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四元数值函数小波变换和正交分解

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摘　要：文章研究了四元数值函数小波变换的一些性质. 小波变换是 Hardy 空间(或者共轭 Hardy 空间)到 $L^2(U,H,\frac{dadb}{a^2})$等距同构算子,通过应用特殊函数的性质得到了 $L^2(U,H,\frac{dadb}{a^2})$的正交分解.

关键词：四元数值函数；小波变换；特殊函数；正交分解
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Quaternionic-valued wavelet transform and orthogonal decomposition

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Abstract: In this paper, we study some properties of the quaternionic-valued wavelet transform in one dimension. The wavelet transform gives an isometric operator from the Hardy space (conjugate Hardy space) to $L^2(U,H,\frac{dadb}{a^2})$. In addition, we obtain an orthogonal decomposition of $L^2(U,H,\frac{dadb}{a^2})$ by using the properties for special functions.

Key words: quaternionic-valued function; wavelet transform; special function; orthogonal decomposition
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Grossman, et al.[1] introduced the wavelet transform in the one-dimensional case, where the group G is the affine group of "$ax+b$". It consists of $\{(a,b): a>0, b\in R\}$ with the group law $(a,b)(a_1,b_1)=(aa_1, ab_1+b)$. We know that it is a locally compact nonunimodular group with left Haar measure $d\mu_L(a,b)=\frac{dadb}{a^2}$. Along with this, Jiang, et al.[2] studied admissible wavelets and the decomposition of function space associated with the $SL(2,R)$. They also researched admissible wavelets and the decomposition of function space in more than one dimension[3-4].

The research for the theory of quaternions and quaternionic-valued functions has received considerable attention. A lot of authors studied some properties of quaternions and the theory of quaternionic-valued function space. He, et al.[5-6] established the continuous wavelet transform theory of $L^2(R,H)$ under the affine group. Cheng, et al.[7] obtained the Plancherel theorem of the quaternion Fourier transform of square integrable space. Akila, et al.[8] in-

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troduced a new convolution of quaternionic-valued functions and presented a natural modification of the definition of wavelet transform on the space of square integrable quaternionic-valued functions on R. They got some nice properties of the wavelet transform. This paper, uses this new definition of wavelet transform to discuss some properties of quaternionic-valued wavelet transform in one dimension, such as linearity, scaling, translation and reconstruction formula. At last, we describe an orthogonal decomposition of $L^2(U,H,\frac{dadb}{a^2})$ by using the properties for special functions.

1 Preliminaries

In this section, we first recall fundamental facts of quaternions which can be found in Refs. [9-10]. Let

$$H=\{a+bi+cj+dk: a,b,c,d\in R\},$$

where $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$ and $i^2=j^2=k^2=-1$. For any $q\in H$, it can be written as $q=a+ib+jc+kd=(a+ib)+j(c-id)=u+jv$, then its conjugation is

$$q^C=a-ib-jc-kd=(a-ib)-j(c-id)=\bar{u}-jv.$$

Obviously, $|q|^2=|u|^2+|v|^2$ and $(pq)^C=q^Cp^C$ for $\forall q_1,q_2\in H$,

$$\langle q_1,q_2\rangle_H=q_1q_2^C=(u_1+jv_1)(\bar{u}_2-jv_2)=(u_1\bar{u}_2+\bar{v}_1v_2)+j(v_1\bar{u}_2-\bar{u}_1v_2).$$

Clearly, $\langle\cdot,\cdot\rangle_H$ can be regarded as the inner product on H[5]. Quaternionic-valued function spaces $L^2(R,H)$ are defined by

$$L^2(R,H):=\{f(x)=f_1(x)+jf_2(x),\ f_1(x),f_2(x)\in L^2(R,C)\},$$

where f satisfies

$$\|f\|_{L^2(R,H)}:=\left(\int_R(|f_1(x)|^2+|f_2(x)|^2)dx\right)^{\frac{1}{2}}<+\infty.$$

Let $f(x)=f_1(x)+jf_2(x)$, $g(x)=g_1(x)+jg_2(x)\in L^2(R,H)$. The inner product $\langle\cdot,\cdot\rangle_{L^2(R,H)}$ is defined by $\langle f,g\rangle_{L^2(R,H)}:=\int_R\langle f,g\rangle_H dx=\int_R f(x)(g(x))^C dx=\int_R[f_1(x)\overline{g_1(x)}+\overline{f_2(x)}g_2(x)+j(f_2(x)\overline{g_1(x)}-\overline{f_1(x)}g_2(x))]dx$.

The quaternionic-valued function space $L^P(R,H)$ and $L^\infty(R,H)$ is the set of $f(x)=f_1(x)+jf_2(x)$, where f satisfies

$$\|f\|_{L^P(R,H)}:=\left(\int_R|f(x)|^P dx\right)^{\frac{1}{P}}<+\infty,$$

$$\|f\|_{L^\infty(R,H)}:=esssup_{x\in R}|f(x)|<+\infty.$$

For any $F(x)=f_1(x)+jf_2(x)\in L^2(R,H)$, we define the Fourier transform by

$$\hat{F}(\xi):=\hat{f}_1(\xi)+j\hat{f}_2(\xi),\ \xi\in R,$$

where $\hat{f}_k(\xi):=\int_R f_k(x)e^{-2\pi ix\xi}dx$, $k=1,2$.

The notions $\tilde{f}$ and $\hat{f}$ are used to denote the functions $\tilde{f}=\overline{f_1}(x)-j\check{f}_2(x)$, $\check{f}(x)=\check{f}_1(x)+j\check{f}_2(x)$, $\forall x\in R$, where $\overline{f_k}(x)$ is the complex conjugate of $f_k(x)$, $\check{f}_k(x)=f_k(-x)$, $k=1,2$, and

$$f_a(x)=\frac{1}{\sqrt{a}}f(\frac{x}{a}),\ \forall x\in R,a>0.$$

Now let us introduce natural convolution of quaternionic-valued functions. For any $f,g\in L^2(R,H)$,

$$(f\circledast g)(x):=(f_1*g_1)(x)-(\check{\bar{f}}_2*g_2)(x)+j((\check{\bar{f}}_1*g_2)(x)+f_2*g_1)(x)).$$

We know that it satisfies $(f\circledast g)\hat{}(\xi)=\hat{f}\hat{g}$ from Ref. [8]. By the above definition of quaternion convolution, we can naturally define the wavelet transform on $L^2(R,H)$.

Given an admissible wavelet φ, the admissible wavelet transform is defined by

$$W_\varphi: L^2(R,H)\to L^2(U,H,\frac{dadb}{a^2}),$$

$$f(x)\to[W_\varphi f](a,b)=C_\varphi^{-\frac{1}{2}}(f\circledast\tilde{\check{\varphi}}_a)(b),$$

where $U=\{(a,b):a>0,b\in R\}$

$$L^2(U,H,\frac{dadb}{a^2})=\{f(a,b):\|f\|_{L^2(U,H,\frac{dadb}{a^2})}=\left(\int_U\frac{|f(a,b)|^2}{a^2}dadb\right)^{\frac{1}{2}}<+\infty\}.$$

$[W_\varphi f](a,b)=C_\varphi^{-\frac{1}{2}}(f\circledast\tilde{\check{\varphi}}_a)(b)=$

$$C_{\varphi}^{-\frac{1}{2}}[(f_1 * [\check{\overline{\varphi}}_1]_a)(b) + (\check{\overline{f}}_2 * [\varphi_2]_a)(b) - j((\check{\overline{f}}_1 * [\varphi_2]_a)(b) - (f_2 * [\check{\overline{\varphi}}_1]_a)(b))] = [T_{\varphi_1} f_1](a,b) + [\overline{T_{\varphi_2} f_2}](a,b) + j\{-[\overline{T_{\varphi_2} f_1}](a,b) + [T_{\varphi_1} f_2](a,b)\}.$$

$T_{\varphi} f$ is the continuous wavelet transform of a square integrable complex valued function f on R, this definition comes from Ref. [8].

If $f \in L^2(R,H)$, we use the fact that $\hat{\check{f}} = \overline{\hat{f}}$, then $\hat{\check{f}} = \hat{\check{f}}_1 - j\hat{\check{f}}_2 = \overline{\hat{f}}_1 - j\hat{f}_2 = (\hat{f})^C$.

2 Some properties of quaternionic-valued wavelet transform

In Ref. [8], Akila, et al. obtained Parseval' identity for quaternionic-valued wavelet transform. If $f, g \in L^2(R,H)$,

$$\langle W_{\varphi} f, W_{\varphi} g\rangle_{L^2(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2})} = C_{\varphi}\langle \hat{f}, \hat{g}\rangle_{L^2(R,H)} = \int_R \int_0^{\infty} \hat{f}(\xi) \mid \hat{\varphi}(a\xi) \mid^2 (g(\xi))^C \frac{\mathrm{d}a\mathrm{d}\xi}{a} = \int_0^{\infty} \int_0^{\infty} \hat{f}(\xi) \mid \hat{\varphi}(a_1) \mid^2 (g(\xi))^C \frac{\mathrm{d}a_1\mathrm{d}\xi}{a_1} + \int_{-\infty}^{0} \int_{-\infty}^{0} \hat{f}(\xi) \mid \hat{\varphi}(a_1) \mid^2 (g(\xi))^C \frac{\mathrm{d}a_1\mathrm{d}\xi}{\mid a_1 \mid},$$

which implies that W_{φ} is an isometry from the Hardy space (or conjugate Hardy space) to a subspace of $L^2(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2})$, if and only if

$$C_{\varphi} = \int_0^{\infty} \mid \hat{\varphi}(\xi) \mid^2 \frac{\mathrm{d}\xi}{\xi} = \int_{-\infty}^{0} \mid \hat{\varphi}(\xi) \mid^2 \frac{\mathrm{d}\xi}{\mid \xi \mid} < \infty \tag{1}$$

If wavelet function φ satisfies Eq. (1), we say φ is an admissible wavelet.

Theorem 1　Let $f, g \in L^2(R,H)$, $\varphi, \psi \in L^2(R,H)$ are admissible wavelets. Then, we have

(1) (Linearity)

$[W_{\varphi}(\mu f + \gamma g)](a,b) = \mu[W_{\varphi} f](a,b) + \gamma[W_{\varphi} g](a,b)$,

$[W_{\mu\varphi+\gamma\psi} f](a,b) = [W_{\varphi} f](a,b)\mu^C + [W_{\varphi} f](a,b)\gamma^C$, $\forall \mu, \gamma \in H$.

(2) (Translation)

$[W_{\varphi} f(\cdot - x_0)](a,b) = [W_{\varphi} f](a, b - x_0)$,

$[W_{\sigma_{x_0}\varphi} f](a,b) = [W_{\varphi} f](a, b + a x_0)$, where $\sigma_{x_0}\varphi(x) = \varphi(x - x_0)$.

(3) (Scaling)

$$[W_{\varphi} f(c\cdot)](a,b) = \frac{1}{c^{\frac{1}{2}}}[W_{\varphi} f](ac, bc),$$

$[W_{D^c\varphi} f](a,b) = \frac{1}{c^{\frac{1}{2}}}[W_{\varphi} f](ac, b)$, where $D^c\varphi(x) = \frac{1}{c}\varphi\left(\frac{x}{c}\right), c > 0$.

(4) (Parity)

$[W_{\tilde{\varphi}} f](a,b) = [W_{\varphi} f](a, -b)$, where $\tilde{f}(x) = f(-x)$.

Proof　We only give the proof of (3), and other proofs are analogous.

$$[T_{\varphi_1} f_1(c\cdot)](a,b) = \int_{-\infty}^{\infty} f_1(cx) a^{-\frac{1}{2}} \overline{\varphi_1}\left(\frac{x-b}{a}\right)\mathrm{d}x = \int_{-\infty}^{\infty} f_1(y)(ac)^{-\frac{1}{2}} c^{-\frac{1}{2}} \overline{\varphi_1}\left(\frac{y-bc}{ac}\right)\mathrm{d}y = c^{-\frac{1}{2}}[T_{\varphi_1} f_1](ac, bc),$$

$$[W_{\varphi} f(c\cdot)](a,b) = c^{-\frac{1}{2}}\{[T_{\varphi_1} f_1](ac,bc) + [\overline{T_{\varphi_2} f_2}](ac,bc) - j\{\overline{T_{\varphi_2} f_1}](ac,bc) + j[T_{\varphi_1} f_2](ac,bc)\} = \frac{1}{c^{\frac{1}{2}}}[W_{\varphi} f](ac,bc),$$

$$[T_{D^c\varphi_1} f_1](a,b) = \int_{-\infty}^{\infty} f_1(x) c^{-1} a^{-\frac{1}{2}} \overline{\varphi_1}\left(\frac{x-b}{ac}\right)\mathrm{d}x = \int_{-\infty}^{\infty} f_1(x) c^{-\frac{1}{2}}(ac)^{-\frac{1}{2}} \overline{\varphi_1}\left(\frac{x-b}{ac}\right)\mathrm{d}x = c^{-\frac{1}{2}}[T_{\varphi_1} f_1](ac, b),$$

$$[W_{D^c\varphi} f](a,b) = [T_{D^c\varphi_1} f_1](a,b) + [\overline{T_{D^c\varphi_2} f_2}](a,b) + j\{-[\overline{T_{D^c\varphi_2} f_1}](a,b) + [T_{D^c\varphi_1} f_2](a,b)\} = \frac{1}{c^{\frac{1}{2}}}[W_{\varphi} f](ac,b).$$

Lemma 1[11]　Let $\{a_j\}$ be a sequence of positive reals, $0 \leqslant \theta \leqslant 1$. Then

$$\sum_{j=1}^{N} a_j^{\theta} \leqslant N^{1-\theta}\left(\sum_{j=1}^{N} a_j\right)^{\theta}.$$

Theorem 2　Let $f \in L^2(R,H)$, φ is an admis-

sible wavelet. Then $W_\varphi f(x) \in L^P(U,H,\frac{dadb}{a^2})$, $2 \leqslant P \leqslant +\infty$.

Proof For $P=2$, we have $\| W_\varphi f \|_{L^2(U,H,\frac{dadb}{a^2})} = C_\varphi^{\frac{1}{2}} \| f \|_{L^2(R,H)}$.

For $P=\infty$,

$$|(f \circledast \check{\bar{\varphi}}_a)| = |f_1 * [\check{\bar{\varphi}}_1]_a)(b) + (\check{\bar{f}}_2 * [\varphi_2]_a)(b) - j((\check{\bar{f}}_1 * [\varphi_2]_a)(b) - (f_2 * [\check{\bar{\varphi}}_1]_a)(b)| \leqslant$$
$$\| f_1 \|_2 \| \varphi_1 \|_2 + \| f_2 \|_2 \| \varphi_2 \|_2 + \| f_1 \|_2 \| \varphi_2 \|_2 + \| f_2 \|_2 \| \varphi_1 \|_2 =$$
$$(\| f_1 \|_2 + \| f_2 \|_2)(\| \varphi_1 \|_2 + \| \varphi_2 \|_2) \leqslant 2 \| f \|_{L^2(R,H)} \| \varphi \|_{L^2(R,H)} \text{ (by Lemma 1).}$$

So $\| W_\varphi f \|_{L^\infty(U,H,\frac{dadb}{a^2})} \leqslant 2 \| f \|_{L^2(R,H)} \| \varphi \|_{L^2(R,H)}$.

By Riesz-Thorin interpolation theorem, we obtain that W_φ is of type $(2,P)$, $2 \leqslant P \leqslant +\infty$.

In the following content, we define

$$\Phi = \{\varphi \in L^2(R,H): C_\varphi = \int_0^\infty |\hat{\varphi}(\xi)|^2 \frac{d\xi}{\xi} = \int_{-\infty}^0 |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} = 1\}.$$

Theorem 3 For $\varphi \in \Phi$, let τ_φ be the operator from $L^2(U,H,\frac{dadb}{a^2})$ to $L^2(R,H)$ defined by

$$(\tau_\varphi F)(x) = \int_0^\infty (F(a,\cdot) \circledast \varphi_a)(x) \frac{da}{a^2}.$$

Then τ_φ is bounded, and $\tau_\varphi W_\varphi$ is the identity on $L^2(R,H)$.

Proof For $F \in L^2(U,H,\frac{dadb}{a^2})$, $\xi \in R$, a direct computation gives

$$|(\tau_\varphi F)(x)|^2 = \left| \int_0^\infty (F(a,\cdot) \circledast \varphi_a)(x) \frac{da}{a^2} \right| = \left| \int_0^\infty (F(a,\cdot)\hat{}(\xi) \hat{\varphi}_a(\xi) \frac{da}{a^2} \right|^2 \leqslant$$
$$\int_0^\infty \left| (F(a,\cdot)\hat{}(\xi) \right|^2 \frac{da}{a^2} \int_0^\infty \left| \hat{\varphi}_a(\xi) \right|^2 \frac{da}{a^2} = \int_0^\infty \left| (F(a,\cdot)\hat{}(\xi) \right|^2 \frac{da}{a^2},$$

which implies

$$\| (\tau_\varphi F)(x) \|_{L^2(R,H)} = \| (\tau_\varphi F)\hat{}(\xi) \|_{L^2(R,H)} \leqslant$$
$$(\int_{-\infty}^\infty \int_0^\infty |(F(a,\cdot)\hat{}(\xi)|^2 \frac{da d\xi}{a^2})^{\frac{1}{2}} \leqslant (\int_{-\infty}^\infty \int_0^\infty |F(a,b)|^2 \frac{dadb}{a^2})^{\frac{1}{2}} =$$
$$\| F \|_{L^2(U,H,\frac{dadb}{a^2})}.$$

So we have

$$\| \tau_\varphi \| \leqslant 1.$$

Now for $L^2(R,H)$, we can get

$$(\tau_\varphi W_\varphi f)\hat{}(\xi) = \int_0^\infty (F(a,\cdot) \circledast \varphi_a)\hat{}(\xi) \frac{da}{a^2} = \int_0^\infty (F(a,\cdot)\hat{}(\xi) \hat{\varphi}_a(\xi) \frac{da}{a^2} =$$
$$\int_0^\infty \hat{f}(\xi) \hat{\bar{\varphi}}_a(\xi) \hat{\varphi}_a(\xi) \frac{da}{a^2} = \int_0^\infty \hat{f}(\xi) (\hat{\varphi}_a(\xi))^C \hat{\varphi}_a(\xi) \frac{da}{a^2} =$$
$$\int_0^\infty \hat{f}(\xi) |\hat{\varphi}_a(\xi)|^2 \frac{da}{a^2} = \hat{f}(\xi).$$

Thus $\tau_\varphi W_\varphi$ is the identity on $L^2(R,H)$.

For $\varphi \in \Phi$, $A'_\varphi = W_\varphi L^2(R,H)$ is a subspace of $L^2\left(U,H,\frac{dadb}{a^2}\right)$, which is isometric to $L^2(R,H)$. Now we define the orthogonal projection P_φ from $L^2\left(U,H,\frac{dadb}{a^2}\right)$ onto A'_φ, then we have the following theorem.

Theorem 4 For φ, A'_φ, P_φ defined as above, we have for $F \in L^2\left(U,H,\frac{dadb}{a^2}\right)$,

$$P_\varphi F(a,b) = \int_0^\infty (F(a_1,\cdot) \circledast \varphi_{a_1}) \circledast \check{\bar{\varphi}}_a(b) \frac{da_1}{a_1^2}.$$

Proof Let

$$Q_\varphi F(a,b) = \int_0^\infty (F(a_1,\cdot) \circledast \varphi_{a_1}) \circledast \check{\bar{\varphi}}_a(b) \frac{da_1}{a_1^2},$$

for $F = W_\varphi f \in A'_\varphi$, we can prove that Q_φ is equal to P_φ.

$$(Q_\varphi F)\hat{}(a,\xi) = \int_0^\infty ((F(a_1,\cdot) \circledast \varphi_{a_1}) \circledast \check{\bar{\varphi}}_a)\hat{}(\xi) \frac{da_1}{a_1^2} =$$
$$\int_0^\infty \hat{F}(a_1,\xi) \hat{\varphi}_{a_1}(\xi) \hat{\bar{\varphi}}_a(\xi) \frac{da_1}{a_1^2} = \int_0^\infty \hat{F}(a_1,\xi) \hat{\varphi}_{a_1}(\xi) (\hat{\varphi}_a(\xi))^C \frac{da_1}{a_1^2} =$$

$$\int_0^\infty \hat{f}(\xi)\overset{\vee}{\tilde{\varphi}}_{a_1}(\xi)\hat{\varphi}_{a_1}(\xi)(\hat{\varphi}_a(\xi))^C\frac{\mathrm{d}a_1}{a_1^2} =$$

$$\int_0^\infty \hat{f}(\xi)(\hat{\varphi}_{a_1}(\xi))^C\hat{\varphi}_{a_1}(\xi)(\hat{\varphi}_a(\xi))^C\frac{\mathrm{d}a_1}{a_1^2} =$$

$$\hat{f}(\xi)\ (\hat{\varphi}_a(\xi))^C = (F(a,\cdot)\hat{}\,(\xi).$$

Clearly, $Q_\varphi F = F = P_\varphi F$.

On the other hand, for $F \perp A'_\varphi$, and any $g \in L^2(R,H)$. Applying Fubini's theorem, then we have

$$\langle Q_\varphi F, W_\varphi g\rangle = \int_0^\infty\int_{-\infty}^\infty\int_0^\infty \hat{F}(a_1,\xi)\hat{\varphi}_{a_1}(\xi)\cdot$$

$$(\hat{\varphi}_a(\xi))^C\frac{\mathrm{d}a_1}{a_1^2}(\hat{g}(\xi)(\hat{\varphi}_a(\xi))^C)^C\mathrm{d}\xi\frac{\mathrm{d}a}{a^2} =$$

$$\int_0^\infty\int_{-\infty}^\infty\int_0^\infty \hat{F}(a_1,\xi)a_1^{\frac{1}{2}}\hat{\varphi}(a_1\xi)(a^{\frac{1}{2}}\hat{\varphi}(a\xi))^C\cdot$$

$$\frac{\mathrm{d}a_1}{a_1^2}(\hat{g}(\xi)(a^{\frac{1}{2}}\hat{\varphi}(a\xi))^C)^C\mathrm{d}\xi\frac{\mathrm{d}a}{a^2} =$$

$$\int_0^\infty\int_{-\infty}^\infty\int_0^\infty \hat{F}(a_1,\xi)a_1^{\frac{1}{2}}\hat{\varphi}(a_1\xi)(a^{\frac{1}{2}}\hat{\varphi}(a\xi))^C\cdot$$

$$a^{\frac{1}{2}}\hat{\varphi}\ (a\xi)\frac{\mathrm{d}a}{a^2}(\hat{g}(\xi))^C\mathrm{d}\xi\frac{\mathrm{d}a}{a_1^2} =$$

$$\int_0^\infty\int_{-\infty}^\infty\int_0^\infty \hat{F}(a_1,\xi)a_1^{\frac{1}{2}}\hat{\varphi}(a_1\xi)\,|\,\hat{\varphi}(a\xi)\,|^2\cdot$$

$$\frac{\mathrm{d}a}{a}\ (\hat{g}(\xi))^C\mathrm{d}\xi\frac{\mathrm{d}a_1}{a_1^2} = \langle F, W_\varphi g\rangle_{L^2(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2})} = 0.$$

This shows

$$Q_\varphi F \perp A'_\varphi.$$

Noticing that $Q_\varphi F(a,b) = (w \circledast \overset{\vee}{\tilde{\varphi}}_a)(b)$, where

$$w(x) = \int_0^\infty (F(a_1,\cdot)\circledast\varphi_{a_1})(x)\frac{\mathrm{d}a_1}{a_1^2} = \tau_\varphi F(x),$$

we know that $Q_\varphi F \in A'_\varphi$, therefore we have

$$Q_\varphi F = 0 = P_\varphi F.$$

So $Q_\varphi = P_\varphi$, we complete the proof.

Theorem 5　For φ, $\Psi \in \Phi$ and $(\hat{\varphi}(\xi))^C\hat{\psi}(\xi)$ is real function, A'_φ is orthogonal to A'_ψ, if and only if

$$\int_0^\infty \frac{(\hat{\varphi}(\xi))^C\hat{\psi}(\xi)}{\xi}\mathrm{d}\xi = \int_{-\infty}^0 \frac{(\hat{\varphi}(\xi))^C\hat{\psi}(\xi)}{|\xi|}\mathrm{d}\xi = 0.$$

Proof　For $f,g \in L^2(R,H)$, we have

$$\langle W_\varphi f, W_\psi g\rangle = \langle f\circledast\overset{\vee}{\tilde{\varphi}}_a, g\circledast\overset{\vee}{\tilde{\psi}}_a\rangle = \int_0^\infty\int_{-\infty}^\infty (f\circledast\overset{\vee}{\tilde{\varphi}}_a)\hat{}\,\cdot$$

$$(\xi)((g\circledast\overset{\vee}{\tilde{\psi}}_a)\hat{}\,(\xi))^C\mathrm{d}\xi\frac{\mathrm{d}a}{a^2} =$$

$$\int_0^\infty\int_{-\infty}^\infty \hat{f}(\xi)a^{\frac{1}{2}}(\hat{\varphi}(a\xi))^C\,a^{\frac{1}{2}}\hat{\psi}(a\xi)\cdot$$

$$(g(\xi))^C\mathrm{d}\xi\frac{\mathrm{d}a}{a^2} =$$

$$\int_{-\infty}^\infty \hat{f}(\xi)(\hat{g}(\xi))^C\mathrm{d}\xi\int_0^\infty(\hat{\varphi}(a\xi))^C\hat{\psi}(a\xi)\frac{\mathrm{d}a}{a} =$$

$$\int_0^\infty \hat{f}(\xi)(\hat{g}(\xi))^C\mathrm{d}\xi\int_0^\infty(\hat{\varphi}(\xi'))^C\hat{\psi}(\xi')\frac{\mathrm{d}\xi'}{\xi'} +$$

$$\int_{-\infty}^0 \hat{f}(\xi)(\hat{g}(\xi))^C\mathrm{d}\xi\cdot$$

$$\int_{-\infty}^0(\hat{\varphi}(\xi'))^C\hat{\psi}(\xi')\frac{\mathrm{d}\xi'}{|\xi'|}.$$

$\langle W_\varphi f, W_\psi g\rangle = 0$ for any $f,g \in L^2(R,H)$, if and only if

$$\int_0^\infty \frac{(\hat{\varphi}(\xi))^C\hat{\psi}(\xi)}{\xi}\mathrm{d}\xi = \int_{-\infty}^0 \frac{(\hat{\varphi}(\xi))^C\hat{\psi}(\xi)}{|\xi|}\mathrm{d}\xi = 0.$$

3　Orthogonal decomposition of $L^2\left(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2}\right)$

Now we need to find an orthonormal basis for $L^2(0,\infty)$. For $Z^+ = \{0,1,2\cdots\}$, $\nu > -1$, $m,n \in Z^+$,

$$\int_0^\infty \mathrm{e}^{-x}x^\nu L_m^{(\nu)}L_n^{(\nu)}\mathrm{d}x = \Gamma(\nu+1)\binom{\nu+m}{m}\delta_{mn},$$

where $L_m^{(\nu)}(x) = \sum_{l=0}^m\binom{\nu+m}{m-l}\frac{(-x)^l}{l!}$ are the Laguerre polynomials[12]. Let AW denote the space consisting of the admissible wavelets whose Fourier transforms are supported in $[0,+\infty)$ and let $AW^- = \{\varphi: \varphi^- \in AW\}$.

Let φ_m be functions on R, whose Fourier transforms are given by

$$\hat{\varphi}_m(\xi) =$$

$$\begin{cases} 2^{\frac{\nu}{2}}\Gamma(\nu+1)^{-\frac{1}{2}}\binom{\nu+m}{m}^{-\frac{1}{2}}\xi^{\frac{\nu+1}{2}}\mathrm{e}^{-\xi}L_m^{(\nu)}(2\xi), & \xi \geqslant 0 \\ 0, & \xi < 0. \end{cases}$$

Then $\{\xi^{-\frac{1}{2}}\hat{\varphi}_m(\xi)\}_{m\in Z^+}$ are orthonormal bases of $L^2(0,\infty)$. And $\hat{\varphi}_m^-(\xi) = \hat{\varphi}_m(-\xi)$. At the same time, we have

$$f(x) = \int_0^\infty((f\circledast\hat{\varphi}_{m,a})\circledast\varphi_{m,a})(x)\frac{\mathrm{d}a}{a^2},$$

$$h(x) = \int_0^\infty((h\circledast\overset{\vee}{\tilde{\varphi}}_{m,a}^-)\circledast\varphi_{m,a}^-)(x)\frac{\mathrm{d}a}{a^2},$$

where $\varphi_{m,a}(x)=\frac{1}{\sqrt{a}}\varphi_m\left(\frac{x}{a}\right)$.

We define

$$H^+(R,H)=\{f\in L^2(R,H):\operatorname{supp}\hat{f}\subset[0,\infty)\},$$
$$H^-(R,H)=\{f\in L^2(R,H):\operatorname{supp}\hat{f}\subset[-\infty,0)\}.$$

Obviously,

$$L^2(R,H)=H^+(R,H)\oplus H^-(R,H).$$

We have

$$AW=\operatorname{span}\{\varphi_m\}_{m\geqslant 0},$$
$$AW^-=\operatorname{span}\{\varphi_m^-\}_{m\geqslant 0}.$$

We define the subspaces A_{φ_m} and $A_{\varphi_m}^-$ of $L^2\left(U,H,\frac{dadb}{a^2}\right)$ by

$$A_{\varphi_m}=\{f\circledast\check{\varphi}_{m,a}(x):f\in H^+(R,H)\},$$
$$A_{\varphi_m}^-=\{f\circledast\check{\varphi}_{m,a}^-(x):f\in H^-(R,H)\}.$$

Any real wavelet function f satisfies Eq. (1), we have

$$\hat{f}(\xi)=\sum_m r_m\hat{\varphi}_m(\xi).$$

Next, let us take the wavelet function φ_m as defined above and perform the orthogonal decomposition of $L^2\left(U,H,\frac{dadb}{a^2}\right)$.

Theorem 6 Let φ_m and $A_{\varphi_m}=W_{\varphi_m}H^+(R,H)$ ($A_{\varphi_m}^-=W_{\varphi_m^-}H^-(R,H)$) defined as above. Then we have the orthogonal decomposition of $L^2\left(U,H,\frac{dadb}{a^2}\right)$ as

$$L^2\left(U,H,\frac{dadb}{a^2}\right)=\oplus_{m=0}^{\infty}(A_{\varphi_m}\oplus A_{\varphi_m}^-).$$

Proof For each $F(a,b)\in L^2\left(U,H,\frac{dadb}{a^2}\right)$, we have

$$\int_{-\infty}^{\infty}\int_0^{\infty}|F(a,b)|^2\frac{dadb}{a^2}=\int_{-\infty}^{\infty}\int_0^{\infty}|\hat{F}(a,\xi)|^2\frac{dad\xi}{a^2}<\infty.$$

We need only consider the following functions: For $f\in H^+(R,H)$,

$$\hat{F}(a,\xi)=f(\xi)g(a,\xi)\tag{2}$$

where $g(a,\xi)$ is real-valued, satisfying

$$\int_0^{\infty}|g(a,\xi)|^2\frac{da}{a^2}<+\infty\tag{3}$$

$f(\xi)$ is quaternionic-valued, satisfying

$$\int_0^{\infty}|f(\xi)|^2d\xi+\infty.$$

Note that

$$\int_0^{\infty}|g(a,\xi)|^2\frac{da}{a^2}=\int_0^{\infty}|t(a,\xi)|^2\frac{da}{a}=\int_0^{\infty}\frac{|t_1(\xi)|^2}{\xi}d\xi<+\infty\tag{4}$$

Where the first equation is obtained by

$$t(a,\xi)=\frac{g(a,\xi)}{a^{\frac{1}{2}}}\tag{5}$$

the second is by

$$\hat{t}_1(a\xi)=t(a,\xi)\tag{6}$$

Thus by Eq. (4), we know that $t_1(x)$ is an admissible wavelet, so

$$\hat{t}_1(\xi)=\sum_m r_m\hat{\varphi}_m(\xi),$$

where $r_m\in R$. Hence

$$t(a,\xi)=\sum_m r_m\hat{\varphi}_m(a\xi).$$

By Eqs. (5) and (2), we get

$$g(a,\xi)=\sum_m a^{\frac{1}{2}}r_m\hat{\varphi}_m(a\xi),$$
$$\hat{F}(a,\xi)=\sum_m\hat{f}(\xi)a^{\frac{1}{2}}r_m\hat{\varphi}_m(a\xi).$$

So

$$F(a,\xi)=\sum_m r_m W_{\varphi_m}f(a,b)\tag{7}$$

We have a similar proof when $f\in H^-$. In this case, we will write the result directly,

$$F^-(a,\xi)=\sum_m r_m W_{\varphi_m^-}f(a,b)\tag{8}$$

From Formulas (7) ~ (8) and Theorem 5, we get

$$L^2\left(U,H,\frac{dadb}{a^2}\right)=\oplus_{m=0}^{\infty}(A_{\varphi_m}\oplus A_{\varphi_m}^-).$$

Now we give another way to prove that this orthogonal decomposition is completely decomposed.

$$\sum_m\|P_{\varphi_m}F\|^2_{L^2(U,H,\frac{dadb}{a^2})}=\sum_m\int_0^{\infty}\int_{-\infty}^{\infty}|(P_{\varphi_m}F)\hat{}(a,\xi)|^2d\xi\frac{da}{a^2}=\sum_m\int_0^{\infty}\int_{-\infty}^{\infty}\left|\int_0^{\infty}\hat{F}(a_1,\xi)a_1^{\frac{1}{2}}\hat{\varphi}_m(a_1\xi)\times a^{\frac{1}{2}}(\hat{\varphi}(a\xi))^C\frac{da_1}{a_1^2}\right|^2d\xi\frac{da}{a^2}=\sum_m\int_{-\infty}^{\infty}\left|\int_0^{\infty}\hat{F}(a_1,\xi)a_1^{-1}\hat{\varphi}_m(a_1\xi)a_1^{-\frac{1}{2}}da_1\right|^2d\xi=$$

$$\sum_m \int_0^{\infty} \left| \int_0^{\infty} \hat{F}(a_1,\xi) a_1^{-1} \hat{\varphi}_m(a_1\xi) a_1^{-\frac{1}{2}} \mathrm{d}a_1 \right|^2 \mathrm{d}\xi +$$

$$\sum_m \int_{-\infty}^{0} \left| \int_0^{\infty} \hat{F}^-(a_1\xi) a_1^{-1} \hat{\varphi}_m^-(a_1\xi) a_1^{-\frac{1}{2}} \mathrm{d}a_1 \right|^2 \mathrm{d}\xi =$$

$$\int_0^{\infty} \sum_m \left(\int_0^{\infty} \left| \hat{F}(a_1,\xi) a_1^{-1} \right|^2 \mathrm{d}a_1 \cdot \int_0^{\infty} \left| \hat{\varphi}_m(a_1\xi) a_1^{-\frac{1}{2}} \right|^2 \mathrm{d}a_1 \right) \mathrm{d}\xi +$$

$$\int_{-\infty}^{0} \sum_m \left(\int_0^{\infty} \left| \hat{F}^-(a_1,\xi) a_1^{-1} \right|^2 \mathrm{d}a_1 \cdot \int_0^{\infty} \left| \hat{\varphi}_m^-(a_1\xi) a_1^{-\frac{1}{2}} \right|^2 \mathrm{d}a_1 \right) \mathrm{d}\xi =$$

$$\int_0^{\infty} \int_0^{\infty} \left| \hat{F}(a_1,\xi) a_1^{-1} \right|^2 \mathrm{d}a_1 \mathrm{d}\xi +$$

$$\int_{-\infty}^{0} \int_0^{\infty} \left| \hat{F}^-(a_1,\xi) a_1^{-1} \right|^2 \mathrm{d}a_1 \mathrm{d}\xi =$$

$$\int_0^{\infty} \int_0^{\infty} \left| \hat{F}(a,\xi) \right|^2 a^{-2} \mathrm{d}a\mathrm{d}\xi +$$

$$\int_{-\infty}^{0} \int_0^{\infty} \left| \hat{F}^-(a,\xi) \right|^2 a^{-2} \mathrm{d}a\mathrm{d}\xi = \| F \|^2_{L^2(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2})}.$$

Thus we have given a full decomposition of $L^2\left(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2}\right)$ by continuous wavelet transform,

$$L^2\left(U,H,\frac{\mathrm{d}a\mathrm{d}b}{a^2}\right) = \oplus_{m=0}^{\infty} P_{\varphi_m} F(R,H) = \oplus_{m=0}^{\infty} (A_{\varphi_m} \oplus A_{\varphi_m}^-).$$

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