&\beta (\| \nabla \widetilde{\boldsymbol{M}} \|_{L_2}^2 + \| \Delta \widetilde{\boldsymbol{M}} \|_{L_2}^2) \leqslant \\
&C \| \widetilde{\boldsymbol{M}} \|_{L^2} (\| \tilde{\boldsymbol{v}} \|_{L^2} + \| \widetilde{\boldsymbol{M}} \|_{L^2} + \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}) + \\
&C (\| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} + \| \nabla \tilde{\boldsymbol{v}} \|_{L^2} + \\
&\| \tilde{\boldsymbol{v}} \|_{L^2} + \| \Delta \widetilde{\boldsymbol{M}} \|_{L^2}) \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2} \leqslant \\
&C (\| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}^2 + \| \widetilde{\boldsymbol{M}} \|_{L^2}^2 + \| \tilde{\boldsymbol{v}} \|_{L^2}^2) + \\
&\frac{\mu}{6} \| \nabla \tilde{\boldsymbol{v}} \|_{L^2}^2 + \frac{\beta}{2} \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}^2 + \frac{\beta}{2} \| \Delta \widetilde{\boldsymbol{M}} \|_{L^2}^2 . \qquad (39)
\end{aligned}
$$

Using Eqs. (32), (35) and (39), we can obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} (\| \tilde{\boldsymbol{v}} \|_{L_2}^2 + \| \widetilde{\boldsymbol{F}} \|_{L_2}^2 + \| \widetilde{\boldsymbol{M}} \|_{L_2}^2 + \| \nabla \widetilde{\boldsymbol{M}} \|_{L_2}^2) + \\
&\mu \| \nabla \tilde{\boldsymbol{v}} \|_{L_2}^2 + \kappa \| \nabla \widetilde{\boldsymbol{F}} \|_{L_2}^2 + \\
&\beta (\| \nabla \widetilde{\boldsymbol{M}} \|_{L_2}^2 + \| \Delta \widetilde{\boldsymbol{M}} \|_{L_2}^2) \leqslant \\
&C (\| \tilde{\boldsymbol{v}} \|_{L^2}^2 + \| \widetilde{\boldsymbol{F}} \|_{L^2}^2 + \| \widetilde{\boldsymbol{M}} \|_{L_2}^2 + \| \nabla \widetilde{\boldsymbol{M}} \|_{L^2}^2).
\end{aligned}
$$

Using $(\tilde{\boldsymbol{v}} , \widetilde{\boldsymbol{F}} , \widetilde{\boldsymbol{M}}) |_{t=0} = 0$ and the Gronwall's lemma, we can obtain

$$
\tilde{\boldsymbol{v}} = 0 , \widetilde{\boldsymbol{F}} = 0 , \widetilde{\boldsymbol{M}} = 0
$$

on an interval $[0, T]$.

Therefore, we have completed the proof of uniqueness.

5 Conclusion and prospect

In the third section, we begin by presenting the perturbation system of the evolutionary model for magnetoviscoelasticity. This system is used to approximate the solution and establish a uniform estimate of the model on $\mathbb{T}^2$ and $\mathbb{T}^3$. We then employ the extension theorem to obtain a uniform estimate of the model on $\mathbb{R}^2$ and $\mathbb{R}^3$, and obtain the local existence of the solution. In the fourth section, the uniqueness of the solution is demonstrated by using Gronwall's inequality.

We prove that the necessary condition for the existence and uniqueness of the solution is $| \boldsymbol{M}_0 | = 1$. Without this condition, we cannot obtain a uniform energy estimate of $\nabla^2 \boldsymbol{M}$ in H^n. The conclusions of this paper are helpful in establishing both the low-order and high-order L^2 decay estimates of the model.

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