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加权 Morrey 空间上的极大交换子

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摘　要: 设 M 是极大函数算子，$[b,M](f)(x)=b(x)Mf(x)-M(bf)(x)$ 是其交换子. 设 C_b 为极大交换子. 文章研究了极大函数的交换子 $[b,M]$ 和极大交换子 C_b 在齐型空间上的加权 Morrey 空间上的有界性. 此外，还得到了极大交换子 C_b 的下界估计.

关键词: 交换子；极大函数；加权 Morrey 空间
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The maximal commutators on weighted Morrey spaces

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Abstract: Let M be the maximal function and $[b,M](f)(x)=b(x)Mf(x)-M(bf)(x)$ be the commutator of maximal functions. Let C_b be the maximal commutator. In this paper, we study the estimates for the commutator of maximal functions $[b,M]$ and the maximal commutators C_b on weighted Morrey spaces on spaces of homogeneous type. The lower bound of the maximal commutator C_b is also obtained.

Key words: commutators; maximal function; weighted Morrey spaces
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0　Introduction

In their remarkable result[1], Coifman—Rochberg—Weiss showed that the commutator of Riesz transforms is bounded on $L^p(R^n)$ if and only if the symbol b is in the BMO space. See also the subsequent result in Refs[2-7]. In Ref. [8], Bastero—Milman—Ruiz characterized the class of functions for which the commutator with the Hardy—Littlewood maximal function and the maximal sharp function are bounded on $L^p(R^n)$. Recently, in Ref. [9] Agcayazi, et al also studied the unweighted version of the maximal commutator $C_b(f)$ on R^n by using a different approach, and this was extended to a space of homogeneous type by Fu, et al in Ref. [10].

In this paper, we aim to provide a quantitative estimate for the commutator of maximal functions $[b, M]$ and the maximal commutator C_b on weighted Morrey spaces on spaces of homogeneous type. To be more precise, let (X,d,μ) be a space of homogeneous type. The Hardy-Littlewood maximal function $Mf(x)$ on X is defined as

$$Mf(x) = \sup_{B:\, x\in B}\frac{1}{\mu(B)}\int_B |f(y)|\,\mathrm{d}\mu(y),$$

where the supremum is taken over all balls $B\subset X$. The commutator of maximal functions $[b,M]$ is defined by $[b,M](f)(x)=b(x)Mf(x)-M(bf)(x)$. The maximal commutator C_b on X with the symbol $b(x)$ is defined by

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$$C_bf(x) = \sup_{B:\, x\in B}\frac{1}{\mu(B)}\int_B |\, b(x) - b(y)\,| \cdot |\, f(y)\,|\,\mathrm{d}\mu(y),$$

where the supremum is taken over all balls $B\subset X$. Let $p\in(1,\infty)$, $\kappa\in(0,1)$ and $w\in A_p(X)$, the weighted Morrey space $L_w^{p,\kappa}(X)$ is defined by

$$L_w^{p,\kappa}(X) = \{f\in L^1_{loc}(X): \|f\|_{L_w^{p,\kappa}(X)} < \infty\},$$

where

$$\|f\|_{L_w^{p,\kappa}(X)} = \sup_B\left\{\frac{1}{w(B)^{\kappa}}\int_B |\, f(x)\,|^p w(x)\,\mathrm{d}\mu(x)\right\}^{\frac{1}{p}}.$$

The main result of this paper is as follows.

Theorem 1 Let $p\in(1,\infty)$, $\kappa\in(0,1)$ and $w\in A_p(X)$. Suppose $b\in L^1_{loc}(X)$. Then the C_b has the following boundedness characterization:

(1) If $b\in BMO(X)$, then C_b is bounded on $L_w^{p,\kappa}(X)$;

(2) If C_b is bounded on $L_w^{p,\kappa}(X)$, then $b\in BMO(X)$.

Theorem 2 Let $p\in(1,\infty)$, $\kappa\in(0,1)$ and $w\in A_p(X)$. Suppose $b\in BMO(X)$ to be a non-negative locally integrable function. Then there exists a positive constant C such that for all $f\in L_w^{p,\kappa}(X)$,

$$\|[b,M]f\|_{L_w^{p,\kappa}(X)} \leqslant C\|b\|_{BMO(X)}\|f\|_{L_w^{p,\kappa}(X)}.$$

Throughout the paper, the letter “C” will denote (possibly different) constants that are independent of the essential variables.

1 Definitions and preliminary results

Let μ be a measure on X and let d be a metric on X. Then we call topological space X to be a space of homogeneous type if it satisfies the doubling property, that is, there exists a constant $C\geqslant 1$, such that for all balls $B(x,r)=\{y\in X: d(y,x)<r\}$

$$\mu(B(x,2r))\leqslant C\mu(B(x,r))<\infty.$$

For the definition of homogeneous type space, one can see Ref. [11], Chapter 3.

Using the doubling property, we can obtain that there exist $C, n>0$ such that

$$\mu(B(x,\lambda r))\leqslant C\lambda^n\mu(B(x,r))$$

holds for all $\lambda>1$. The parameter n is a measure of the dimension of the space.

Let w be a nonnegative locally integrable function on X. For $1<p<\infty$, we say w is an A_p weight, written $w\in A_p$, if

$$[w]_{A_p} = \sup_B\left(\frac{1}{\mu(B)}\int_B w(x)\,\mathrm{d}\mu(x)\right)\cdot \left(\frac{1}{\mu(B)}\int_B w(x)^{\frac{1}{1-p}}\mathrm{d}\mu(x)\right)^{p-1} < \infty.$$

Here the suprema are taken over all balls $B\subset X$. The quantity $[w]_{A_p}$ is called the $A_p$constant of w. Next we note that for $w\in A_p$ the measure $w(x)\,\mathrm{d}\mu(x)$ is a doubling measure on X.

A function $b\in L^1_{loc}(X)$ belongs to the BMO space $BMO(X)$ if

$$\|b\|_{BMO(X)} = \sup_B\frac{1}{\mu(B)}\int_B |b(x)-b_B|\,\mathrm{d}\mu(x) < \infty,$$

where the sup is taken over all quasi-metric balls $B\subset X$ and $b_B = \dfrac{1}{\mu(B)}\displaystyle\int_B b(x)\,\mathrm{d}\mu(x)$.

2 The proof of the main results

In order to prove Theorem 1, we need the following lemma.

Lemma 1[10] Let $b\in BMO(X)$. Then there exists a positive constant C such that for all $f\in L^1_{loc}(X)$,

$$C_b(f)(x)\leqslant C\|b\|_{BMO(X)}M^2f(x),\ x\in X.$$

Proof of Theorem 1

(1) Assume $b\in BMO(X)$, by Lemma 1 and the fact that M is bounded on $L_w^{p,\kappa}(X)$, we can see that, for $p\in(1,\infty)$, $\kappa\in(0,1)$, $w\in A_p$ and $f\in L_w^{p,\kappa}(X)$,

$$\|C_b(f)\|_{L_w^{p,\kappa}(X)}\leqslant C\|b\|_{BMO(X)}\|M^2f\|_{L_w^{p,\kappa}(X)}\leqslant C\|b\|_{BMO(X)}\|f\|_{L_w^{p,\kappa}(X)}.$$

(2) Assume that C_b is bounded on $L_w^{p,\kappa}(X)$, next we will show that $b\in BMO(X)$. For any fixed $B\subset X$, we have

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B |\, b(x)-b_B\,|\,\mathrm{d}\mu(x)\leqslant\\ &C\inf_c\frac{1}{\mu(B)}\int_B |\, b(x)-c\,|\,\mathrm{d}\mu(x)\leqslant\\ &C\inf_{y\in B}\frac{1}{\mu(B)}\int_B |\, b(x)-b(y)\,|\,\mathrm{d}\mu(x)\leqslant\\ &C\frac{1}{w(B)}\int_B\left(\frac{1}{\mu(B)}\int_B |\, b(x)-\right.\\ &\left.b(y)\,|\,\mathrm{d}\mu(x)\right)w(y)\,\mathrm{d}\mu(y)\leqslant\end{aligned}$$

$$C\frac{1}{w(B)}\int_B C_b(\chi_B)(y)w(y)\,\mathrm{d}\mu(y),$$

where $\chi_B(x)=\begin{cases}1, x\in B\\0, x\notin B.\end{cases}$ Then by Hölder's inequality, we have

$$\frac{1}{\mu(B)}\int_B |b(x)-b_B|\,\mathrm{d}\mu(x)\leqslant$$

$$C\Big(\frac{1}{w(B)}\int_B |C_b(\chi_B)(y)|^p w(y)\,\mathrm{d}\mu(y)\Big)^{\frac{1}{p}}\leqslant$$

$$Cw(B)^{\frac{\kappa-1}{p}}\|C_b(\chi_B)\|_{L_w^{p,\kappa}}.$$

Note that C_b is bounded on $L_w^{p,\kappa}(X)$. This implies that

$$\frac{1}{\mu(B)}\int_B |b(x)-b_B|\,\mathrm{d}\mu(x)\leqslant$$

$$C\|C_b\|_{L_w^{p,\kappa}\to L_w^{p,\kappa}}w(B)^{\frac{\kappa-1}{p}}\|\chi_B\|_{L_w^{p,\kappa}}.$$

From the definition of $L_w^{p,\kappa}(X)$, it is easy to know that $\|\chi_B\|_{L_w^{p,\kappa}}\leqslant Cw(B)^{\frac{1-\kappa}{p}}$. Then we have

$$\frac{1}{\mu(B)}\int_B |b(x)-b_B|\,\mathrm{d}\mu(x)\leqslant$$

$$C\|C_b\|_{L_w^{p,\kappa}\to L_w^{p,\kappa}}w(B)^{\frac{\kappa-1}{p}}w(B)^{\frac{1-\kappa}{p}}\leqslant$$

$$C\|C_b\|_{L_w^{p,\kappa}\to L_w^{p,\kappa}}.$$

This implies that $b\in BMO(X)$, and

$$\|b\|_{BMO(X)}\leqslant C\|C_b\|_{L_w^{p,\kappa}\to L_w^{p,\kappa}}.$$

The proof of Theorem 1 is complete.

In order to prove Theorem 2, we need the following lemma.

Lemma 2[10] Let $b\in BMO(X)$ be a non-negative locally integrable function. Then there exists a positive constant C such that for all $f\in L_{loc}^1(X)$,

$$|[b,M]f(x)|\leqslant CC_b(f)(x).$$

Proof of Theorem 2

Let $p\in(1,\infty)$, $\kappa\in(0,1)$ and $w\in A_p(X)$. From Lemma 2 and Theorem 1, we have

$$\|[b,M]f\|_{L_w^{p,\kappa}(X)}\leqslant C\|C_b(f)\|_{L_w^{p,\kappa}(X)}\leqslant$$

$$C\|b\|_{BMO(X)}\|f\|_{L_w^{p,\kappa}(X)}$$

for all $f\in L_w^{p,\kappa}(X)$. The proof of Theorem 1 is complete.

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