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折叠点下曲线的相交重数

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摘　要：解析几何中最重要的基本问题之一是求两条代数曲线的交点数。Bézout 定理表明，m 次代数曲线和 n 次代数曲线有 mn 个交点计算重数，除非它们有共同的分量，否则交点个数不超过 mn。在局部情况下，Liang 分别在$\mathbb{R}^2$和 $P_{\mathbb{R}}^2$ 中介绍了两条代数曲线在一点的相交重数。由于相交重数与折叠点有密切联系，而线性变换(或射影变换)又不改变$\mathbb{R}^2$(或 $P_{\mathbb{R}}^2$)中曲线的相交重数，所以探讨变换后折叠点重数的变化规律具有一定的研究意义。文章分别研究了$\mathbb{R}^2$和 $P_{\mathbb{R}}^2$ 中曲线的相交重数，利用$\mathbb{R}^2$(或 $P_{\mathbb{R}}^2$)中的线性变换(或射影变换)给出曲线在一点相交重数变换关系的等价性。

关键词：代数曲线；相交重数；折叠点

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The intersection multiplicity of curves under the fold point

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Abstract: In analytic geometry, one of the most important fundamental problems is to find the number of intersection points of two algebraic curves. Bézout's theorem states that two algebraic curves of degrees m and n intersect at mn points, counting multiplicities, and cannot meet at more than mn points unless they have a component in common. In the local case, Liang introduced the intersection multiplicity of two algebraic curves at some point in $\mathbb{R}^2$ and $P_{\mathbb{R}}^2$ respectively. Since the intersection multiplicity is closely related to the fold point, and linear transformation (or projective transformation) preserves the intersection multiplicity of curves in $\mathbb{R}^2$ (resp. $P_{\mathbb{R}}^2$), it is of certain research significance to discuss the change rule of the multiplicity of the fold point after transformation. In this paper, we study the intersection multiplicity of curves at a point in $\mathbb{R}^2$ and $P_{\mathbb{R}}^2$, respectively. We give the equivalence of transformation relation of the intersection multiplicity of curves at a point by linear transformation (resp. projective transformation) in $\mathbb{R}^2$ (resp. $P_{\mathbb{R}}^2$).

Key words: algebraic curve; intersection multiplicity; the fold point

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Analytic geometry or Cartesian geometry is important in algebra. It establishes the correspondence between the algebraic equations and the geometric curves, for example, an algebraic curve is the graph of a polynomial equation in two variables x and y in the real plane $\mathbb{R}^2$ (resp. the projective plane $P^2_{\mathbb{R}}$). In analytic geometry, one of the most important fundamental problems is to find the number of intersection points of two algebraic curves. Bézout's theorem states that two algebraic curves of degrees m and n intersect in mn points counting multiplicities and cannot meet in more than mn points when they have no component in common[1]. Work on analyzing the local properties of a curve at some point by calculus the intersection multiplicity of curves. The first question is how to find the intersection of the curves. Hilmar used Euclid's algorithm for polynomials to find the points of intersection of two algebraic plane curves[2]. The question that follows is what is the property of multiplicities of curves at intersecting points. Walker[3] proved that if P is a point of multiplicity r and multiplicity s of curves $F=0$ and $G=0$, respectively, then F and G intersect at P at least rs times, and exactly rs times when the curves F and G do not have common tangents at point P. Avagyan proved the same result as the above by means of operators with partial derivatives[4-5]. Liang[6] introduced the similar result by means of the fold point in $P^2_{\mathbb{R}}$. In this paper, we consider the intersection multiplicity of curves at some point in $\mathbb{R}^2$ and $P^2_{\mathbb{R}}$, respectively, and we mainly give the equivalence of transformation relation of intersection multiplicity of curves at some point by linear transformation (resp. projective transformation) and the fold point in $\mathbb{R}^2$ (resp. $P^2_{\mathbb{R}}$).

Let $f(x,y)=0$ and $g(x,y)=0$ be two algebraic curves in $\mathbb{R}^2$, we denote the intersection multiplicity of f and g at point p by $I_p(f,g)$, which is the number of times that the curves $f(x,y)=0$ and $g(x,y)=0$ intersect at point p[7]. We have a similar definition of intersection multiplicity of curves at some point in $P^2_{\mathbb{R}}$. There are also some other definitions of the intersection multiplicity of algebraic curves at some point in (cf. [3, 8-12]). We can connect the intersection multiplicity in $\mathbb{R}^2$ with the intersection multiplicity in $P^2_{\mathbb{R}}$ by homogenizing polynomials[6-7]. Using the factorization theorem of polynomials, we notice that the intersection multiplicity of two curves at a point is closely related to the fold point. According to the definition of the fold point and the property of linear transformation which preserves the intersection multiplicity of curves in $\mathbb{R}^2$, we give the equivalence of transformation relation of intersection multiplicity of curves at a point by linear transformation. We can make similar conclusions by projective transformation in $P^2_{\mathbb{R}}$.

This paper is organized as follows. In Section 1, we introduce some properties of multiplicity of curves at the intersection points and the fold point of curves in $\mathbb{R}^2$. Then we give the equivalence of transformation relation of intersection multiplicity of curves at a point by linear transformation in $\mathbb{R}^2$. In Section 2, as a generalization, we use projective transformation and the fold point to obtain similar conclusions as Section 2 in $P^2_{\mathbb{R}}$.

1 Intersection multiplicity of curves under the fold point in $\mathbb{R}^2$

In this section, we introduce some properties of the intersection multiplicity of algebraic curves at a point in $\mathbb{R}^2$. When we study the intersection multiplicity of curves, we will introduce the fold point of curves and linear transformation which is a linear change of coordinates while preserving the intersection multiplicity of curves.

Let $f(x,y)=0$ and $g(x,y)=0$ be algebraic curves (abbreviate to curves) which intersect at a point p in $\mathbb{R}^2$. The definition of intersection multiplicity of f and g at p is the number of times that the curves $f=0$ and $g=0$ intersect at the point p, denoted by $I_p(f,g)$.

Proposition 1[7] Let $f(x,y)=0$, $g(x,y)=0$ and $h(x,y)=0$ be curves and p a point in $\mathbb{R}^2$. Then

(i) $I_p(f,g)$ is a nonnegative integer or ∞, and $I_p(f,g)=I_p(g,f)$.

(ii) $I_p(f,g)\geqslant 1$ if and only if $f(p)=0$ and $g(p)=0$.

(iii) $I_p(f,g)=I_p(f,g+fh)$.

(iv) $I_p(f,gh)=I_p(f,g)+I_p(f,h)$, and $I_p(f,gh)=I_p(f,g)$ if $h(p)\neq 0$.

In linear geometry, linear transformations play a central role. A linear transformation of $\mathbb{R}^2$ is a map $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$(x,y)^{\mathrm{T}}\to A(x,y)^{\mathrm{T}},$$

where $A\in GL(2,R)$ is an invertible 2×2 matrix, and T is a bijection which maps points to points. Following Ref. [6], we can know that linear transformations are linear changes of coordinates and preserve the intersection multiplicity of curves, that is, for any curve $f(x,y)=0$ and $g(x,y)=0$ in $\mathbb{R}^2$, which intersect at point p and satisfy that

$$I_p(f,g)=I_{p'}(f',g'), \tag{1}$$

where $T(f)=T(f')$, $T(g)=T(g')$ and $T(p)=T(p')$.

Let $f(x,y)$ be a nonzero polynomial and $f_d(x,y)$ be the sum of the terms of degree d in $f(x,y)$. Therefore, we can write

$$f_d(x,y)=(a_1x+b_1y)^{s_1}(a_2x+b_2y)^{s_2}\cdots(a_jx+b_jy)^{s_j}r(x,y) \tag{2}$$

for distinct lines $a_ix+b_iy=0$ uniquely, where s_i are non-negative integers and $r(x,y)$ is a polynomial which has no linear factors. Thus, we can write

$$f=f_d+f_{d+1}+\cdots+f_n. \tag{3}$$

We can start by talking about the intersection multiplicity of a curve $f(x,y)=0$ and a line $l(x,y)=0$ at point p. From Proposition 1, (i) and (iii), we have the following lemma and definition which talks about the fold point.

Lemma 1[6] Let $f(x,y)=0$ be a curve that contains the origin $o=(0,0)\in\mathbb{R}^2$, and d is the smallest degree of the terms in f. Assume that $l=0$ is a line through the origin o. Then $I_o(l,f)>d$ if l is a factor of f_d and $I_o(l,f)=d$ if l is not a factor of f_d.

Definition 1[7] Let $f(x,y)=0$ be a curve and p a point in $\mathbb{R}^2$. We say that the point p is a d-fold point of $f=0$ if there is a non-negative integer d, such that there are at most d distinct lines intersect f at p more than d times, and all other lines intersect f at p exactly d times.

Lemma 2 Let $f(x,y)=0$ be a curve and p a point in $\mathbb{R}^2$. Assume that p is a d-fold point of $f=0$. Then $T(p)$ is a d-fold point of $T(f)$ for any linear transformation T.

Proof Assume that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation. We know that there exist infinite lines $\daleth:=\{l_i=0,i=1,2,\cdots\}$ through the point p. Since the point p is a d-fold point of $f(x,y)=0$ in $\mathbb{R}^2$, there are at most d distinct lines $\daleth':=\{l_i'=0,i\in\{1,\cdots,d\}\}\subsetneq\daleth$ such that

$$I_p(f,l_i')>d$$

by Definition 1. Thus we have

$$I_p(f,l_i')=d$$

for any $l_i\in\daleth\setminus\daleth'$. Since linear transformations preserve intersection multiplicities by Eq. (1), it follows that

$$I_{T(p)}(T(f),T(l_i'))>d$$

and

$$I_{T(p)}(T(f),T(l_i))=d,$$

where $T(l_i')\in T(\daleth')\subsetneq T(\daleth)$ and $T(l_i)\in T(\daleth)\setminus T(\daleth')$. It means that there are at most d distinct lines intersect $T(f)$ at $T(p)$ more than d times, and all other lines intersect $T(f)$ at $T(p)$ exactly d times. Hence, $T(p)$ is a d-fold point of $T(f)$ for any linear transformation.

Theorem 1 Let $f(x,y)=0$ and $g(x,y)=0$ be curves and p a point in $\mathbb{R}^2$. Assume that p is a d-fold point of $f=0$ and an e-fold point of $g=0$. Then $T(p)$ is a $(d+e)$-fold point of $T(fg)=0$ for any linear transformation T.

Proof Assume that T is a linear transformation. It is obvious that there exist infinite lines $\aleph:=\{T(l_i)=0,i=1,2,\cdots\}$ through the point p. Since p

is a d-fold point of $f=0$, by Lemma 2, we have that $T(p)$ is a d-fold point of $T(f)=0$. Therefore, there are at most d distinct lines $\aleph'_1:=\{T(l'_i)=0, i\in\{1,\cdots,d\}\}\subsetneq\aleph$ such that

$$I_{T(p)}(T(f),T(l'_i))>d. \quad (4)$$

In addition, we have

$$I_{T(p)}(T(f),T(l'_j))=d \quad (5)$$

by Definition 1, where $T(l'_j)\in\aleph\setminus\aleph'_1$. Similarly, since p is an e-fold point of $g=0$, we have that $T(p)$ is an e-fold point of $T(g)=0$ by Lemma 2. It follows that there are at most e distinct lines $\aleph'_2:=\{T(l''_i)=0, i\in\{1,\cdots,e\}\}\subsetneq\aleph$ such that

$$I_{T(p)}(T(f),T(l''_i))>e. \quad (6)$$

Furthermore, we have

$$I_{T(p)}(T(f),T(l''_j))=e \quad (7)$$

for any $T(l''_j)\in\aleph\setminus\aleph'_2$. It is easy to know that there are at most $d+e$ elements in $\aleph'_1\cup\aleph'_2$, by Proposition 1, Eqs. (4) and (6), we have

$$I_{T(p)}(T(fg),T(\zeta))=I_{T(p)}(T(f),T(\zeta))+I_{T(p)}(T(g),T(\zeta))>d+e,$$

where $T(\zeta)\in\aleph'_1\cup\aleph'_2$. That is, there are at most $d+e$ distinct lines which intersect $T(fg)$ at point $T(p)$ more that $d+e$ times. Since the number of elements in $\aleph$is infinite, and there are at most $d+e$ elements in $\aleph'_1\cup\aleph'_2$, it follows $\aleph\setminus(\aleph'_1\cup\aleph'_2)$ must not be an empty set. Therefore, by Proposition 1, Eqs. (5) and (7), we obtain that

$$I_{T(p)}(T(fg),T(\zeta'))=I_{T(p)}(T(f),T(\zeta'))+I_{T(p)}(T(g),T(\zeta'))=d+e$$

for any line $T(\zeta')\in\aleph\setminus\aleph'_1\cup\aleph'_2$, that is, except for the lines in $\aleph'_1\cup\aleph'_2$ all intersect $T(fg)$ at point $T(p)$ $d+e$ times. Thus, when $T(p)$ is a d-fold point of $T(f)=0$ and an e-fold point of $T(g)=0$, then $T(p)$ is a $d+e$-fold point of $T(fg)=0$ for any linear transformation.

Corollary 1 Let $g_1(x,y)=0,\cdots,g_n(x,y)=0$ be curves and p a point in $\mathbb{R}^2$. Assume that p is an a_i-fold point of $g_i=0$ for $1\leqslant i\leqslant n$. Then $T(p)$ is a $\sum_{i=1}^{n}a_i$ -fold point of $T(\prod_{i=1}^{n}g_i)$.

2 Intersection multiplicity of curves under the fold point in $P^2_{\mathbb{R}}$

We can extend the intersection multiplicity of curves from $\mathbb{R}^2$ to $P^2_{\mathbb{R}}$ by homogenizing polynomials. We know that $P^2_{\mathbb{R}}$ is $(R^3-\{(0,0,0)\})/\sim$, where $\sim$ is the equivalence relation defined by $(x,y,z)\sim(x',y',z')$ if there exists a nonzero $\lambda\in\mathbb{R}$, such that $(x,y,z)=(\lambda x',\lambda y',\lambda z')$. For any homogeneous polynomial $F(x,y,z)$ in $P^2_{\mathbb{R}}$, suppose $f(x,y)=F(x,y,1)$, then a point (x,y) of $\mathbb{R}^2$ which lies on the curve $f(x,y)=0$ if and only if the point $(x,y,1)$ of $P^2_{\mathbb{R}}$ lies on the curve $F(x,y,z)=0$. In this section, we introduce some properties of the intersection multiplicity of algebraic curves at a point in $P^2_{\mathbb{R}}$. When we study the intersection multiplicity of curves at a point, we will introduce the fold point of curves and projective transformation which is a linear change of coordinates while preserving the intersection multiplicity of curves in $P^2_{\mathbb{R}}$.

Let $F(x,y,z)=0$ and $G(x,y,z)=0$ be curves in $P^2_{\mathbb{R}}$. Similar as the definition of $I_p(f,g)$ in $\mathbb{R}^2$, we can give the intersection multiplicity of curves $F(x,y,z)=0$ and $G(x,y,z)=0$ at a point $P\in P^2_{\mathbb{R}}$, that is, the number of times that curves $F(x,y,z)=0$ and $G(x,y,z)=0$ intersect at the point P, denoted by $I_P(F,G)$.

Lemma 3[7] Let $F(x,y,z)=0$ and $G(x,y,z)=0$ be curves in $P^2_{\mathbb{R}}$, and set $f(x,y)=F(x,y,1)$, $g(x,y)=G(x,y,1)$. Then for any point $(a,b,1)\in P^2_{\mathbb{R}}$, we have

$$I_{(a,b,1)}(F(x,y,z),G(x,y,z))=I_{(a,b)}(f(x,y),g(x,y)).$$

Similar as Proposition 1, we can give the following properties of $I_P(F,G)$.

Proposition 2[7] Let $F(x,y,z)=0$, $G(x,y,z)=0$ and $H(x,y,z)=0$ be curves and P a point in $P^2_{\mathbb{R}}$. Then

(ⅰ) $I_P(F,G)$ is a nonnegative integer or ∞, and $I_P(F,G)=I_P(G,F)$.

(ii) $I_P(F,G) \geqslant 1$ if and only if $F(P)=0$ and $G(P)=0$.

(iii) $I_P(F,G)=I_P(F,G+FH)$ if $G+FH$ is homogeneous.

(iv) $I_P(F,GH)=I_P(F,G)+I_P(F,H)$, and $I_P(F,GH)=I_P(F,G)$ if $H(P)\neq 0$.

A projective transformation is a linear map $T:P_{\mathbb{R}}^2 \to P_{\mathbb{R}}^2$ defined by

$$(x,y,z)^{\mathrm{T}} \to A(x,y,z)^{\mathrm{T}},$$

where $A \in GL(3,\mathbb{R})$ is an invertible 3×3 matrix. By Ref. [7], we know that projective transformations preserve intersection multiplicities given in the following.

Lemma 4[7] Let $F(x,y,z)=0$ and $G(x,y,z)=0$ be curves and P a point in $P_{\mathbb{R}}^2$, T is a projective transformation that maps $P(x,y,z)$ (resp. $F=0$, $G=0$) to $P'(x',y',z')$ (resp. $F'=0$, $G'=0$). Then

$$I_P(F(x,y,z),G(x,y,z))=$$
$$I_{P'}(F'(x',y',z'),G'(x',y',z')).$$

By Lemma 1 and Lemma 3, we have the following definition.

Definition 2[6] Let $F(x,y,z)=0$ be a curve and P a point in $P_{\mathbb{R}}^2$. We say that the point P is a d-fold point of $F=0$ if there is a non-negative integer d, such that there are at most d distinct lines which intersect F at P more than d times, and all other lines intersect F at P exactly d times.

Lemma 5 Let $F(x,y,z)=0$ be a curve and P a point in $P_{\mathbb{R}}^2$. If P is a d-fold point of $F=0$, then $T(P)$ is a d-fold point of $T(F)$ for any projective transformation T.

Proof Let $\daleth := \{L_i=0, i=1,2,\cdots\}$ be the set of lines through the point P and T is any projective transformation. Suppose that the point P is a d-fold point of $F(x,y,z)=0$ in $P_{\mathbb{R}}^2$, by Definition 2, there are at most d distinct lines $\daleth' := \{L_i'=0, i\in\{1,\cdots,d\}\} \subsetneq \daleth$ such that

$$I_P(F,L_i')>d$$

and

$$I_P(F,L_i)=d$$

for any $L_i \in \daleth\setminus\daleth'$. Since projective transformations preserve intersection multiplicities by Lemma 4, we have

$$I_{T(P)}(T(F),T(L_i'))>d$$

and

$$I_{T(P)}(T(F),T(L_i))=d,$$

where $T(L_i')\in T(\daleth')\subsetneq T(\daleth)$ and $T(L_i)\in T(\daleth)\setminus T(\daleth')$. It means that there are at most d distinct lines which intersect $T(F)$ at $T(P)$ more than d times, and all other lines intersect $T(F)$ at $T(P)$ exactly d times. Hence, we obtain that $T(P)$ is a d-fold point of $T(F)$ for any projective transformation T.

Theorem 2 Let $F(x,y,z)=0$ and $G(x,y,z)=0$ be curves and P a point in $P_{\mathbb{R}}^2$. If P is a d-fold point of $F=0$ and an e-fold point of $G=0$, respectively, then $T(P)$ is a $(d+e)$-fold point of $T(FG)=0$ for any projective transformation T.

Proof Let T be any projective transformation and $\aleph := \{T(L_i)=0, i=1,2,\cdots\}$ the set of lines through the point $T(P)$, where L_i is the line through the point $P\in P_{\mathbb{R}}^2$. Suppose that P is a d-fold point of $F=0$ and an e-fold point of $G=0$, then $T(P)$ is a d-fold point of $T(F)=0$ and an e-fold point of $T(G)=0$ following Lemma 5. From Definition 2, we know that there are at most d distinct lines $\aleph_1' := \{T(L_i')=0, i\in\{1,\cdots,d\}\}\subsetneq\aleph$ such that

$$I_{T(P)}(T(F),T(L_i'))>d \tag{8}$$

and

$$I_{T(P)}(T(F),T(L_j'))=d \tag{9}$$

for $T(L_j')\in\aleph\setminus\aleph_1'$. Similarly, there are at most e distinct lines

$$\aleph_2' := \{T(L_m')=0, m\in\{1,\cdots,e\}\}\subsetneq\aleph$$

such that

$$I_{T(P)}(T(F),T(L_m'))>e \tag{10}$$

and

$$I_{T(P)}(T(F),T(L_n'))=e \tag{11}$$

for $T(L_n')\in\aleph\setminus\aleph_2'$.

It's easy to know that there are at most $d+e$ elements in $\aleph'_1 \cup \aleph'_2$, by Proposition 2, we have

$$I_{T(P)}(T(FG), T(\zeta)) = I_{T(P)}(T(F), T(\zeta)) + I_{T(P)}(T(G), T(\zeta)) > d + e,$$

where $T(\zeta) \in \aleph'_1 \cup \aleph'_2$. That is, there are at most $d+e$ distinct lines which intersect $T(FG)$ at point $T(P)$ more that $d+e$ times. Since the number of elements in $\aleph$ is infinite, and there are at most $d+e$ elements in $\aleph'_1 \cup \aleph'_2$, it follows that $\aleph \setminus (\aleph'_1 \cup \aleph'_2)$ must not be an empty set. Therefore, by Proposition 2, we obtain that

$$I_{T(P)}(T(FG), T(\zeta')) = I_{T(P)}(T(F), T(\zeta')) + I_{T(P)}(T(G), T(\zeta')) = d + e$$

for any line $T(\zeta') \in \aleph \setminus (\aleph'_1 \cup \aleph'_2)$, that is, except for the lines in $\aleph'_1 \cup \aleph'_2$ all intersect $T(FG)$ at point $T(P)$ $d+e$ times. Thus, when $T(P)$ is a d-fold point of $T(F)=0$ and an e-fold point of $T(G)=0$, then $T(P)$ is a $(d+e)$-fold point of $T(FG)=0$.

Corollary 2 Let $G_1(x,y)=0, \cdots, G_n(x,y)=0$ be curves and P a point in $P^2_{\mathbb{R}}$. If P is a d_i-fold point of $G_i=0$ for $1 \leqslant i \leqslant n$, respectively, then $T(P)$ is a $\sum_{i=1}^{n} d_i$ -fold point of $T(\prod_{i=1}^{n} G_i) = 0$.

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