第 20 卷　第 1 期　　广州大学学报（自然科学版）　　Vol. 20　No. 1
2021 年　3 月　　Journal of Guangzhou University（Natural Science Edition）　　Mar.　2021

文章编号：1671-4229（2021）01-0023-11

泡型星图网络的 3 限制诊断度

王世英，王　琛
（山西师范大学 数学与计算机科学学院，山西 临汾　041004）

摘　要：互连网络故障诊断的一个新的方法是 g 限制诊断度，该方法限制每个无故障分支至少有（$g+1$）个无故障节点. 作为一种良好的互联网络拓扑结构，n 维泡型星图 BS_n 具有许多良好的性质. 文章证明了在 $n\geqslant5$ 的 PMC 模型和在 $n\geqslant12$ 的 MM* 模型下 BS_n 的 3-限制诊断度是 $8n-20$.

关键词：互联网络；连通度；诊断度；泡型星图
中图分类号：O 157.5　　**文献标志码：**A

The 3-extra diagnosability of bubble-sort star graph networks

WANG Shi-ying, *WANG Chen*
（School of Mathematics and Computer Science, Shanxi Normal University, Linfen 041004, China）

Abstract: The g-extra diagnosability of G is a new fault diagnosis method for interconnection networks, which limits every good component to at least（$g+1$）fault free nodes. As a desirable topology structure of interconnection networks, the n-dimensional bubble-sort star graph BS_n has some important properties. In this paper, we prove that the 3-extra diagnosability of BS_n is $8n-20$ under the PMC model for $n\geqslant5$ and under the MM* model for $n\geqslant12$.

Key words: interconnection network; connectivity; diagnosability; bubble-sort star graph
CLC number: O 157.5　　**Document code**: A

Interconnection networks（networks for short）can be regarded as network topologies. A network is represented by a graph where vertices represent processors and edges represent communication links between two processors. We can exchange graphs and networks. For the system, it is of great significance to study the topological properties of its network. Because the processors may fail and cause faults in the system, node fault identification is very important for such systems. To deal with faults, we first identify the fault processors from the fault-free processors. The process of identifying faults is called system diagnosis.

If all faulty processors can be identified without replacement and the number of faulty processors does not exceed t, then this system is t-diagnosable. The diagnosability $t(G)$ is the maximum number of t where G is t-diagnosable[1-4]. Dahbura, et al.[2] proposed an algorithm with time complex $O(n^{2.5})$ about a t-diagnosable system. Some diagnosis models for identifying fault processors are introduced. The main method is the PMC diagnosis model[5]. The diagnosis of the system is that two connected processors test each other. Maeng, et al.[6]（MM model）pro-

---

Foundation items: National Natural Science Foundation of China（61772010）

Biography: WANG Shi-ying（1961—）, male, professor, doctoral supervisor. E-mail: wangsy@sxnu.edu.cn, wangshiying@htu.edu.cn

Citation: WANG S Y, WANG C. The 3-extra diagnosability of bubble-sort star graph networks[J]. Journal of Guangzhou University（Natural Science Edition）, 2021, 20(1): 23-33.

pose another diagnosis model. In the MM model, a node sends the same task to its 2 neighbors, and then compares their responses. Sengupta, et al. ① put forward a special example of the MM model, called the MM* model, where each node must test any pair of its neighbors. Numerous studies have been investigated under the PMC model, MM model or MM* model (see [3-4,7-11]). In 2005, Lai, et al.[4] introduced conditional diagnosability of systems, which is also called restricted diagnosability. They assume that all neighbors of any vertex in the system are not included in the fault set. The restricted diagnosability of the system has received much attention[12-16]. In 2012, Peng, et al.[10] proposed a g-good neighbor diagnosability (also known as g-good neighbor conditional diagnosability). This method of system fault diagnosis requires each good node to have at least g good neighbors. In the PMC model, Peng, et al.[10] studied the g-good-neighbor diagnosability of an n-dimensional hypercube. In the MM* model, Wang, et al.[17] studied the diagnosability of g-good neighbors of the n-dimensional hypercube. The g-good-neighbor diagnosability of the system has received much attention, too [11,16,18-21]. In 2016, Zhang, et al.[22] proposed g-extra diagnosability. It is a new method of system fault-diagnosis, which limits each fault-free component to have at least $(g+1)$ fault-free nodes. Zhang, et al.[22] studied the g-extra diagnosability of the n-dimensional hypercube with the PMC model and the MM* model. In 2016, Wang, et al.[23] studied the 2-extra diagnosability of the bubble-sort star graph BS_n in PMC model and the MM* model.

The bubble-sort star graph is studied after studying the star graph and the bubble-sort graph. The star graph and the bubble-sort graph have been proved to be an important viable candidate for interconnecting a multiprocessor system[24]. A feature of the star graph includes a low degree of node, small diameter, symmetry, and high degree of fault-tolerance. For details, see [7,25-37]. In Refs. [38-39], the diagnosability of a star graph under the PMC model and the MM model is studied. Lin, et al.[9] showed that the conditional diagnosability of the star graph S_n under the comparison diagnosis model is $3n-7$. Guo, et al.[40] showed that the conditional diagnosability of the bubble-sort star graph BS_n under the MM model is $6n-15$ for $n\geqslant 6$ and under the PMC model is $8n-21$ for $n\geqslant 5$. Using the PMC model and the MM* model, Wang, et al.[23] studied the 2-extra diagnosability of BS_n. Guo, et al.[41] studied the extra connectivity of BS_n.

As a favorable topology structure of interconnection networks, the n-dimensional bubble-sort star graph BS_n has many good properties. In this paper, we prove that the 3-extra diagnosability of BS_n is $8n-20$ under the PMC model with $n\geqslant 5$ and under the MM* model with $n\geqslant 12$.

1 Preliminaries

In this section, there are some definitions and notations needed for our discussion; the n-dimensional bubble-sort star graph BS_n, the PMC model and the MM* model are introduced.

1.1 Notations

A graph G is k-regular if $d_G(v)=k$ for each v in $V(G)$. The connectivity $\kappa(G)=\min\{|S|: G-S$ is disconnected$\}$. The g-good-neighbor set $F=\{F\subseteq V: |N(v)\cap(V\backslash F)|\geqslant g$ for every vertex $v\in V\backslash F\}$. For the g-good-neighbor set F, if $G-F$ is disconnected, then F is a g-good-neighbor cut of G. The g-good-neighbor connectivity of G is $\kappa^{(g)}(G)=\min\{|F|$: F is a g-good-neighbor cut of $G\}$. For $F\subseteq V$, if every component of $G-F$ has at least $(g+1)$ vertices, then F is called a g-extra set, in addition, if $G-F$ is disconnected, then F is a g-extra cut of G. If a connected graph G has a g-extra cut, then G is g-extra connected. The g-extra connectivity of G is $\tilde{\kappa}^{(g)}(G)$

---

① Sengupta A, Dahbura A T. On self-diagnosable multiprocessor systems: Diagnosis by the comparison approach[J]. IEEE Transactions on Computers, 1992, 41(11): 1386-1396.

$= \min\{|F|: F$ is a g-extra cut$\}$. Some symbols are define in Table 1. Other definitions and symbols are given in Ref. [42].

Table 1　Some symbols in the graph[42]

Symbol	Meaning
$G[V']$	the subgraph induced by V'
$d_G(v)$	the number of edges incident with v
$\delta(G)$	the minimum degrees of vertices of G
$N_G(v)$	the set of vertices adjacent to v
$N_G(S)$	$\cup_{v\in S} N_G(v)\backslash S$
$\kappa(G)$	connectivity
$\kappa^{(g)}(G)$	g-good-neighbor connectivity
$\kappa^{(g)}(G)$	g-extra connectivity

1.2　The PMC model and the MM* model

In the PMC model[5], to diagnose a system G, two adjacent nodes in G have the ability to test each other. For $uv \in E(G)$, the test performed by u on v is represented by the ordered pair (u, v). The result of testing (u, v) is 1 (resp. 0) if u testing v is faulty (resp. fault-free). We usually assume that the test results are reliable (resp. unreliable) if u is fault-free (resp. faulty). The test allocation T of system G is the test set of each adjacent vertex pair. It is expressed as directed test graph $(V(G), L)$, where $(u, v) \in L$ means that $uv \in E(G)$. Syndrome is the set of test results for all tests T. A fault set is made up of all fault processors. Given a syndrome σ, if σ satisfies $u \in V\backslash F$ and $\sigma(u, v) = 1$ for any $(u, v) \in L$ iff $v \in F$, then $F \subseteq V(G)$ is consistent with σ. This means that F may be faulty processors. Because the test results generated by the faulty processor are not reliable. A set F of faulty vertices may produce many different syndromes. However, different fault sets may cause the same syndrome. Let $\sigma(F)$ be the set of all syndromes consistent with it. Let $F_1, F_2 \in V(G)$, $F_1 \neq F_2$. If $\sigma(F_1) \cap \sigma(F_2) = \varnothing$, then they are distinguishable and we say (F_1, F_2) is a distinguishable pair, otherwise, they are indistinguishable and (F_1, F_2) is an indistinguishable pair.

In the MM model[6], the same test task is sent from one processor to a pair of processors and their responses are compared to perform the diagnosis. There are three assumptions: (a) all faults are invariable; (b) the faulty processor produces incorrect output; (c) the comparison performed by the fault processor is not reliable. The comparison of the system $G = (V, E)$ is transformed into a multigraph, which is represented by $M(V(G), L)$ where L is the marked edge set. $(u, v)_w \in L$ denotes a comparison in which w compares u and v, which means that uw, $vw \in E(G)$. The syndrome σ^* represents the set of all comparison results in $M(V(G), L)$. If the results $(u, v)_w$ are different, then $\sigma^*((u, v)_w) = 1$. Otherwise, $\sigma^*((u, v)_w) = 0$. In the MM* model, if uw, $vw \in E(G)$, then $(u, v)_w \in L$. Consistent and indistinguishable are similar to those in PMC.

Let F_1 and F_2 be a distinct pair of g-good-neighbor subsets with $|F_1| \leqslant t$ and $|F_2| \leqslant t$ in V. If F_1 and F_2 are distinguishable for each distinct pair (F_1, F_2), then the system $G = (V, E)$ is g-good-neighbor t-diagnosable. The g-good-neighbor diagnosability $t_g(G) = \max\{t$: G is g-good-neighbor t-diagnosable$\}$.

Proposition 1[10]　For any given system G, $t_g(G) \leqslant t_{g'}(G)$ if $g \leqslant g'$.

In a system $G = (V, E)$, a set $F \subseteq V$ is called a conditional set if it does not contain all the neighbor vertices of any vertex in G. A system G is conditional t-diagnosable if every two distinct conditional subsets F_1, $F_2 \in V$ with $|F_1| \leqslant t$, $|F_2| \leqslant t$, are distinguishable. The conditional diagnosability $t_c(G) = \max\{t$: G is conditional t-diagnosable$\}$. By [8], $t_c(G) \geqslant t(G)$.

Theorem 1[18]　For a system $G = (V, E)$, $t(G) = t_0(G) \leqslant t_1(G) \leqslant t_c(G)$.

In Ref. [18], Wang, et al. proved $t_1(B_n) = 2n - 3$ in the PMC model for $n \geqslant 4$. In Ref. [43], Zhou, et al. proved $t_c(B_n) = 4n - 11$ for $n \geqslant 4$ in the PMC model. Therefore, $t_1(G) < t_c(G)$ when $n \geqslant 5$ and $t_1(G) = t_c(G)$ when $n = 4$. In $G = (V, E)$, F in V is the g-extra set if every component of $G - F$ has more than g nodes. G is g-extra t-diagnosable iff for each pair g-extra vertex subsets F_1, $F_2 \subseteq V(G)$ and $F_1 \neq F_2$ such that $|F_i| \leqslant t$, F_1 and F_2 are distinguishable. The g-extra diagnosability of G, denoted by $\tilde{t}_g(G)$, is the maximum number of t such that G is g-extra t-diagnosable.

Proposition 2[23] For any given system G, $\tilde{t}_g(G) \leqslant \tilde{t}_{g'}(G)$ if $g \leqslant g'$.

Theorem 2[23] For a system $G=(V,E)$, $t(G)=\tilde{t}_0(G) \leqslant \tilde{t}_g(G) \leqslant t_g(G)$.

Theorem 3[23] For a system $G=(V,E)$, $\tilde{t}_1(G)=t_1(G)$.

1.3 The bubble-sort star graph

In the permutation $\begin{pmatrix}1 & 2 & \cdots & n\\ p_1 & p_2 & \cdots & p_n\end{pmatrix}$, $i \to p_i$. For convenience, $\begin{pmatrix}1 & 2 & \cdots & n\\ p_1 & p_2 & \cdots & p_n\end{pmatrix}$ denoted by $p_1 p_2 \cdots p_n$. Every permutation can be denoted by a product of cycles[44]. For example, $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = (132)$. Specially, $\begin{pmatrix}1 & 2 & \cdots & n\\ 1 & 2 & \cdots & n\end{pmatrix} = (1)$. The product $\sigma\tau$ of two permutations is the composition of function τ followed by σ, that is, $(12)(13)=(132)$. There are no symbols defined here in [45].

Let $[n]=\{1,2,\cdots,n\}$, and let S_n be the symmetric group on $[n]$ containing all permutations $p=p_1p_2\cdots p_n$ of $[n]$. $\{(1i): 2 \leqslant i \leqslant n\}$ is a generating set for S_n. Thus, $\{(1,i): 2 \leqslant i \leqslant n\} \cup \{(i,i+1): 2 \leqslant i \leqslant n-1\}$ is a generating set for S_n. The n-dimensional bubble-sort star graph BS_n[40,45] is the graph with vertex set $V(BS_n)=S_n$ in which two vertices $uv \in E(BS_n)$ iff $u=v(1,i)$, $2 \leqslant i \leqslant n$, or $u=v(i,i+1)$, $2 \leqslant i \leqslant n-1$. Clearly, BS_n is a $(2n-3)$-regular graph with $n!$ vertices. The graphs BS_n for $n=2,3,4$ are depicted in Fig. 1.

Note that BS_n is a special Cayley graph. BS_n has the following useful properties.

Proposition 3 For any integer $n \geqslant 1$, BS_n is $(2n-3)$-regular, vertex transitive.

Proposition 4 For any integer $n \geqslant 2$, BS_n is bipartite.

Theorem 4[44] Every nonidentity permutation in the symmetric group is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2.

Proposition 5[45] Let BS_n be the bubble-sort star graph. If $uv \in E(BS_n)$, then $|N(u) \cap N(v)|=0$. If $uv \notin E(BS_n)$, then $|N(u) \cap N(v)| \leqslant 3$.

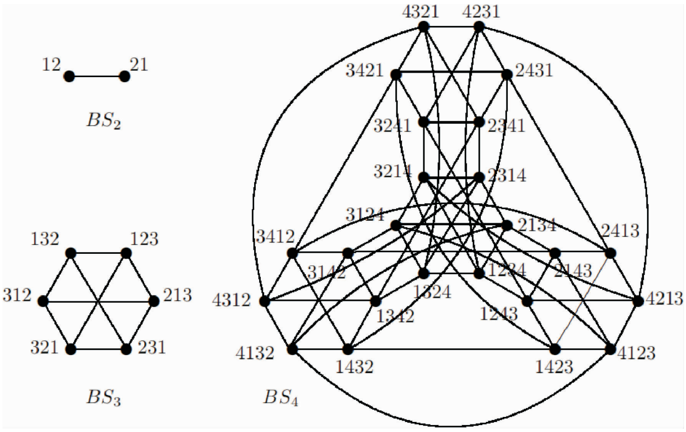

Fig. 1 The bubble-sort star graphs BS_2, BS_3 and BS_4

Lemma 1① Let $A=\{(1),(14),(15),(34)\}$. If $n \geqslant 5$, $F_1=N(A)$, $F_2=A \cup N(A)$, then $|F_1|=8n-23$, $|F_2|=8n-19$, F_1 is a 3-extra cut of BS_n, and BS_n-F_1 has two components BS_n-F_2 and $BS_n[A]$.

Theorem 5① For $n \geqslant 5$, the 3-extra connectivity of the bubble-sort star graph BS_n is $8n-23$, i.

---

① Wang S. The tightly super 3-extra connectivity of bubble-sort star graphs [J]. RAIRO-Theoretical Informatics and Applications (to appear).

e., $\tilde{\kappa}^{(3)}(BS_n)=8n-23$. The 3-extra connectivity of BS$_4$ is 8, i.e., $\tilde{\kappa}^{(3)}(BS_4)=8$.

A connected graph G is super g-extra connected if for each minimum g-extra cut F, $G-F$ has a component with $g+1$ vertices. Besides, $G-F$ exactly contains 2 components, where one component has $g+1$ vertices, then G is called to be tightly $|F|$ super g-extra connected.

Theorem 6① For $n\geqslant 5$, the bubble-sort star graph BS_n is tightly $(8n-23)$ super 3-extra connected.

2 The 3-extra diagnosability of the bubble-sort star graph under the PMC

Theorem 7[2,11] A system $G=(V,E)$ is g-extra t-diagnosable in PMC model iff G has $uv\in E(G)$ where $u\in V(G-F_1-F_2)$ and $v\in F_1\backslash F_2$ or $v\in F_2\backslash F_1$ for g-extra sets F_1, $F_2\in V$, $F_1\neq F_2$, $|F_1|\leqslant t$ and $|F_2|\leqslant t$ (See Fig. 2).

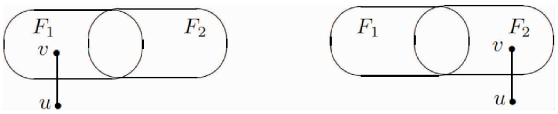

Fig. 2 Illustration of a distinguishable pair (F_1, F_2) under the PMC model

Theorem 8[46] Let $G=(V,E)$ be a g-extra connected graph, and let $V\neq F_1\cup F_2$ for each distinct pair of g-extra subsets F_1 and F_2 of G with $|F_1|\leqslant\tilde{\kappa}^{(g)}(G)+g$ and $|F_2|\leqslant\tilde{\kappa}^{(g)}(G)+g$. If there is connected subgraph H of G with $|V(H)|=g+1$ such that $N(V(H))$ is a minimum g-extra cut of G, then the g-extra diagnosability of G is $\tilde{\kappa}^{(g)}(G)+g$ under the PMC model.

Corollary 1 Let $n\geqslant 5$. Then the 3-extra diagnosability of the bubble-sort star graph BS_n under the PMC model is $8n-20$.

Proof Let F_1 and F_2 be each distinct pair of 3-extra subsets F_1 and F_2 of BS_n with $|F_1|\leqslant 8n-20$ and $|F_2|\leqslant 8n-20$. Assume that $V(BS_n)=F_1\cup F_2$. By the definition of the symmetric group S_n, $n!=|S_n|=|F_1\cup F_2|=|F_1|+|F_2|-|F_1\cap F_2|\leqslant|F_1|+|F_2|\leqslant 2(8n-20)=16n-40$. Since $n\geqslant 5$, $n!>16n-40$, a contradiction. Therefore, $V(BS_n)\neq F_1\cup F_2$.

Let A be defined in Lemma 1. Then $|A|=3+1=4$ and $N(A)$ is a 3-extra cut of BS_n. By Theorem 4, $N(A)$ is a minimum 3-extra cut of BS_n. By Theorem 8, the 3-extra diagnosability of BS_n is $\tilde{\kappa}^{(3)}(BS_n)+3=8n-28+3=8n-20$ under the PMC model.

□

3 The 3-extra diagnosability of the bubble-sort star graph under the MM*

Theorem 9[2,11] A system $G=(V,E)$ is g-extra t-diagnosable in the MM* model iff for g-extra sets F_1, $F_2\in V$, $F_1\neq F_2$, $|F_1|\leqslant t$ and $|F_2|\leqslant t$, they have one of 3 conditions: ① G has $uw, vw\in E(G)$ where $u,w\in V(G-F_1-F_2)$ and $v\in F_1\backslash F_2$ or $v\in F_2\backslash F_1$. ② G has $uw, vw\in E(G)$ where $u,v\in F_1\backslash F_2$ and $w\in V(G-F_1-F_2)$. ③ G has $uw, vw\in E(G)$ where $u,v\in F_2\backslash F_1$ and $w\in V(G-F_1-F_2)$ (See Fig. 3).

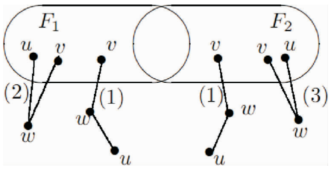

Fig. 3 Illustration of a distinguishable pair (F_1, F_2) under the MM$*$ model.

Lemma 2 Let $n\geqslant 5$. Then the 3-extra diagnosability of the bubble-sort star graph BS_n under the MM* model, $\tilde{t}_3(BS_n)\leqslant 8n-20$.

Proof Let A be defined as above, and let $F_1=N(A)$, $F_2=A\cup N(A)$. By Lemma 1, $|F_1|=8n-23$, $|F_2|=8n-19$, F_1 is a 3-extra cut of BS_n, and BS_n-F_1 has two components BS_n-F_2 and $BS_n[A]$. Therefore, $|V(BS_n-F_2)|\geqslant 4$ and $|V(BS_n[A])|=4$. Thus, F_1 and F_2 are both 3-extra sets of

BS_n with $|F_1|=8n-23$ and $|F_2|=8n-19$. Since $A=F_1\Delta F_2$ and $N_{BS_n}(A)=F_1\subset F_2$, there is no edge of BS_n between $V(BS_n)\setminus(F_1\cup F_2)$ and $F_1\Delta F_2$. Therefore, both F_1 and F_2 are not satisfied in any one condition in Theorem 9, and BS_n is not 3-extra $(8n-19)$-diagnosable in the MM* model. Therefore, we deduce that $\tilde{t}_3(BS_n)\leqslant 8n-20$. □

A component of a graph G is odd according as it has an odd number of vertices. We denote by $o(G)$ the number of odd components of G.

Lemma 3 ([42]Tutte′s Theorem) A graph $G=(V,E)$ has a perfect matching if and only if $o(G-S)\leqslant|S|$ for all $S\subseteq V$.

Lemma 4 Let $n\geqslant 12$. Then the 3-extra diagnosability of the bubble-sort star graph BS_n under the MM* model, $\tilde{t}_3(BS_n)\geqslant 8n-20$.

Proof By the definition of $\tilde{t}_3(G)$, we only need to establish that BS_n is 3-extra $(8n-20)$-diagnosable. According to Theorem 9, proof by contradiction, assume that there exist two 3-extra sets F_1, $F_2\in V(BS_n)$, $F_1\neq F_2$, $|F_1|\leqslant 8n-20$ and $|F_2|\leqslant 8n-20$, but (F_1,F_2) does not satisfy any one of the three conditions of Theorem 9. Assume that $F_2\setminus F_1\neq\emptyset$. According to the similar discussion to $V(BS_n)\neq F_1\cup F_2$ in Corollary 1, $V(BS_n)\neq F_1\cup F_2$ can be obtained.

Claim 1 $BS_n-F_1-F_2$ does not contain isolated vertex.

Proof by contradiction, assume that there exists at least 1 isolated vertex $w_1\in BS_n-F_1-F_2$. Because F_1 is one 3-extra set, there is a vertex $u\in F_2\setminus F_1$ such that $uw_1\in E(BS_n)$. Meanwhile, (F_1,F_2) can not be applied in Theorem 9. So $F_2\setminus F_1$ has at most a vertex u such that $uw_1\in E(BS_n)$. Thus, only u in $F_2\setminus F_1$ satisfies $uw_1\in E(BS_n)$. If $F_1\setminus F_2=\emptyset$, then $F_1\subseteq F_2$. Since F_2 is a 3-extra set, every component G_i of $BS_n-F_1-F_2=BS_n-F_2$ has $|V(G_i)|\geqslant 4$. Therefore, $BS_n-F_1-F_2$ does not contain isolated vertex. Thus, $F_1\setminus F_2\neq\emptyset$. Similarly, we can deduce that there exists just one vertex $a\in F_1\setminus F_2$ such that $aw_1\in E(BS_n)$. Let $W\subseteq S_n\setminus(F_1\cup F_2)$ be the set of isolated vertices in $BS_n-F_1-F_2$, and let $H=BS_n-F_1-F_2-W$. Then for any $w\in W$, there are $(2n-5)$ neighbors in $F_1\cap F_2$. By Propositions 4, $|W|\leqslant\frac{n!}{2}$. Note that $14n-35<\frac{n!}{2}$. Suppose that $V(H)=\emptyset$. Then $n!=|V(BS_n)|=|F_1\cup F_2|+|W|=|F_1|+|F_2|-|F_1\cap F_2|+|W|\leqslant 2(8n-20)-(2n-5)+|W|=14n-35+|W|$ and hence $|W|\geqslant n!-(14n-35)>\frac{n!}{2}$, a contradiction to that $|W|\leqslant\frac{n!}{2}$. So $V(H)\neq\emptyset$. Since the vertex set pair (F_1,F_2) does not satisfy the condition (1) of Theorem 9, and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1\Delta F_2$. Note $F_2\setminus F_1\neq\emptyset$. If $F_1\cap F_2=\emptyset$, then this is a contradiction to that BS_n is connected. Therefore, $F_1\cap F_2\neq\emptyset$. Thus, $F_1\cap F_2$ is a cut of BS_n. Since F_1 is a $3-$extra set of BS_n, we have that every component H_i of H has $|V(H_i)|\geqslant 4$ and every component B_i of $BS_n[W\cup(F_2\setminus F_1)]$ has $|V(B_i)|\geqslant 4$. Since F_2 is a 3-extra set of BS_n, we have that every component B_i' of $BS_n[W\cup(F_1\setminus F_2)]$ has $|V(B_i')|\geqslant 4$. Note that $BS_n-(F_1\cap F_2)$ has two parts (for convenience): H and $BS_n[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)]$. Let $\mathcal{B}_i$ be a component of $BS_n[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)]$ and let $b_i\in V(\mathcal{B}_i)$. If $b_i\in W$, then there is a component G_i of $BS_n[(F_2\setminus F_1)\cup W]$ $(|V(G_i)|\geqslant 3)$ and a component B_i of $BS_n[(F_1\setminus F_2)\cup W]$ $(|V(B_i)|\geqslant 4)$ such that $b_i\in V(G_i)$ and $b_i\in V(B_i)$. It follows that $G_i\cup B_i$ is connected in $BS_n[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)]$ and $b_i\in V(G_i\cup B_i)$. Since connection is an equivalence relation on the vertex set $W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)$, $\mathcal{B}_i=(G_i\cup B_i)$ holds. Therefore, $|V(\mathcal{B}_i)|\geqslant 5>4$. If $b_i\in(F_2\setminus F_1)$, then there is a component G_i of $BS_n[(F_2\setminus F_1)\cup W]$ $(|V(G_i)|\geqslant 4)$ such that $b_i\in V(G_i)$. It follows that G_i is connected in $BS_n[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)]$ and $b_i\in V(G_i)$. Since connection is an equivalence relation on the vertex set $W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)$, we have that G_i is a subgraph of $\mathcal{B}_i$. Therefore, $|V(\mathcal{B}_i)|\geqslant 4$. Similarly, if $b_i\in(F_1\setminus F_2)$, then $|V(\mathcal{B}_i)|\geqslant 4$. Therefore, $F_1\cap$

F_2 is a 3-extra cut of BS_n. By Theorem 5, $|F_1\cap F_2|\geqslant 8n-23$.

Since $|F_1|\leqslant 8n-20$, $|F_2|\leqslant 8n-20$ and $|F_1\cap F_2|\geqslant 8n-23$, we have $|F_2\setminus F_1|=|F_2|-|F_1\cap F_2|\leqslant 8n-20-(8n-23)=3$ and $|F_1\setminus F_2|\leqslant 3$.

Let $\mathcal{B}_i$ be a component of $BS_n[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)]$ and let $b_i\in V(\mathcal{B}_i)$. If $b_i\in W$, from the above discussion, then $|V(\mathcal{B}_i)|\geqslant 5>4$. Suppose that $b_i\in(F_2\setminus F_1)$. Since F_1 is a 3-extra set, every component G_i of $BS_n[(F_2\setminus F_1)\cup W]$ has $|V(G_i)|\geqslant 4$. Since $|F_2\setminus F_1|\leqslant 3$, there is a component G_j of $BS_n[(F_2\setminus F_1)\cup W]$ ($|V(G_j)|\geqslant 4$) such that $b_i,w_i\in V(G_j)$ and $w_i\in W$. From the above discussion, $|V(\mathcal{B}_i)|\geqslant 5>4$. Similarly, if $b_i\in(F_1\setminus F_2)$, then $|V(\mathcal{B}_i)|\geqslant 5>4$. We consider the following cases.

Case 1 $|F_2\setminus F_1|=3$.

In this case, $|F_1\cap F_2|=8n-23$. By Theorem 5, $F_1\cap F_2$ is a minimum 3-extra cut of BS_n. By Theorem 6, BS_n is tightly $(8n-23)$ super 3-extra connected, i.e., $BS_n-(F_1\cap F_2)$ has two components, one of which is a connected subgraph of order 4. Since every component G_i of $BS_n[(F_2\cap F_1)\cup W]$ has $|V(G_i)|\geqslant 5$, we have that $|V(BS_n-F_1-F_2)|=4$. By Lemma 3, $|W|\leqslant o(G-(F_1\cup F_2))\leqslant|F_1\cup F_2|\leqslant|F_1|+|F_2|\leqslant 8n-20+8n-20=16n-40$. Thus, $n!=|V(BS_n)|=|V(BS_n-F_1-F_2)|+|F_2\setminus F_1|+|F_1\setminus F_2|+|W|+|F_2\cap F_1|\leqslant 4+3+3+16n-40+8n-23=24n-53$, a contradiction to $n\geqslant 5$.

Case 2 $|F_2\setminus F_1|=2$.

In this case, $|F_1\cap F_2|=8n-23=\tilde{\kappa}^{(3)}(BS_n)$ or $|F_1\cap F_2|=8n-22$. Suppose $|F_1\cap F_2|=8n-23=\tilde{\kappa}^{(3)}(BS_n)$. Similarly to above, we have that $n!/2=|V(BS_n)|=|V(BS_n-F_1-F_2-W)|+|F_2\setminus F_1|+|F_1\setminus F_2|+|W|+|F_2\cap F_1|\leqslant 4+2+3+16n-40+8n-23=24n-54$, a contradiction to $n\geqslant 5$. Suppose $|F_1\cap F_2|=8n-22$.

Let $x,y\in F_2\setminus F_1$. Suppose $xy\notin E(BS_n)$. Since F_1 is a 3-extra set of BS_n, we have that every component B_i of $BS_n[W\cup(F_2\setminus F_1)])$ has $|V(B_i)|\geqslant 4$. Suppose that $x,y\in V(B_i)$. Then there is an $xy-$path in B_i. It's only possible that there are edges between $\{x,y\}$ and W in $BS_n[W\cup(F_2\setminus F_1)])$. Therefore, the $xy-$path is the xwy, where $w\in W$. This contradicts the fact that there is just a vertex of $F_2\setminus F_1$ such that it is adjacent to a vertex of W. Therefore, $BS_n[W\cup(F_2\setminus F_1)])$ has two components B_1 and B_2 such that $x\in V(B_1)$ and $y\in V(B_2)$. Since $|V(B_1)|\geqslant 4$, $|W|\geqslant 3$. Since every vertex of W is adjacent to just a vertex of $F_2\setminus F_1$ and $|V(B_2)|\geqslant 4$, we have that $|W|\geqslant 6$. If $|F_1\setminus F_2|=3$, then $|F_1\cap F_2|=8n-23$, a contradiction. Therefore, $1\leqslant|F_1\setminus F_2|\leqslant 2$. If $|F_1\setminus F_2|=1$, then, by Proposition 5, $|W|=6$. Let $\{z\}=F_1\setminus F_2$. We have that $|N(\{x,y\})\cap(F_1\cap F_2)|+|N(z)\cap(F_1\cap F_2)|+|N(W)\setminus((F_2\setminus F_1)\cup(F_1\setminus F_2))|\geqslant 2(2n-6)+2n-9+6(2n-5)-24=18n-75>8n-22$ when $n\geqslant 6$, a contradiction to $|F_1\cap F_2|=8n-22$. Let $F_1\setminus F_2=\{u,v\}$. Suppose that $uv\notin E(BS_n)$.

By Proposition 5, $|N(x)\cap N(u)\cap W|\leqslant 3$ and $|N(x)\cap N(v)\cap W|\leqslant 3$. Since every vertex of W is adjacent to just a vertex of $F_1\setminus F_2$, $|(N(x)\cap N(u)\cap W)\cup(N(x)\cap N(v)\cap W)|=|(N(x)\cap N(u)\cap W)|+|(N(x)\cap N(v)\cap W)|$. Since F_2 is a 3-extra set of BS_n, $|N(u)\cap W|\geqslant 3$ and $|N(v)\cap W|\geqslant 3$. Therefore, $|(N(x)\cap N(u)\cap W)\cup(N(x)\cap N(v)\cap W)|=|(N(x)\cap N(u)\cap W)|+|(N(x)\cap N(v)\cap W)|\leqslant 6$. Similarly, $|(N(y)\cap N(u)\cap W)\cup(N(y)\cap N(v)\cap W)|=|(N(y)\cap N(u)\cap W)|+|(N(y)\cap N(v)\cap W)|\leqslant 6$. Since $(N(x)\cap W)\cap(N(y)\cap W)=\varnothing$, $6\leqslant|(N(\{x,y,u,v\})\cap W|\leqslant 12$.

Let $f(m)=m(2n-3)-\dfrac{3m(m-1)}{2}-4$. When $m<\dfrac{2n-5.5}{3}$, $f'(m)>0$ for $n\geqslant 12$. Therefore, $f(m)$ is an increasing function on $[6,12]$ for $n\geqslant 12$, the minimum $f(6)=12n-67$. Note that $N(W)\subseteq(F_1\cup F_2)$ and $N(W)\setminus\{x,y,u,v\}\subseteq(F_1\cap F_2)$. Let $|W|=m$. By Proposition 5, $|(N(W)|\geqslant m(2n-3)-\dfrac{3m(m-1)}{2}$ in $BS_n[W\cup(F_1\cup F_2)]$.

Therefore, $|(N(W)\cap(F_1\cap F_2)|\geqslant f(m)>8n-22$ when $n\geqslant 12$, a contradiction. Suppose that $uv\in E(BS_n)$. We consider the following subcases.

Case a1 $N(\{u\})\cap W\neq\varnothing$ and $N(\{v\})\cap W=\varnothing$.

By Proposition 5, $|N(x)\cap N(u)\cap W|\leqslant 3$ and $|N(y)\cap N(u)\cap W|\leqslant 3$. Since every vertex in W is adjacent to u in $BS_n[W\cup(F_1\setminus F_2)]$, $|W|=6$. From the above discussion, there is a contradiction.

Case a2 $N(\{u\})\cap W\neq\varnothing$ and $N(\{v\})\cap W\neq\varnothing$.

By Proposition 5, $|N(x)\cap N(u)\cap W|\leqslant 3$ and $|N(x)\cap N(v)\cap W|\leqslant 3$. If $|N(x)\cap N(u)\cap W|\neq 0$, then, by Proposition 4, $|N(x)\cap N(v)\cap W|=0$. Let $|N(x)\cap N(u)\cap W|\neq 0$. Then $|N(x)\cap N(v)\cap W|=0$. Since every vertex of W is adjacent to just a vertex of $F_1\setminus F_2$, $|N(x)\cap N(u)\cap W|=3$.

Since $N(\{v\})\cap W\neq\varnothing$, $1\leqslant|N(y)\cap N(v)\cap W|\leqslant 3$. Note that $|N(y)\cap N(u)\cap W|=0$ and every vertex of W is adjacent to just a vertex of $F_1\setminus F_2$. Therefore, $|N(y)\cap N(v)\cap W|=3$ and $|W|=6$. From the above discussion, there is a contradiction.

Suppose $xy\in E(BS_n)$. From the above discussion, $1\leqslant|F_1\setminus F_2|\leqslant 2$. We consider the following subcases.

Case b1 $N(\{x\})\cap W\neq\varnothing$ and $N(\{y\})\cap W=\varnothing$.

Let $|F_1\setminus F_2|=1$ and $\{u\}=F_1\setminus F_2$. Since F_1 is a 3-extra set of BS_n, $|N(x)\cap W|\geqslant 2$. By Proposition 5, $|N(x)\cap N(u)\cap W|\leqslant 3$. Since F_2 is a 3-extra set of BS_n, $|N(u)\cap W|\geqslant 3$. Therefore, $|W|=3$. We have that $|N(\{x,y,u\})\cap(F_1\cap F_2)|+|N(W)\setminus((F_2\setminus F_1)\cup(F_1\setminus F_2))|\geqslant 3(2n-5)+(2n-4)+(2n-7)+(2n-6)-9=14n-41>8n-22$ when $n\geqslant 4$, a contradiction to $|F_1\cap F_2|=8n-22$. Let $|F_1\setminus F_2|=2$ and $\{u,v\}=F_1\setminus F_2$.

Suppose that $uv\notin E(BS_n)$. By Case a1, there is a contradiction.

Suppose that $uv\in E(BS_n)$. Recall that $|N(x)\cap W|\geqslant 2$. If $N(\{u\})\cap W\neq\varnothing$ and $N(\{v\})\cap W\neq\varnothing$, then there is a 5-cycle, a contradiction to Proposition 4. If $N(\{u\})\cap W\neq\varnothing$ and $N(\{v\})\cap W=\varnothing$, then $2\leqslant|W|\leqslant 3$. Similarly to above, there is a contradiction.

Case b2 $N(\{x\})\cap W\neq\varnothing$ and $N(\{y\})\cap W\neq\varnothing$.

Let $|F_1\setminus F_2|=1$ and $\{u\}=F_1\setminus F_2$. Since every vertex of W is adjacent to just a vertex of $F_2\setminus F_1$, $(N(\{x\})\cap W)\cap(N(\{y\})\cap W)=\varnothing$. For $w_1\in N(\{x\})\cap W$ and $w_2\in N(\{y\})\cap W$, w_1u, $w_2u\in E(BS_n)$ and hence xw_1uw_2yx is a 5-cycle, a contradiction to Proposition 4. Let $|F_1\setminus F_2|=2$ and $\{u,v\}=F_1\setminus F_2$.

Suppose that $uv\notin E(BS_n)$. By Case a2, there is a contradiction. Suppose that $uv\in E(BS_n)$.

By Proposition 5, $|N(x)\cap N(u)\cap W|\leqslant 3$ and $|N(x)\cap N(v)\cap W|\leqslant 3$. If $N(x)\cap N(u)\cap W\neq\varnothing$ and $N(x)\cap N(v)\cap W\neq\varnothing$, then there is a 5-cycle, a contradiction. Let $N(x)\cap N(u)\cap W\neq\varnothing$. Then $|N(x)\cap N(u)\cap W|\leqslant 3$. Similarly, $|N(y)\cap N(3)\cap W|\leqslant 3$. Therefore, $2\leqslant|W|\leqslant 6$. Similarly to above, there is a a contradiction.

Case 3 $|F_2\setminus F_1|=1$.

In this case, $|F_1\cap F_2|=8n-23=\tilde{\kappa}^{(3)}(BS_n)$ or $|F_1\cap F_2|=8n-22$ or $|F_1\cap F_2|=8n-21$. We consider the following subcases.

Case 3.1 $|F_1\cap F_2|=8n-23$.

Similarly to above, there is a contradiction.

Case 3.2 $|F_1\cap F_2|=8n-22$.

Since $|F_1|\leqslant 8n-20$, $1\leqslant|F_1\setminus F_2|\leqslant 2$. When $|F_1\setminus F_2|=2$, we discuss in Case 2. Let $|F_1\setminus F_2|=1$. Let $x\in F_2\setminus F_1$. Since F_1 is a 3-extra set of BS_n, we have that every component B_i of $BS_n[W\cup(F_2\setminus F_1)]$ has $|V(B_i)|\geqslant 4$. Since $x\in F_2\setminus F_1$, $BS_n[W\cup(F_2\setminus F_1)]$ has a component B_1. Let $x\in V(B_1)$. Since $|V(B_1)|\geqslant 4$, $|W|\geqslant 3$ and there are $a,b,c\in W$ such that $xa,xb,xc\in E(BS_n)$. Let $y\in F_1\setminus F_2$. Since F_2 is a 3-extra set of BS_n, $ya,yb,yc\in E(BS_n)$ and $|W|=3$. Therefore, $|F_2\cap F_1|\geqslant 3(2n-5)+2(2n-6)-3=10n-30>8n-22=|F_1\cap F_2|$, a contradiction to $n\geqslant 7$.

Case 3.3 $|F_1\cap F_2|=8n-21$.

In this case, $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. We discuss it in Case 3.2. Claim 1 is complete.

Let $u \in V(BS_n - F_1 - F_2)$. Then u is adjacent to $BS_n - F_1 - F_2$ from Claim 1. Since (F_1, F_2) can not be applied in Theorem 9, according to the condition (1) in Theorem 9, adjacent vertices $u, w \in V(BS_n - F_1 - F_2)$ does not satisfy $uw \in E(BS_n)$ and $vw \in E(BS_n)$ where $v \in F_1 \Delta F_2$. We deduce that u is not adjacent to any vertex of $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. According to the generality of u, $V(BS_n - F_1 - F_2)$ is not connected to $F_1 \Delta F_2$.

If $F_1 \cap F_2 = \varnothing$, then this is a contradiction that BS_n is connected. Therefore, $F_1 \cap F_2 \neq \varnothing$ and $F_1 \cap F_2$ is a cut of BS_n. Since $F_2 \setminus F_1 \neq \varnothing$ and F_1 is a 3-extra set, we have that every component H_i of $BS_n - F_1 - F_2$ has $|V(H_i)| \geqslant 4$ and every component G_i of $BS_n([F_2 \setminus F_1])$ has $|V(G_i)| \geqslant 4$. Suppose that $F_1 \setminus F_2 = \varnothing$. Then $F_1 \cap F_2 = F_1$. Since F_1 is a 3-extra set of BS_n, we have that $F_1 \cap F_2 = F_1$ is a 3-extra set of BS_n. Since there is no edge between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_2 \setminus F_1$, we have that $F_1 \cap F_2 = F_1$ is a 3-extra cut of BS_n. Suppose that $F_1 \setminus F_2 \neq \varnothing$. Similarly, every component B_i of $LTQ_n([F_1 \setminus F_2])$ has $|V(B_i)| \geqslant 4$. Therefore, $F_1 \cap F_2$ is a 3-extra cut of BS_n. By Theorem 5, we have $|F_1 \cap F_2| \geqslant 8n - 23$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geqslant 4 + (8n - 23) = 8n - 19$, which contradicts $|F_2| \leqslant 8n - 20$. Therefore, BS_n is 3-extra $(8n-20)$-diagnosable and $\tilde{t}_3(BS_n) \geqslant 8n - 20$. The proof is complete.

Combining Lemma 2 and Lemma 4, we have the following theorem.

Theorem 10 Let $n \geqslant 12$. Then the 3-extra diagnosability of the bubble-sort star graph BS_n under the MM* model is $8n - 20$.

4 Conclusions

In this paper, we investigate the problem of the 3-extra diagnosability of the bubble-sort star graph BS_n under the PMC model and the MM* model. It is proved that 3-extra diagnosability of BS_n is $8n - 20$ under the PMC model for $n \geqslant 5$ and under the MM* model for $n \geqslant 12$. The above results show that the 3-extra diagnosability is several times larger than the classical diagnosability of BS_n depending on the condition: 3-extra. The work will help engineers to develop more different measures of 3-extra diagnosability based on application environment, network topology, network reliability, and statistics related to fault patterns.

References:

[1] Barsi F, Grandoni F, Maestrini P. A theory of diagnosability of digital systems[J]. IEEE Transactions on Computers, 1976, 25(6): 585-593.

[2] Dahbura A T, Masson G M. An $O(n^{2.5})$ fault identification algorithm for diagnosable systems[J]. IEEE Transactions on Computers, 1984, 33(6): 486-492.

[3] Fan J. Diagnosability of crossed cubes under the comparison diagnosis model[J]. IEEE Transactions on Parallel and Distributed Systems, 2002, 13(10): 1099-1104.

[4] Lai P L, Tan J M, Chang C P, et al. Conditional diagnosability measures for large multiprocessor systems[J]. IEEE Transactions on Computers, 2005, 54(2): 165-175.

[5] Preparata F P, Metze G, Chien R T. On the connection assignment problem of diagnosable systems[J]. IEEE Transactions on Computers, 1967, EC-16: 848-854.

[6] Maeng J, Malek M. A comparison connection assignment for self-diagnosis of multiprocessor systems[C]//Proceeding of 11th International Symposium on Fault-Tolerant Computing, Los Alamitos, CA: IEEE Computer Society, 1981: 173-175.

[7] Chang N W, Hsieh S Y. Structural properties and conditional diagnosability of star graphs by using the PMC model[J]. IEEE Transactions on Parallel and Distributed Systems, 2014, 25(11): 3002-3011.

[8] Hsieh S Y, Kao C Y. The conditional diagnosability of k-ary n-ubes under the comparison diagnosis model[J]. IEEE Trans-

actions on Computers, 2013, 62(4): 839-843.

[9] Lin C K, Tan J M, Hsu L H, et al. Conditional diagnosability of Cayley graphs generated by transposition trees under the comparison diagnosis model[J]. Journal of Interconnection Networks, 2008, 9(1/2): 83-97.

[10] Peng S L, Lin C K, Tan J M, et al. The *g*-good-neighbor conditional diagnosability of hypercube under PMC model[J]. Applied Mathematics and Computation, 2012, 218(21): 10406-10412.

[11] Yuan J, Liu A, Ma X, et al. The *g*-good-neighbor conditional diagnosability of *k*-ary *n*-cubes under the PMC model and MM^* model[J]. IEEE Transactions on Parallel and Distributed Systems, 2015, 26(4): 1165-1177.

[12] Lin L, Zhou S, Xu L, et al. The extra connectivity and conditional diagnosability of alternating group networks[J]. IEEE Transactions on Parallel and Distributed Systems, 2015, 26(8): 2352-2362.

[13] Lin L, Xu L, Zhou S. Relating the extra connectivity and the conditional diagnosability of regular graphs under the comparison model[J]. Theoretical Computer Science, 2016, 618: 21-29.

[14] Lin L, Xu L, Zhou S, et al. The extra, restricted connectivity and conditional diagnosability of split-star networks[J]. IEEE Transactions on Parallel and Distributed Systems, 2016, 27(2): 533-545.

[15] Xu L, Lin L, Zhou S, et al. The extra connectivity, extra conditional diagnosability and *t/m*-diagnosability of arrangement graphs[J]. IEEE Transactions on Reliability, 2016, 65(3): 1248-1262.

[16] Xu X, Zhou S, Li J. Reliability of complete cubic networks under the condition of *g*-good-neighbor[J]. The Computer Journal, 2017, 60(5): 625-635.

[17] Wang S, Han W. The *g*-good-neighbor conditional diagnosability of *n*-dimensional hypercubes under the MM * model[J]. Information Processing Letters, 2016, 116: 574-577.

[18] Wang M, Guo Y, Wang S. The 1-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the PMC model and MM^* model[J]. International Journal of Computer Mathematics, 2017, 94(3): 620-631.

[19] Wang M, Lin Y, Wang S. The 2-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the PMC model and MM * model[J]. Theoretical Computer Science, 2016, 628: 92-100.

[20] Wang S, Wang Z, Wang M. The 2-good-neighbor connectivity and 2-good-neighbor diagnosability of bubble-sort star graph networks[J]. Discrete Applied Mathematics, 2017, 217: 691-706.

[21] Yuan J, Liu A, Qin X, et al. *g*-Good-neighbor conditional diagnosability measures for 3-ary *n*-cube networks[J]. Theoretical Computer Science, 2016, 622: 144-162.

[22] Zhang S, Yang W. The *g*-extra conditional diagnosability and sequential *t/k*-diagnosability of hypercubes[J]. International Journal of Computer Mathematics, 2016, 93(3): 482-497.

[23] Wang S, Wang Z, Wang M. The 2-extra connectivity and 2-extra diagnosability of bubble-sort star graph networks[J]. The Computer Journal, 2016, 59(12): 1839-1856.

[24] Sheldon B A, Krishnamurthy B. A group-theoretic model for symmetric interconnection networks[J]. IEEE Transactions on Computers, 1989, 38(4): 555-566.

[25] Day K, Triphthi A. A comparative study of topological properties of hypercubes and star graphs[J]. IEEE Transactions on Parallel and Distributed Systems, 1994, 5(1): 31-38.

[26] Hsieh S Y. Embedding longest fault-free paths onto star graphs with more vertex faults[J]. Theoretical Computer Science, 2005, 337(1/3): 370-378.

[27] Hu S C, Yang C B. Fault tolerance on star graphs[C]//Proceedings of the First Aizu International Symposium on Parallel Algorithms/Architecture Synthesis, Piscataway: IEEE, 1995: 176-182.

[28] Huang C W, Huang H L, Hsieh S Y. Edge-bipancyclicity of star graphs with faulty elements[J]. Theoretical Computer Science, 2011, 412: 6938-6947.

[29] Latifi S. A study of fault tolerance in star graph[J]. Information Processing Letters, 2007, 102(5): 196-200.

[30] Latifi S, Saberinia E, Wu X. Robustness of star graph network under link failure[J]. Information Sciences, 2008, 178(3): 802-806.

[31] Li T K, Tan J M, Hsu L H. Hyper hamiltonian laceability on edge fault star graph[J]. Information Sciences, 2004, 165

(1/2): 59-71.

[32] Li X, Xu J. Generalized measures for fault tolerance of star networks[J]. Networks, 2014, 63(3): 225-230.

[33] Rescigno A A. Vertex-disjoint spanning trees of the star network with applications to fault-tolerance and security[J]. Information Sciences, 2001, 137(1/4): 259-276.

[34] Tsai P Y, Fu J S, Chen G H. Fault-free longest paths in star networks with conditional link faults[J]. Theoretical Computer Science, 2009, 410(8/10): 766-775.

[35] Walker D, Latifi S. Improving bounds on link failure tolerance of the star graph[J]. Information Sciences, 2010, 180(13): 2571-2575.

[36] Wan M, Zhang Z. A kind of conditional vertex connectivity of star graphs[J]. Applied Mathematics Letters, 2009, 22(2): 264-267.

[37] Yang Y, Wang S. Conditional connectivity of star graph networks under embedding restriction[J]. Information Sciences, 2012, 199: 187-192.

[38] Kavianpour A. Sequential diagnosability of star graphs[J]. Computers and Electrical Engineering, 1996, 22(1): 37-44.

[39] Zheng J, Latifi S, Regentova E, et al. Diagnosability of star graphs under the comparison diagnosis model[J]. Information Processing Letters, 2005, 93(1): 29-36.

[40] Guo J, Lu M. Conditional diagnosability of bubble-sort star graphs[J]. Discrete Applied Mathematics, 2016, 201(11): 141-149.

[41] Guo J, Lu M. The extra connectivity of bubble-sort star graphs[J]. Theoretical Computer Science, 2016, 645: 91-99.

[42] Bondy J A, Murty U S R. Graph theory[M]. New York: Springer, 2007.

[43] Zhou S, Wang J, Xu X, et al. Conditional fault diagnosis of bubble sort graphs under the PMC model[J]. Intelligence Computation and Evolutionary Computation, 2013, AISC 180: 53-59.

[44] Thomas W H. Algebra[M]. New York: Springer-Verlag, 1974.

[45] Cai H, Liu H, Lu M. Fault-tolerant maximal local-connectivity on bubble-sort star graphs[J]. Discrete Applied Mathematics, 2015, 181: 33-40.

[46] Wang S, Wang M. The *g*-good-neighbor and *g*-extra diagnosability of networks[J]. Theoretical Computer Science, 2019, 773: 107-114.

【责任编辑: 陈　钢】