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具有 Dzyaloshinskii-Moriya 相互作用和 V-flow 项的不可压缩 Navier-Stokes-Landau-Lifshitz 方程的局部存在唯一性

黄贵活, 王光武, 叶婷婷, 邱 臻
(广州大学 数学与信息科学学院, 广东 广州 510006)

摘 要: 文章证明了在$\mathbb{R}^2$和$\mathbb{R}^3$中具有 Dzyaloshinskii-Moriya 相互作用和 V-flow 项的不可压缩 Navier-Stokes-Landau-Lifshitz 方程组的 Cauchy 问题存在唯一局部解。文章所提出的方法依赖于用扰动抛物系统和平行传输来逼近此系统。

关键词: 不可压缩 incompressible Navier-Stokes-Landau-Lifshitz 方程; Dzyaloshinskii-Moriya 相互作用; 局部解
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Local existence and uniqueness of the incompressible Navier-Stokes-Landau-Lifshitz equations with the Dzyaloshinskii-Moriya interaction and V-flow term

HUANG Gui-huo, WANG Guang-wu, YE Ting-ting, QIU Zhen
(School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China)

Abstract: In this paper, we prove that there exists a unique local solution for the Cauchy problem of a system of the incompressible Navier-Stokes-Landau-Lifshitz equations with the Dzyaloshinskii-Moriya interaction and V-flow term in $\mathbb{R}^2$ and $\mathbb{R}^3$. Our methods rely upon approximating the system with a perturbed parabolic system and parallel transport.

Key words: incompressible Navier-Stokes-Landau-Lifshitz equations; Dzyaloshinskii-Moriya interaction; local solution

1 Introduction

In this paper, we are concerned with the incompressible Navier-Stokes-Landau-Lifshitz equations (NSLL) with the Dzyaloshinskii-Moriya (DM) inter-action and V-flow term initial-value problem:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \mu\Delta u - \lambda\nabla\cdot(\nabla d\odot\nabla d), \\ d_t + u\cdot\nabla d + \alpha d\times(\Delta d + V\cdot\nabla d + \nabla\times d) + \\ \quad \beta d\times(d\times(\Delta d + V\cdot\nabla d + \nabla\times d)) = d\times f, \\ \nabla\cdot u = 0, \\ (u,d)|_{t=0} = (u_0, d_0). \end{cases}$$

Here, $u:\mathbb{R}^m\times[0,T]\to\mathbb{R}^m$, $m=2,3$, represents the velocity field vector, $d:\mathbb{R}^m\times[0,T]\to\mathbb{S}^2\subset\mathbb{R}^3$ represents

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Biography: HUANG Gui-huo(1998—), male, postgraduate. E-mail:hgh935@163.com
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the magnetization field vector, where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2=\{x\in\mathbb{R}^3:|x|=1\}$, thus we set $|d|=1$. p is the pressure function, which is usually determined through the equations of state. μ and λ are positive constants, which represent the shear viscosity and the competition between kinetic energy and potential energy. The constant $\beta>0$ is called the Gilbert damping coefficient which is a gyroscopic ratio. This torque is most commonly termed non-adiabatic and α characterizes its strength. The notation $\times$ is the cross product for vectors in $\mathbb{R}^3$. The term $\nabla d\odot\nabla d$ denotes the matrix whose $(i,j)-th$ entry is given by $\partial_{x_i}d\cdot\partial_{x_j}d$. $V=V(x)$ is a smooth vector field on Ω and

$$\Delta_V d:=\Delta d+V\cdot\nabla d:=\Delta d+\nabla_V d$$

is the so-called $V-$Laplacian with respect to the metric on Ω. This system is a coupled parabolic system which consists of incompressible Navier-Stokes equations and Landou-Lifshitz-Gilbert equations with DM interaction in a strongly coupled way.

If d is a constant vector in $\mathbb{S}^2$ and $f=0$, the system becomes the incompressible Navier-Stokes equations. There is considerable literature on the equations. The most representative result is that in the 1930s, Leray[1] proved that the global classical solution of the Navier-Stokes equation is unique in $\mathbb{R}^2$, and he also proved that the equation has weak solutions in some sense. Later, Hopf[2] extended these results. And more representative results can be found in Refs. [3-6]. Recently, Tao[7] constructed a result of a smooth solution for an averaged Navier-Stokes equation which blow-up in finite time.

If $u=0$, $V=0$, $f=0$, $\lambda=0$ and neglecting the DM term, the system becomes the Landau-Lifshitz equation:

$$d_t=-\alpha d\times\Delta d-\beta d\times d\times\Delta d.$$

The Landau-Lifshitz equation was proposed by physicists Landau and Lifshitz in 1935 when they studied the dispersion theory of ferromagnet permeability. In 1993, Guo, et al.[8] proved a global existence of solutions for the Landau-Lifshitz equation of the ferromagnetic spin chain from an $m-$dimensional manifold M into the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ and constructed the relations between harmonic maps and the solutions of the Landau-Lifshitz equation. In 1998, Ding, et al.[9] proved that one-dimensional Schrödinger flow has a unique global smooth solution. In 2001, Ding, et al.[10] showed that there exists a unique local smooth solution for the Cauchy problem of the Schrödinger flow for maps from a compact Riemannian manifold into a complete Kähler manifold. The same year, Carbou, et al.[11] proved local existence, global existence with small data and uniqueness of regular solutions for the Landau-Lifshitz equations in R^3, and in Ref. [12] he proved global existence of regular solutions for the Landau-Lifschitz equation in the R^2. In 2011, Carbou, et al.[13] established global existence of the weak solutions for the Landau-Lifschitz equation with magnetostriction.

If $V=0$, $f=0$ and neglecting the DM term, the system can be regarded as the incompressible Navier-Stokes-Landau-Lifshitz equations. In 2010, Fan, et al.[14] first proposed the study of the Navier-Stokes-Landau-Lifshitz equation, and they proved the regularity criteria for a smooth solution of the Navier-Stokes-Landau-Lifshitz system in Besov spaces and the multiplier spaces. Zhai, et al.[15] obtained the global existence of a unique solution for this system without any small conditions imposed on the third component of the initial velocity field. Wang, et al.[16] proved the existence and uniqueness of the weak solution for the incompressible Navier-Stokes-Landau-Lifshitz equations in R^2 with finite energy. Wang, et al.[17] investigated the global existence of the weak solutions to the quantum Navier-Stokes-Landau-Lifshitz equations with density dependent viscosity in R^2. Wei, et al.[18] obtained the global solutions to the Navier-Stokes-Landau-Lifshitz system, and they also obtained the time decay rates of the higher-order spatial derivatives of the solutions. The improvement of this result obtained by Duan, et al.[19]. Qiu, et al.[20] established a blowup criterion for the R^2 incompressible Navier-Stokes-Landau-Lifshitz system with finite positive initial density. Recently, Wang, et al.[21] proved global existence of the smooth solution for the incompressible Navier-Stokes-Landau-Lifshitz equation with small initial data in $\mathbb{T}^2$ or $\mathbb{R}^2$ and $\mathbb{R}^3$.

In this paper, we will prove the local existence and uniqueness by parabolic approximation, which has been shown to be successful in the study of the Schrödinger flow[10].

Next, to facilitate the presentation of our main results, we shall first introduce some main notations and definitions of intrinsic Sobolev space and extrinsic Sobolev space. Let

$\mathbb{Z}_+ = \{0,1,2,\cdots\}$ and $[q]$ be the integer part of a positive number q. For $k \in \mathbb{Z}_+$, $p \in [1, \infty]$. Let $H^k(\mathbb{R}^m)$, $W^{k,p}(\mathbb{R}^m)$ denotes the usual Sobolev spaces of functions on $\mathbb{R}^m$. ∇ denotes the usual derivative for functions on $\mathbb{R}^m$. Then for $Q \in \mathbb{S}^2$, we define the extrinsic Sobolev spaces

$$W_Q^{k,p} = \{f:\mathbb{R}^m \to \mathbb{R}^3 : |f(x)| = 1 \text{ a.e. and } f - Q \in W^{k,p}\},$$

with the induced distance $d_Q^{k,p}(f,g) = \|f-g\|_{W_Q^{k,p}}$. For simplicity of notation, let $\|f\|_{W_Q^{k,p}} = d_Q^{k,p}(f,Q)$ and further denotes $H_Q^k := W_Q^{k,2}$. Next we introduce the intrinsic Sobolev spaces as follows. For smooth maps d from (M,g) to $\mathbb{S}^2$, the pullback bundle $d^*T\mathbb{S}^2$ is the vector bundle over (M, g) whose fiber at $x \in M$ is the tangent space $T_{d(x)}\mathbb{S}^2$. D denotes the induced covariant derivative in $d^*T\mathbb{S}^2$. Then, the intrinsic norm of vector bundle ∇d is defined by

$$\|\nabla d\|_{\mathbf{W}^{k,p}(M)}^p = \sum_{i=0}^{k} \int_M |D^i \nabla d|^p dvol_g,$$

where $p \in [1, \infty)$. For $p = \infty$, we also define

$$\|\nabla d\|_{\mathbf{W}^{k,\infty}(M)} = \max\{\|D^i \nabla d\|_{L^\infty} : 0 \leqslant i \leqslant k\}.$$

For simplicity, we denote $\mathbf{H}^k := \mathbf{W}^{k,2}$.

Next, we state the following main result.

Theorem 1 Let Ω be $\mathbb{R}^m$, $m = 2,3$. Suppose $V \in L^2((0, T), H^{k+1}(\Omega))$, $f \in L^2((0,T), H^{k-1}(\Omega))$. The Cauchy problem with $(u_0, d_0) \in H^k(\Omega) \times H_Q^{k+1}(\Omega)$, for any integer $k \geqslant \left[\frac{m}{2}\right] + 1$, admits a unique local solution (u,d) satisfying

$$\sup_{0 \leqslant t \leqslant T} (\|u\|_{H^k(\Omega)}^2 + \|\nabla d\|_{H_Q^k(\Omega)}^2) + \mu \int_0^T \|\nabla u\|_{H^k(\Omega)}^2 ds \leqslant C(T, \|u_0\|_{H^k(\Omega)}, \|\nabla d_0\|_{H_Q^k(\Omega)}).$$

The proof method of Theorem 1 mainly refers to Refs. [10,22-24]. We will prove the local existence of system with finite data by approximation of a perturbed parabolic system. By standard parabolic argument, it is easy to find that the perturbed parabolic system admits a local solution $(u_\varepsilon, d_\varepsilon)$ on some time interval $[0, T_\varepsilon)$ for every $\varepsilon > 0$. Then, we derive the uniform estimates of $(u_\varepsilon, d_\varepsilon)$ and a lower bound for the life span T_ε, and obtain the solution of on M and $\varepsilon \to 0$, where M is a flat closed m-dimensional Riemannian manifold.

The organization of this paper is the following. In Section 2, we recall some lemmas which should be used in the following sections. In Section 3, we apply the approximating scheme and obtain the uniform bound for energy and then give the proof of local existence. In Section 4, we prove the uniqueness of the solution by parallel transport.

2 Preliminaries

In this section, we introduce the interpolation inequality for sections on vector bundles, equivalent relation between $\|\nabla d\|_{H_Q^{k-1}}$ and $\|\nabla d\|_{\mathbf{H}^{k-1}}$, and some lemmas which should be used in the following sections.

Lemma 1[10] Suppose $s \in C^\infty(E)$ is a section where E is a vector bundle over a closed m-dimensional Riemannian manifold M. Then, we have

$$\|D^j s\|_{L^p(M)} \leqslant C \|s\|_{\mathbf{W}^{k,q}(M)}^a \|s\|_{L^r(M)}^{1-a},$$

where $1 \leqslant p,q,r \leqslant \infty$, and $j/k \leqslant a \leqslant 1$ ($j/k \leqslant a < 1$ if $q = m/(k-j) \neq 1$) are numbers such that

$$\frac{1}{p} = \frac{j}{m} + \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{k}{m}\right).$$

The constant C only depends on M and the number j,k,q,r, a. Moreover, if $M = \mathbb{T}^m = \mathbb{R}^m/(R \cdot \mathbb{Z})^m$, then the constant C does not depend on the diameter $R \geqslant 1$.

Lemma 2 (Gronwall inequality(differential form)).

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \leqslant \phi(t)\eta(t) + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \leqslant \exp\left\{\int_0^t \phi(s) ds\right\}\left[\eta(0) + \int_0^t \psi(s) ds\right]$$

for all $0 \leqslant t \leqslant T$.

(ii) Inparticular, if

$$\eta' \leqslant \phi\eta, \text{ on } [0,T], \text{ and } \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0,T].$$

Lemma 3[8] Let $f: R^+ \to R^+$ be a nondecreasing continuous function such that $f > 0$ on $(0, \infty)$ and $\int_1^\infty \frac{1}{f} dx < \infty$. Let y be a continuous function which is nonnegative on R^+ and let g be a nonnegative function in $L_{loc}^1(R^+)$. We assume that there exists a positive number $y_0 > 0$ such that for all $t > 0$, we have the inequality:

$$y(t) \leqslant y_0 + \int_0^t g(s) ds + \int_0^t f(y(s)) ds.$$

Then, there exists a positive number T^* depending only on

y_0 and f, such that for all $T<T^*$, there holds true

$$\sup_{0\leqslant t\leqslant T} y(t)\leqslant C(T,y_0)$$

for some constant $C(T,y_0)$.

Lemma 4[10]　Assume that $k>m/2$, (M,g) is a closed Riemannian manifold. Then, there exists a constant $C=C(\mathbb{S}^2,k)$ such that for all maps $\phi\in C^\infty(M,\mathbb{S}^2)$,

$$\|\nabla d\|_{H_Q^{k-1}(M)}\leqslant C\sum_{l=1}^{k}\|Dd\|^l_{\boldsymbol{H}^{k-1}(M)}$$

and

$$\|Dd\|_{\boldsymbol{H}^{k-1}(M)}\leqslant C\sum_{l=1}^{k}\|\nabla d\|^l_{H_Q^{k-1}(M)}.$$

Finally, we introduce the density property of Sobolev spaces $H_Q^k(\mathbb{R}^m,\mathbb{S}^2)$.

Lemma 5[10]　Let $k>m/2$ and $d\in H_Q^k(\mathbb{R}^m,\mathbb{S}^2)$. Then, there exists a sequence of map $d_i-Q\in H^k(\mathbb{R}^m,\mathbb{S}^2)\cap C_0^\infty(\mathbb{R}^m,\mathbb{R}^3)$ such that $d_i\to d$ in $H_Q^k(\mathbb{R}^m,\mathbb{S}^2)$.

3　Local existence

In this section, we will prove the local existence of smooth solutions of the initial value problem to the incompressible Navier-Stokes-Landau-Lifshitz equations with the Dzyaloshinskii-Moriya interaction and V-flow term

$$\begin{cases} u_t+u\cdot\nabla u+\nabla p=\mu\Delta u-\lambda\nabla\cdot(\nabla d\odot\nabla d),\\ d_t+u\cdot\nabla d+\alpha d\times(\Delta d+V\cdot\nabla d+\nabla\times d)+\\ \quad\beta d\times(d\times\Delta(d+V\cdot\nabla d+\nabla\times d))=d\times f,\\ \nabla\cdot u=0,\\ (u,d)|_{t=0}=(u_0,d_0)\in C^\infty(M\times M,\mathbb{R}^m\times\mathbb{S}^2), \end{cases}\tag{1}$$

where M is a flat closed m-dimensional Riemannian manifold. Then, we will use the smooth solutions (u_i,d_i) on $\mathbb{T}_i^{2m}=\mathbb{R}^{2m}/(2R_i\cdot\mathbb{Z})^{2m}$ to obtain the smooth solution of system and complete the proof of Theorem 1.

Since $(\mathbb{S}^2,J,h)$ is a compact Kähler manifold with complex structure J and Kähler metric h. Then, the operation $d\times F$ acts as a complex number in the tangent plane, rotating F by ninety degrees for any $F\in T_d\mathbb{S}^2$. Therefore, $J(d)F=d\times F$. Then, the term $d\times\Delta d$ can be rewritten as

$$J(d)D_k\partial_k d,$$

and since $|d|=1$ and d is the normal to the tangent plane $T_d\mathbb{S}^2$, we have

$$D_k(\partial_k d)=\Delta d-\langle\Delta d,d\rangle d=\Delta d+|\nabla d|^2 d=-d\times d\times\Delta d,$$

where we implicitly sum over repeated indices. Then, we can rewrite the equations and solve the following perturbed problem:

$$\begin{cases} u_t+u\cdot\nabla u+\nabla p=\mu\Delta u-\lambda\nabla\cdot(\nabla d\odot\nabla d),\\ \nabla\cdot u=0,\\ d_t=-u\cdot\nabla d-\alpha J(d)D_k\partial_k d-\\ \quad\alpha J(d)(V\cdot\nabla d+\nabla\times d)+(\varepsilon+\beta)D_k\partial_k d+\\ \quad\beta(V\cdot\nabla d+\nabla\times d)+J(d)f,\\ (u,d)|_{t=0}=(u_0,d_0)\in C^\infty(M\times M,\mathbb{R}^m\times\mathbb{S})^2, \end{cases}\tag{2}$$

where $\varepsilon>0$ small. Hence, we have

Lemma 6　Let $m_0=\left[\dfrac{m}{2}\right]+1=2$, and let $u_0\in C^\infty(M,\mathbb{R}^m)$, $d_0\in C^\infty(M,\mathbb{S}^2)$, $V\in L^2((0,T),H^{n+1})$, $f\in L^2((0,T),H^{n-1})$. There exists a constant $T=T(\mathbb{S}^2,\|u_0\|_{H^2(M)},\|\nabla d_0\|_{\boldsymbol{H}^2(M)})>0$, independent of $\varepsilon\in(0,1]$, such that if $(u,d)\in C^\infty(M\times[0,T_\varepsilon])$ is a solution of Eq. (2) with $\varepsilon\in(0,1]$, then

$$T_\varepsilon\geqslant T(\|u_0\|_{H^2(M)},\|\nabla d_0\|_{\boldsymbol{H}^2(M)})$$

and

$$\|u(t)\|^2_{H^k(M)}+\mu\|\nabla u\|^2_{L^2([0,T],H^k(M))}+\|\nabla d\|^2_{\boldsymbol{H}^k(M)}\leqslant C(k,\|u_0\|_{H^k(M)},\|\nabla d_0\|_{\boldsymbol{H}^k(M)}),t\in[0,T]$$

for all $k\geqslant m_0$.

Proof　For simplicity, we denote $(u,d):=(u_\varepsilon,d_\varepsilon)$ to be a solution of Eq. (2), and denote $H^l:=H^l(M)$, $\boldsymbol{H}^l:=\boldsymbol{H}^l(M)$ for any integer $l\geqslant 0$. Then it is easy to obtain that the energy

$$E(u,d):=\frac{1}{2}\|u\|^2_{L^2}+\frac{1}{2}\|\nabla d\|^2_{L^2}+\mu\int_0^t\|\nabla u\|^2_{L^2}\mathrm{d}s+\left(\frac{\beta}{4}+\varepsilon\right)\int_0^t\|D_k\partial_k d\|^2_{L^2}\mathrm{d}s$$

is uniformly bounded for $t\in[0,T_\varepsilon)$. In fact, multi-plying u − equation by u, and integrating by parts, we get

$$\langle u_t,u\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2},$$

$$\langle u\cdot\nabla u,u\rangle=\int_M u\cdot\nabla\left(\frac{1}{2}|u|^2\right)\mathrm{d}x=\int_M\nabla\cdot u\frac{1}{2}|u|^2\mathrm{d}x=0,$$

$$\langle\nabla P,u\rangle=-\langle P,\nabla\cdot u\rangle=0,$$

$$\langle\mu\Delta u,u\rangle=-\mu\|\nabla u\|^2_{L^2},$$

$$-\lambda\langle\nabla\cdot(\nabla d\odot\nabla d),u\rangle=-\lambda\int_M\sum_{i,j,k=1}^{3}\partial_i(\partial_i d_j\partial_k d_j)u_k\mathrm{d}x=\lambda\int_M\sum_{i,j,k=1}^{3}\partial_i d_j\partial_k d_j\partial_i u_k\mathrm{d}x.$$

Applying D to d - equation and multiplying the resulting identities by $\lambda\nabla d$, and we use the fact $D_k\partial_t d = D_t\partial_k d$, one gets

$$\lambda\langle\nabla d, D\partial_t d\rangle = \lambda\langle\nabla d, D_t\nabla d\rangle = \frac{1}{2}\lambda\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d\|_{L^2}^2,$$

$$-\lambda\langle D(u\cdot\nabla d),\nabla d\rangle = -\lambda\int_M\sum_{i,j,k=1}^{3}(\partial_k u_i\partial_i d_j\partial_k d_j + u_i D_i\partial_k d_j\partial_k d_j)\mathrm{d}x = -\lambda\int_M\sum_{i,j,k=1}^{3}\partial_k u_i\partial_i d_j\partial_k d_j\mathrm{d}x,$$

$$-\lambda\alpha\langle\nabla d, D(J(d)D_k\partial_k d)\rangle = -\lambda\alpha\int_M\sum_{i=1}^{3}\partial_i d\cdot D_i(J(d)D_k\partial_k d)\mathrm{d}x = \lambda\alpha\int_M D_k\partial_k d\cdot J(d)D_k\partial_k d\mathrm{d}x = 0,$$

$$-\lambda\alpha\langle\nabla d, D(J(d)(V\cdot\nabla d+\nabla\times d))\rangle = \lambda\alpha\int_M D_k\partial_k d\cdot J(d)(V\cdot\nabla d+\nabla\times d)\mathrm{d}x \leqslant \lambda\alpha\|D_k\partial_k d\|_{L^2}\|J(d)\|_{L^\infty}\|(V\cdot\nabla d+\nabla\times d)\|_{L^2} \leqslant \frac{\beta\lambda}{4}\|D_k\partial_k d\|_{L^2}^2 + \frac{\lambda\alpha^2}{\beta}(\|V\|_{L^\infty}^2+1)\|\nabla d\|_{L^2}^2,$$

$$\lambda\beta\langle\nabla d, D(D_k\partial_k d)\rangle = \lambda\beta\int_M\sum_{i=1}^{3}\partial_i d\cdot D_i(D_k\partial_k d)\mathrm{d}x = -\beta\lambda\|D_k\partial_k d\|_{L^2}^2,$$

$$\lambda\varepsilon\langle\nabla d, D(D_k\partial_k d)\rangle = \lambda\varepsilon\int_M\sum_{i=1}^{3}\partial_i d\cdot D_i(D_k\partial_k d)\mathrm{d}x = -\varepsilon\lambda\|D_k\partial_k d\|_{L^2}^2,$$

$$\lambda\beta\langle\nabla d, D(V\cdot\nabla d+\nabla\times d)\rangle = -\lambda\beta\int_M D_k\partial_k d\cdot(V\cdot\nabla d+\nabla\times d)\mathrm{d}x \leqslant \lambda\beta\|D_k\partial_k d\|_{L^2}\|(V\cdot\nabla d+\nabla\times d)\|_{L^2} \leqslant \frac{\beta\lambda}{4}\|D_k\partial_k d\|_{L^2}^2 + \lambda\beta(\|V\|_{L^\infty}^2+1)\|\nabla d\|_{L^2}^2,$$

$$\lambda\langle\nabla d, D(J(d)f)\rangle = -\lambda\int_M D_k\partial_k d\cdot J(d)f\mathrm{d}x \leqslant \lambda\|D_k\partial_k d\|_{L^2}\|J(d)\|_{L^\infty}\|f\|_{L^2} \leqslant \frac{\beta\lambda}{4}\|D_k\partial_k d\|_{L^2}^2 + \frac{\lambda}{\beta}\|f\|_{L^2}^2.$$

Combining the above equations and inequations yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \frac{1}{2}\lambda\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 + \left(\frac{\beta}{4}+\varepsilon\right)\lambda\|D_k\partial_k d\|_{L^2}^2 \leqslant \frac{\lambda(\alpha^2+\beta^2)}{\beta}(\|V\|_{L^\infty}^2+1)\|\nabla d\|_{L^2}^2 + \frac{\lambda}{\beta}\|f\|_{L^2}^2.$$

Then, using the Gronwall inequality leads to

$$\sup_{0\leqslant t\leqslant T_\varepsilon}\|u\|_{L^2}^2 + \lambda\sup_{0\leqslant t\leqslant T_\varepsilon}\|\nabla d\|_{L^2}^2 + 2\mu\int_0^{T_\varepsilon}\|\nabla u\|_{L^2}^2\mathrm{d}s + \left(\frac{\beta}{2}+2\varepsilon\right)\lambda\int_0^{T_\varepsilon}\|D_k\partial_k d\|_{L^2}^2\mathrm{d}t \leqslant \exp\left\{\frac{2(\alpha^2+\beta^2)}{\beta}\int_0^{T_\varepsilon}(\|V\|_{L^\infty}^2+1)\mathrm{d}s\right\}\times\left\{\|u_0\|_{L^2}^2 + \lambda\|\nabla d_0\|_{L^2}^2 + \frac{2\lambda}{\beta}\int_0^{T_\varepsilon}\|f\|_{L^2}^2\mathrm{d}s\right\}.$$

From the above equation we can obtain

$$E(u,d) \leqslant \sup_{0\leqslant t\leqslant T_\varepsilon}\|u\|_{L^2}^2 + \sup_{0\leqslant t\leqslant T_\varepsilon}\|\nabla d\|_{L^2}^2 + 2\mu\int_0^{T_\varepsilon}\|\nabla u\|_{L^2}^2\mathrm{d}s + \left(\frac{\beta}{2}+2\varepsilon\right)\int_0^{T_\varepsilon}\|D_k\partial_k d\|_{L^2}^2\mathrm{d}t \leqslant \left(1+\frac{1}{\lambda}\right)\exp\left\{\frac{2(\alpha^2+\beta^2)}{\beta}\int_0^{T_\varepsilon}(\|V\|_{L^\infty}^2+1)\mathrm{d}s\right\}\times\left\{\|u_0\|_{L^2}^2 + \lambda\|\nabla d_0\|_{L^2}^2 + \frac{2\lambda}{\beta}\int_0^{T_\varepsilon}\|f\|_{L^2}^2\mathrm{d}s\right\},$$

therefore, we obtain that $E(u,d)$ is bounded.

Next, we fix an $N\geqslant m_0$, and set n to be any integer with $1\leqslant n\leqslant N$ and expressed as derivative order. Suppose that $\boldsymbol{a}$ is a multi-index of length n, i.e., $\boldsymbol{a}=(a_1,\cdots,a_m)$. Then, for $t\leqslant T_\varepsilon$, we have the energy functional:

$$E_n(u,d) := \frac{1}{2}\|\nabla^a u\|_{L^2}^2 + \frac{1}{2}\|D^a\nabla d\|_{L^2}^2 + \mu\int_0^t\|\nabla^a\nabla u\|_{L^2}^2\mathrm{d}s + \left(\frac{\beta}{4}+\varepsilon\right)\int_0^t\|D^a D_k\partial_k d\|_{L^2}^2\mathrm{d}s.$$

Then by Eq. (2), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n = \int_M -\nabla^a u\cdot\nabla^a(u\cdot\nabla u + \lambda\partial_j(Dd\cdot\partial_j d))\mathrm{d}x + \int_M D^a\nabla d\cdot D_t D^a\nabla d\mathrm{d}x + \left(\frac{\beta}{4}+\varepsilon\right)\|D^a D_k\partial_k d\|_{L^2}^2 =: I + II + \left(\frac{\beta}{4}+\varepsilon\right)\|D^a D_k\partial_k d\|_{L^2}^2. \tag{3}$$

First, we estimate I. The following inequality (see Ref. [25] Proposition 3.7) should be used:

$$\|f\cdot g\|_{H^k} \leqslant C\|f\|_{L^\infty}\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_{L^\infty}.$$

Then, when $n\leqslant 2$, integration by parts, we have

(i) $n=1$

$$I \leqslant C\|u\|_{H^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2} + C\|\nabla u\|_{H^1}(\|\nabla d\|_{L^\infty}\|\nabla d\|_{\boldsymbol{H}^1} + \|\nabla d\|_{\boldsymbol{H}^1}\|\nabla d\|_{L^\infty}) \leqslant C\|\nabla u\|_{H^2}\|u\|_{H^2}^2 + C\|\nabla u\|_{H^2}\|\nabla d\|_{\boldsymbol{H}^2}^2,$$

(ii) $n=2$

$$I \leqslant C\|u\|_{H^2}((\|u\|_{L^\infty}\|\nabla u\|_{H^2} + \|u\|_{H^2}\|\nabla u\|_{L^\infty}) + C\|\nabla u\|_{H^2}(\|\nabla d\|_{L^\infty}\|\nabla d\|_{\boldsymbol{H}^2} + \|\nabla d\|_{\boldsymbol{H}^2}\|\nabla d\|_{L^\infty}) \leqslant C\|\nabla u\|_{H^2}\|u\|_{H^2}^2 + C\|\nabla u\|_{H^2}\|\nabla d\|_{\boldsymbol{H}^2}^2.$$

When $n\geqslant 3$, integration by parts, we have

$$I\leqslant C\|u\|_{H^n}(\|u\|_{L^\infty}\|\nabla u\|_{H^n}+\|u\|_{H^n}\|\nabla u\|_{L^\infty})+ C\|\nabla u\|_{H^n}(\|\nabla d\|_{L^\infty}\|\nabla d\|_{\boldsymbol{H}^n}+\|\nabla d\|_{\boldsymbol{H}^n}\|\nabla d\|_{L^\infty})\leqslant C\|\nabla u\|_{H^n}\|u\|_{H^n}^2+C\|\nabla u\|_{H^n}\|\nabla d\|_{\boldsymbol{H}^n}^2.$$

Thus, we have the bounds

$$I\leqslant\begin{cases}C\|\nabla u\|_{H^2}\|u\|_{H^2}^2+C\|\nabla u\|_{H^2}\|\nabla d\|_{\boldsymbol{H}^2}^2,\ n\leqslant 2,\\ C\|\nabla u\|_{H^n}\|u\|_{H^n}^2+C\|\nabla u\|_{H^n}\|\nabla d\|_{\boldsymbol{H}^n}^2,\ n\geqslant 3.\end{cases}\tag{4}$$

Next, we estimate the term II. Exchanging the order of covariant differentiation, we obtain

$$D_tD^{\boldsymbol{a}}\partial_i d=D^{\boldsymbol{a}}D_i\partial_t d+[D_t,D^{\boldsymbol{a}}D_i]d=D^{\boldsymbol{a}}D_i\partial_t d+\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}\partial_t d)D^{\boldsymbol{e}}\partial_i d,\tag{5}$$

where R is the curvature tensor, and over the sum is over all multi-indices $\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ and $\boldsymbol{e}$ with possible zero lengths, except that $|\boldsymbol{c}|>0$ always holds, such that

$$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e})=\sigma(\boldsymbol{a})$$

is a permutation of $\boldsymbol{a}$. Replacing $\partial_t d$ in the second term by the righthand side of d − equation in Eq. (2), the second term can be rewritten as

$$\begin{aligned}&\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}\partial_t d)D^{\boldsymbol{e}}\partial_i d=\\&\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}((\varepsilon+\beta)D_k\partial_k d-\alpha J(d)D_k\partial_k d))D^{\boldsymbol{e}}\partial_i d-\\&\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}(u\cdot\nabla d))D^{\boldsymbol{e}}\partial_i d+\\&\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}(\beta(V\cdot\nabla d+\nabla\times d)-\alpha J(d)(V\cdot\nabla d+\nabla\times d)))D^{\boldsymbol{e}}\partial_i d+\\&\sum D^{\boldsymbol{b}}R(d)(D^{\boldsymbol{c}}d,D^{\boldsymbol{d}}(J(d)f))D^{\boldsymbol{e}}\partial_i d=:\\&Q_1+Q_2+Q_3+Q_4,\end{aligned}\tag{6}$$

and we have

$$|Q_1|\lesssim\sum_{(j_1,\cdots,j_s)\in J_1}|D^{j_1}d|\cdots|D^{j_s}d|,\tag{7}$$

where

$$J_1:=\{j_1,\cdots,j_s\in\mathbb{N}:\ j_1\geqslant j_2\geqslant\cdots\geqslant j_s,\ n+1\geqslant j_i\geqslant 1,j_1+\cdots+j_s=n+3,s\geqslant 3\}.\tag{8}$$

Similarly, we also have

$$|Q_2|\lesssim\sum_{(j_0,\cdots,j_s)\in J_2}|\partial^{j_0}u||D^{j_1}d|\cdots|D^{j_s}d|,\tag{9}$$

where

$$J_2:=\{j_0,\cdots,j_s\in\mathbb{N}:j_1\geqslant j_2\geqslant\cdots\geqslant j_s,\ j_0+\cdots+j_s=n+2,s\geqslant 3,n-1\geqslant j_0\geqslant 0,\ n\geqslant j_i\geqslant 1,\text{ for } s\geqslant i\geqslant 1\}.\tag{10}$$

$$|Q_3|\lesssim\sum_{(j_0,\cdots,j_s)\in J_3}|\partial^{j_0}V||D^{j_1}d|\cdots|D^{j_s}d|+\sum_{(j_1,\cdots,j_s)\in J_3'}|D^{j_1}d|\cdots|D^{j_s}d|,\tag{11}$$

where

$$J_3:=\{j_0,\cdots,j_s\in\mathbb{N}:j_1\geqslant j_2\geqslant\cdots\geqslant j_s,\ j_0+\cdots+j_s=n+2,s\geqslant 3,n-1\geqslant j_0\geqslant 0,\ n\geqslant j_i\geqslant 1,\text{ for } s\geqslant i\geqslant 1\},\tag{12}$$

$$J_3':=\{j_1,\cdots,j_s\in\mathbb{N}:j_1\geqslant j_2\geqslant\cdots\geqslant j_s,\ n\geqslant j_i\geqslant 1,j_1+\cdots+j_s=n+2,s\geqslant 3\}.\tag{13}$$

$$|Q_4|\lesssim\sum_{(j_0,\cdots,j_s)\in J_4}|\partial^{j_0}f||D^{j_1}d|\cdots|D^{j_s}d|,\tag{14}$$

where

$$J_4:=\{j_0,\cdots,j_s\in\mathbb{N}:j_1\geqslant j_2\geqslant\cdots\geqslant j_s,\ j_0+\cdots+j_s=n+1,s\geqslant 2,n-1\geqslant j_0\geqslant 0,\ n\geqslant j_i\geqslant 1,\text{ for } s\geqslant i\geqslant 1\}.\tag{15}$$

For the first term in the righthand side of Eq. (5), we can use Eq. (2) to obtain

$$\begin{aligned}&D^{\boldsymbol{a}}D_i\partial_t d=D^{\boldsymbol{a}}D_i((\varepsilon+\beta)D_k\partial_k d-\alpha J(d)D_k\partial_k d-\\&u\cdot\nabla d-\alpha J(d)(V\cdot\nabla d+\nabla\times d)+\\&\beta(V\cdot\nabla d+\nabla\times d)+J(d)f)=\\&(\varepsilon+\beta)D_kD_kD^{\boldsymbol{a}}\partial_i d-\alpha J(d)D_kD_kD^{\boldsymbol{a}}\partial_i d+\\&u\cdot DD^{\boldsymbol{a}}\partial_i d+\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}\nabla^{\boldsymbol{b}}\partial_i u\cdot D^{\boldsymbol{c}}\nabla d+\\&\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a}),\ |\boldsymbol{b}\geqslant 1|}\nabla^{\boldsymbol{b}}u\cdot D^{\boldsymbol{c}}D_i\nabla d+Q_5+Q_6-\\&\alpha\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}J(d)\nabla^{\boldsymbol{b}}\partial_i V\cdot D^{\boldsymbol{c}}\nabla d-\\&\alpha\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a}),\ |\boldsymbol{b}\geqslant 1|}J(d)\nabla^{\boldsymbol{b}}V\cdot D^{\boldsymbol{c}}D_i\nabla d-\\&\alpha J(d)V\cdot D^{\boldsymbol{a}}D_i\nabla d-\alpha J(d)D^{\boldsymbol{a}}D_i(\nabla\times d)+\\&\beta D^{\boldsymbol{a}}D_i(\nabla\times d)+\beta V\cdot D^{\boldsymbol{a}}D_i\nabla d+\\&\beta\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}\nabla^{\boldsymbol{b}}\partial_i V\cdot D^{\boldsymbol{c}}\nabla d+\\&\beta\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a}),\ |\boldsymbol{b}\geqslant 1|}\nabla^{\boldsymbol{b}}V\cdot D^{\boldsymbol{c}}D_i\nabla d+J(d)D^{\boldsymbol{a}}D_i f,\end{aligned}\tag{16}$$

where Q_5, Q_6 satisfy Eqs. (7) and (9), respectively.

Thus, we can obtain from Eqs. (5), (6) and (16):

$$\begin{aligned}D_tD^{\boldsymbol{a}}\partial_i d=&(\varepsilon+\beta)D_kD_kD^{\boldsymbol{a}}\partial_i d-\alpha J(d)D_kD_kD^{\boldsymbol{a}}\partial_i d+\\&u\cdot DD^{\boldsymbol{a}}\partial_i d+\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}\nabla^{\boldsymbol{b}}\partial_i u\cdot D^{\boldsymbol{c}}\nabla d+\\&\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a}),\ |\boldsymbol{b}\geqslant 1|}\nabla^{\boldsymbol{b}}u\cdot D^{\boldsymbol{c}}D_i\nabla d-\\&\alpha\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}J(d)\nabla^{\boldsymbol{b}}\partial_i V\cdot D^{\boldsymbol{c}}\nabla d-\\&\alpha\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a}),\ |\boldsymbol{b}\geqslant 1|}J(d)\nabla^{\boldsymbol{b}}V\cdot D^{\boldsymbol{c}}D_i\nabla d-\\&\alpha J(d)V\cdot D^{\boldsymbol{a}}D_i\nabla d-\alpha J(d)D^{\boldsymbol{a}}D_i(\nabla\times d)+\\&\beta D^{\boldsymbol{a}}D_i(\nabla\times d)+\beta V\cdot D^{\boldsymbol{a}}D_i\nabla d+\\&\beta\sum_{(\boldsymbol{b},\boldsymbol{c})=\sigma(\boldsymbol{a})}\nabla^{\boldsymbol{b}}\partial_i V\cdot D^{\boldsymbol{c}}\nabla d+\end{aligned}$$

$$\beta \sum_{(b,c)=\sigma(a),\ |b|\geqslant 1} \nabla^b V\cdot D^c D_i \nabla d + J(d)D^a D_i f + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6.$$

Substituting this into II in and by integrating parts, we get

$$\begin{aligned} II = &\int_M -(\varepsilon+\beta)|D_k D^a \partial_i d|^2 + D_k D^a \partial_i d\cdot \alpha J(d) D_k D^a \partial_i d + \\ &D^a \partial_i d\cdot u\cdot D D^a \partial_i d \mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot \Big(\sum_{(b,c)=\sigma(a)} \nabla^b \partial_i u\cdot D^c \nabla d + \\ &\sum_{(b,c)=\sigma(a),\ |b|\geqslant 1} \nabla^b u\cdot D^c D_i \nabla d\Big)\mathrm{d}x - \\ &\int_M D^a \partial_i d\cdot \Big(\alpha \sum_{(b,c)=\sigma(a)} J(d)\nabla^b \partial_i V\cdot D^c \nabla d - \\ &\alpha \sum_{(b,c)=\sigma(a),\ |b|\geqslant 1} J(d)\nabla^b V\cdot D^c D_i \nabla d\Big)\mathrm{d}x - \\ &\int_M D^a \partial_i d\cdot (\alpha J(d) V\cdot D^a D_i \nabla d)\mathrm{d}x - \\ &\int_M D^a \partial_i d\cdot \alpha J(d) D^a D_i(\nabla\times d)\mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot \Big(\beta \sum_{(b,c)=\sigma(a)} \nabla^b \partial_i V\cdot D^c \nabla d + \\ &\beta \sum_{(b,c)=\sigma(a),\ |b|\geqslant 1} \nabla^b V\cdot D^c D_i \nabla d\Big)\mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot (\beta V\cdot D^a D_i \nabla d)\mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot \beta D^a D_i(\nabla\times d)\mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot (Q_1+Q_5)\mathrm{d}x + \int_M D^a \partial_i d\cdot (Q_2+Q_6)\mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot Q_3 \mathrm{d}x + \int_M D^a \partial_i d\cdot Q_4 \mathrm{d}x + \\ &\int_M D^a \partial_i d\cdot J(d) D^a D_i f \mathrm{d}x. \end{aligned}$$

For the first integrand, the second term vanishes since complex structure J and the last term vanishes since $\nabla\cdot u = 0$. Then by Eqs. (7), (9), (11) and (14), we get

$$\begin{aligned} II \leqslant &\int_M -(\varepsilon+\beta)|D_k D^a \partial_i d|^2 \mathrm{d}x + \\ &\sum_{n_1+n_2=n+1,\ n_1\geqslant 1} \int_M |D^{n+1} d||\nabla^{n_1} u||D^{n_2}\nabla d|\mathrm{d}x + \\ &(\alpha+\beta)\sum_{n_3+n_4=n+1,\ n_3\geqslant 1} \int_M |D^{n+1} d||\nabla^{n_3} V||D^{n_4}\nabla d|\mathrm{d}x + \\ &(\alpha+\beta)\int_M |D^{n+1} d||V||D^{n+2}\nabla d|\mathrm{d}x + \\ &(\alpha+\beta)\int_M |D^{n+1} d||D^{n+2}\nabla d|\mathrm{d}x + \\ &\sum_{(j_1,\cdots,j_s)\in J_1} \int_M |D^{n+1} d||D^{j_1} d|\cdots|D^{j_s} d|\mathrm{d}x + \\ &\sum_{(j_0,j_1,\cdots,j_s)\in J_2} \int_M |D^{n+1} d||\partial_{j_0} u||D^{j_1} d|\cdots|D^{j_s} d|\mathrm{d}x + \\ &\sum_{(j_0,j_1,\cdots,j_s)\in J_3} \int_M |D^{n+1} d||\partial_{j_0} V||D^{j_1} d|\cdots|D^{j_s} d|\mathrm{d}x + \\ &\sum_{(j_1,\cdots,j_s)\in J_3'} \int_M |D^{n+1} d||D^{j_1} d|\cdots|D^{j_s} d|\mathrm{d}x + \\ &\sum_{(j_0,j_1,\cdots,j_s)\in J_4} \int_M |D^{n+1} d||\partial_{j_0} f||D^{j_1} d|\cdots|D^{j_s} d|\mathrm{d}x =: \\ &II_0 + II_1 + II_2 + II_3 + II_4 + II_5 + II_6 + II_7 + II_8 + II_9. \end{aligned}$$

We shall estimate each term of the righthand of the above inequality.

Step 1: We estimate the term II_1.

By Hölder's inequality, Sobolev embedding and Lemma 1, when $n\leqslant 2$, we have

(i) $n=1$

$$\begin{aligned} II_1 \leqslant &\|D^2 d\|_{L^2}(\|\nabla^2 u\|_{L^2}\|\nabla d\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|D\nabla d\|_{L^2}) \leqslant \\ &C\|\nabla d\|_{H^1}(\|\nabla u\|_{H^1}\|\nabla d\|_{H^2} + \|\nabla u\|_{H^2}\|\nabla d\|_{H^1}) \leqslant \\ &C\|\nabla d\|_{H^2}^2\|\nabla u\|_{H^2}; \end{aligned}$$

(ii) $n=2$

$$\begin{aligned} II_1 \leqslant &\|D^3 d\|_{L^2}(\|\nabla^3 u\|_{L^2}\|\nabla d\|_{L^\infty} + \|\nabla^2 u\|_{L^4}\|D\nabla d\|_{L^4} + \\ &\|\nabla u\|_{L^\infty}\|D^2\nabla d\|_{L^2}) \leqslant C\|\nabla d\|_{H^2}^2\|\nabla u\|_{H^2}; \end{aligned}$$

when $n\geqslant 3$, we have

$$\begin{aligned} II_1 \leqslant &\|D^{n+1} d\|_{L^2}\Big(\sum_{1\leqslant n_1\leqslant \frac{n+1}{2}} \|\nabla^{n_1} u\|_{L^\infty}\|D^{n+1-n_1}\nabla d\|_{L^2} + \\ &\sum_{\frac{n+1}{2}\leqslant n_1\leqslant n+1} \|\nabla^{n_1} u\|_{L^2}\|D^{n+1-n_1}\nabla d\|_{L^\infty}\Big) \leqslant \\ &C\|\nabla d\|_{H^n}^2\|\nabla u\|_{H^n}. \end{aligned}$$

From the above discussion, we prove the bound

$$II_1 \leqslant \begin{cases} C\|\nabla d\|_{H^2}^2\|\nabla u\|_{H^2}, & \text{if } n\leqslant 2, \\ C\|\nabla d\|_{H^n}^2\|\nabla u\|_{H^n}, & \text{if } n\geqslant 3. \end{cases} \tag{17}$$

Step 2: We estimate the term II_2.

Similar to II_1, we have

$$II_2 \leqslant \begin{cases} C\|\nabla d\|_{H^2}^2\|V\|_{H^3}, & \text{if } n\leqslant 2, \\ C\|\nabla d\|_{H^n}^2\|V\|_{H^{n+1}}, & \text{if } n\geqslant 3, \end{cases} \tag{18}$$

where the constant C depends on n.

Step 3: We estimate the term II_3.

By Höder's inequality, when $n\leqslant 2$, we have

(i) $n=1$

$$\begin{aligned} II_3 \leqslant &C\|D^2 d\|_{L^2}\|V\|_{L^\infty}\|D^3 d\|_{L^2} \leqslant \\ &C\|\nabla d\|_{H^1}\|V\|_{L^\infty}\|\nabla d\|_{H^2} \leqslant \\ &C\|\nabla d\|_{H^2}\|V\|_{L^\infty}\|\nabla d\|_{H^3}; \end{aligned}$$

(ii) $n=2$

$$II_3 \leqslant C\|D^3 d\|_{L^2}\|V\|_{L^\infty}\|D^4 d\|_{L^2} \leqslant C\|\nabla d\|_{H^2}\|V\|_{L^\infty}\|\nabla d\|_{H^3};$$

when $n\geqslant 3$, we have

$$II_3 \leqslant C\|D^{n+1} d\|_{L^2}\|V\|_{L^\infty}\|D^{n+2} d\|_{L^2} \leqslant C\|\nabla d\|_{H^n}\|V\|_{L^\infty}\|\nabla d\|_{H^{n+1}}.$$

From the above discussion, we prove the bound

$$II_3 \leqslant \begin{cases} C\|\nabla d\|_{H^2}\|V\|_{L^\infty}\|\nabla d\|_{H^3}, & \text{if } n\leqslant 2,\\ C\|\nabla d\|_{H^n}\|V\|_{L^\infty}\|\nabla d\|_{H^{n+1}}, & \text{if } n\geqslant 3.\end{cases} \tag{19}$$

Step 4: We estimate the term II_4.

It is easy to obtain its bound by Hölder's inequality:

$$II_4 \leqslant \begin{cases} C\|\nabla d\|_{H^2}\|\nabla d\|_{H^3}, & \text{if } n\leqslant 2,\\ C\|\nabla d\|_{H^n}\|\nabla d\|_{H^{n+1}}, & \text{if } n\geqslant 3.\end{cases} \tag{20}$$

Step 5: We estimate the term II_5.

We note that the integral is the same as Eq. (3.10) in Ref. [10]. Hence, its bound is obtained immediately by the following lemma which was proved in Ref. [10].

Lemma 7[10] If $1\leqslant n\leqslant 2$, then there exists a constant $C=C(M,n)$ such that

$$II_5 \leqslant C\|\nabla d\|_{H^2}^A\|\nabla d\|_{L^2}^B\|\nabla^{n+1} d\|_{L^2},$$

where $A=\left[n+3+\left(\frac{m}{2}-1\right)s-\frac{m}{2}\right]$ and $B=s-A$.

If $n\geqslant 3$, then there exists a constant $C=C(M,n)$ such that

(i) If $j_1=n+1$,

$$II_5 \leqslant C\|\nabla^{n+1} d\|_{L^2}^2\|\nabla d\|_{H^{m_0}}^{m/m_0}\|\nabla d\|_{L^2}^{2-m/m_0}.$$

(ii) If $j_1\leqslant n$,

$$II_5 \leqslant C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}}^A),$$

where $A=A(m,n)$.

Then, we have the bound

$$II_5 \leqslant \begin{cases} C\sum_{s=3}^{n+3}\|\nabla d\|_{H^2}^{s+1}, & \text{if } n\leqslant 2,\\ C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}})^{n+2}, & \text{if } n\geqslant 3.\end{cases} \tag{21}$$

Step 6: We estimate the term II_6.

When $n\leqslant 2$, by Eq. (10), we note that

$$j_1\leqslant 2, j_2,\cdots,j_s\leqslant 1.$$

Then, by Hölder's inequality, we have

$$II_6 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_2}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}u\|_{L^\infty}\|D^{j_1}d\|_{L^2}\cdots\|D^{j_s}d\|_{L^\infty} \leqslant C\|\nabla d\|_{H^2}^{s+1}\|\nabla u\|_{H^2}.$$

When $n\geqslant 3$, we have the following three situations.

(i) If $j_1=n$, by Eq. (10), we know that $s=3$, $j_0=0$, and $j_2=j_3=1$, then, using Hölder's inequality, we estimate II_6 by

$$II_6 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_2, j_1=n}\|D^{n+1}d\|_{L^2}\|u\|_{L^\infty}\|D^n d\|_{L^2}\|Dd\|_{L^\infty}^2 \leqslant C\|\nabla d\|_{H^n}\|u\|_{H^2}\|\nabla d\|_{H^{n-1}}^3.$$

(ii) If $j_1\leqslant n-1$, $j_0\leqslant[n/2]$, from Eq. (10), we obtain

$$j_2\leqslant n-1, j_3,\cdots,j_s\leqslant n-2,$$

then, using Hölder's inequality, II_6 can be bounded by

$$\begin{aligned} II_6 \leqslant & \sum_{(j_0,j_1,\cdots,j_s)\in J_2, j_1\leqslant n-1, j_0\leqslant[n/2]}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}u\|_{L^4}\times \\ & \|D^{j_1}d\|_{L^4}\|D^{j_2}d\|_{L^\infty}\cdots\|D^{j_s}d\|_{L^\infty} \leqslant \\ & C\|\nabla d\|_{H^n}\|u\|_{H^{[n/2]+1}}\|\nabla d\|_{H^{n-1}}\|\nabla d\|_{H^n}\|\nabla d\|_{H^{n-1}}^{s-2} \leqslant \\ & C\|\nabla d\|_{H^n}^2\|u\|_{H^{n-1}}\|\nabla d\|_{H^{n-1}}^{s-1}.\end{aligned}$$

(iii) If $j_1\leqslant n-1$, $j_0>[n/2]$, by Eq. (10), we have

$$j_1,\cdots,j_s\leqslant n-2,$$

then, one get

$$\begin{aligned} II_6 \leqslant & \sum_{(j_0,j_1,\cdots,j_s)\in J_2, j_1\leqslant n-1, j_0>\left[\frac{n}{2}\right]}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}u\|_{L^2}\times \\ & \|D^{j_1}d\|_{L^\infty}\cdots\|D^{j_s}d\|_{L^\infty} \leqslant C\|\nabla d\|_{H^n}\|u\|_{H^{n-1}}\|\nabla d\|_{H^{n-1}}^s.\end{aligned}$$

Then, combined with the above estimates, we prove the following bound:

$$II_6 \leqslant \begin{cases} C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1}\|\nabla u\|_{H^2}, & \text{if } n\leqslant 2,\\ C\|\nabla d\|_{H^n}^2(1+\|u\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+2}, & \text{if } n\geqslant 3.\end{cases} \tag{22}$$

Step 7: We estimate the term II_7.

We note that the integral is similar to II_6, then we have

$$II_7 \leqslant \begin{cases} C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1}\|V\|_{H^3}, & \text{if } n\leqslant 2,\\ C\|\nabla d\|_{H^n}^2(1+\|V\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+2}, & \text{if } n\geqslant 3.\end{cases} \tag{23}$$

Step 8: We estimate the term II_8.

By the similar discussion to II_5, we have the following lemma.

Lemma 8 If $1\leqslant n\leqslant 2$, then there exists a constant $C=C(M,n)$ such that

$$II_8 \leqslant C\|\nabla d\|_{H^2}^A\|\nabla d\|_{L^2}^B\|\nabla^{n+1} d\|_{L^2},$$

where $A=\left[n+2+\left(\frac{m}{2}-1\right)s-\frac{m}{2}\right]$ and $B=s-A$.

If $n\geqslant 3$, then there exists a constant $C=C(M,n)$ such that

$$II_8 \leqslant C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}}^A),$$

where $A=A(m,n)$.

Proof The proof of the lemma is similar to the proof of Lemmas 3.2 and 3.3 (ii) in Ref. [10].

Then, we have the bound

$$II_8 \leqslant \begin{cases} C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1}, & \text{if } n\leqslant 2, \\ C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}})^{n+1}, & \text{if } n\geqslant 3. \end{cases} \quad (24)$$

Step 9: We estimate the term II_9.

We refer to the estimates of II_6. When $n\leqslant 2$, and by J_4, we have

$$j_1\leqslant 2, j_2,\cdots,j_s\leqslant 1.$$

Then by Hölder and Lemma 1, we may estimate II_9 by

$$II_9 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_4}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}f\|_{L^\infty}\times$$
$$\|D^{j_1}d\|_{L^2}\cdots\|D^{j_s}d\|_{L^\infty}\leqslant C\|\nabla d\|_{H^2}\|f\|_{H^3}\|\nabla d\|_{H^2}^s.$$

When $n\geqslant 3$. First, if $j_1=n$, J_4 implies $s=2$, $j_0=0$ and $j_2=1$, then we may use Hölder's inequality and Lemma 1 to bound II_9 by

$$II_9 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_4, j_1=n}\|D^{n+1}d\|_{L^2}\|f\|_{L^\infty}\|D^n d\|_{L^2}\|Dd\|_{L^\infty}\leqslant$$
$$C\|\nabla d\|_{H^n}\|f\|_{H^2}\|\nabla d\|_{H^{n-1}}^2.$$

Second, if $j_1\leqslant n-1$, $j_0\leqslant[n/2]$, from J_4, we obtain

$$j_2,\cdots,j_s\leqslant n-2,$$

then by Hölder's inequality and Lemma 1, II_9 can be bounded by

$$II_9 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_4, j_1\leqslant n-1, j_0\leqslant[\frac{n}{2}]}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}f\|_{L^4}\times$$
$$\|D^{j_1}d\|_{L^4}\|D^{j_2}d\|_{L^\infty}\cdots\|D^{j_s}d\|_{L^\infty}\leqslant$$
$$C\|\nabla d\|_{H^n}\|f\|_{H^{[n/2]+1}}\|\nabla d\|_{H^{n-1}}\|\nabla d\|_{H^{n-1}}^{s-1}\leqslant$$
$$C\|\nabla d\|_{H^n}\|f\|_{H^{n-1}}\|\nabla d\|_{H^{n-1}}^s.$$

Finally, we consider the remainder case $j_0>[n/2]$. By J_4, we get

$$j_1,\cdots,j_s\leqslant n-2.$$

Then, it follows from Hölder's inequality and Lemma 1 that

$$II_9 \leqslant \sum_{(j_0,j_1,\cdots,j_s)\in J_4, j_1\leqslant n-1, j_0>[\frac{n}{2}]}\|D^{n+1}d\|_{L^2}\|\partial^{j_0}f\|_{L^2}\times$$
$$\|D^{j_1}d\|_{L^\infty}\cdots\|D^{j_s}d\|_{L^\infty}\leqslant C\|\nabla d\|_{H^n}\|f\|_{H^{n-1}}\|\nabla d\|_{H^{n-1}}^s.$$

From the above discussion, we get

$$II_9 \leqslant \begin{cases} C\sum_{s=2}^{n+1}\|\nabla d\|_{H^2}^{s+1}\|f\|_{H^3}, & \text{if } n\leqslant 2, \\ C\|\nabla d\|_{H^n}^2(1+\|f\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+1}, & \text{if } n\geqslant 3. \end{cases} \quad (25)$$

Thus, from the above discussion and using Young's inequality, we obtain the bound of the term II when $1\leqslant n\leqslant 2$,

$$\begin{aligned} II \leqslant & -(\varepsilon+\beta)\|\nabla d\|_{H^3}^2 + C\|\nabla d\|_{H^2}^2\|\nabla u\|_{H^2} + \\ & C\|\nabla d\|_{H^2}^2\|V\|_{H^3} + C\|\nabla d\|_{H^2}\|V\|_{L^\infty}\|\nabla d\|_{H^3} + \\ & C\|\nabla d\|_{H^2}\|\nabla d\|_{H^3} + C\sum_{s=3}^{n+3}\|\nabla d\|_{H^2}^{s+1} + \\ & C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1}\|\nabla u\|_{H^2} + C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1}\|\nabla V\|_{H^2} + \\ & C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1} + C\sum_{s=2}^{n+1}\|\nabla d\|_{H^2}^{s+1}\|f\|_{H^3} \leqslant \\ & -\left(\varepsilon+\frac{\beta}{4}\right)\|\nabla d\|_{H^3}^2 + \frac{\mu}{4}\|\nabla u\|_{H^2}^2 + C\|\nabla d\|_{H^2}^4 + \\ & C\|V\|_{H^3}^2 + C(1+\|V\|_{L^\infty}^2)\|\nabla d\|_{H^2}^2 + \\ & C\sum_{s=3}^{n+3}\|\nabla d\|_{H^2}^{s+1} + C\sum_{s=3}^{n+2}(\|\nabla d\|_{H^2}^{s+1})^2 + \\ & C\sum_{s=3}^{n+2}\|\nabla d\|_{H^2}^{s+1} + C\sum_{s=2}^{n+1}(\|\nabla d\|_{H^2}^{s+1})^2 + Cf_{H^3}^2 \leqslant \\ & -\left(\varepsilon+\frac{\beta}{4}\right)\|\nabla d\|_{H^3}^2 + \frac{\mu}{4}\|\nabla u\|_{H^2}^2 + \\ & C(1+\|V\|_{L^\infty}^2)(1+\|\nabla d\|_{H^2}^2)^5 + \\ & C\|V\|_{H^3}^2 + C\|f\|_{H^3}^2, \end{aligned} \quad (26)$$

and when $n\geqslant 3$

$$\begin{aligned} II \leqslant & -(\varepsilon+\beta)\|\nabla d\|_{H^{n+1}}^2 + C\|\nabla d\|_{H^n}^2\|\nabla u\|_{H^n} + \\ & C\|\nabla d\|_{H^n}^2\|V\|_{H^{n+1}} + C\|\nabla d\|_{H^n}\|V\|_{L^\infty}\|\nabla d\|_{H^{n+1}} + \\ & C\|\nabla d\|_{H^n}\|\nabla d\|_{H^{n+1}} + \\ & C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}})^{n+2} + \\ & C\|\nabla d\|_{H^n}^2(1+\|u\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+2} + \\ & C\|\nabla d\|_{H^n}^2(1+\|V\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+2} + \\ & C(1+\|\nabla d\|_{H^n}^2)(1+\|\nabla d\|_{H^{n-1}})^{n+1} + \\ & C\|\nabla d\|_{H^n}^2((1+\|f\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+1} \leqslant \\ & -\left(\varepsilon+\frac{\beta}{4}\right)\|\nabla d\|_{H^{n+1}}^2 + \frac{\mu}{4}\|\nabla u\|_{H^n}^2 + C\|V\|_{H^{n+1}}^2 + \\ & C(1+\|V\|_{L^\infty}^2)\|\nabla d\|_{H^n}^2 + \\ & C(1+\|\nabla d\|_{H^n}^2)^2(1+\|V\|_{H^{n-1}} + \\ & \|f\|_{H^{n-1}}+\|u\|_{H^{n-1}}+\|\nabla d\|_{H^{n-1}})^{n+2}. \end{aligned} \quad (27)$$

Now, we return to bound the energy of u and d. We first consider the case $1\leqslant n\leqslant 2$. Then, Eq. (3), together with Eqs. (4) and (26), we gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)+$$
$$\mu\|\nabla u\|_{H^2}^2+\left(\frac{\beta}{4}+\varepsilon\right)\|\nabla d\|_{\boldsymbol{H}^3}^2\leqslant$$
$$C\|\nabla u\|_{H^2}(\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)+\frac{\mu}{4}\|\nabla u\|_{H^2}^2+$$
$$C(1+\|V\|_{L^\infty}^2)(1+\|\nabla d\|_{\boldsymbol{H}^2}^2)^5+C\|V\|_{H^3}^2+$$
$$C\|\nabla f\|_{H^2}^2\leqslant\frac{\mu}{2}\|\nabla u\|_{H^2}^2+$$
$$C(1+\|V\|_{L^\infty}^2)(1+\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)^5+$$
$$C\|V\|_{H^3}^2+C\|\nabla f\|_{H^2}^2.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)+\mu\|\nabla u\|_{H^2}^2+\left(\frac{\beta}{2}+2\varepsilon\right)\|\nabla d\|_{\boldsymbol{H}^3}^2\leqslant$$
$$C(1+\|V\|_{L^\infty}^2)(1+\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)^5+$$
$$C(\|V\|_{H^3}^2+\|\nabla f\|_{H^2}^2).\tag{28}$$

Using the comparison lemma, we obtain that there exists a time $T^*(\mathbb{S}^2,\|u_0\|_{H^2(M)},\|\nabla d_0\|_{\boldsymbol{H}^2(M)})$, and $K_2(T,\|u_0\|_{H^2(M)},\|\Delta d_0\|_{\boldsymbol{H}^2(M)})>0$ such that for any $T<T^*$, we have

$$\sup_{0\leqslant t\leqslant T}(\|u\|_{H^2}^2+\|\nabla d\|_{\boldsymbol{H}^2}^2)+\mu\int_0^T\|\nabla u\|_{H^2}^2\mathrm{d}s+$$
$$\left(\frac{\beta}{2}+2\varepsilon\right)\int_0^T\|\nabla d\|_{\boldsymbol{H}^3}^2\mathrm{d}s\leqslant K_2.\tag{29}$$

For the higher-order energy of u and d. Then, Eq. (3), together with Eqs. (4) and (27), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^n}^2+\|\nabla d\|_{\boldsymbol{H}^n}^2)+\mu\|\nabla u\|_{H^n}^2+$$
$$\left(\frac{\beta}{4}+\varepsilon\right)\|\nabla d\|_{\boldsymbol{H}^{n+1}}^2\leqslant$$
$$C\|\nabla u\|_{H^n}(\|u\|_{H^n}+\|\nabla d\|_{\boldsymbol{H}^n}^2)+$$
$$\frac{\mu}{4}\|\nabla u\|_{H^n}^2+C\|V\|_{H^{n+1}}^2+C(1+\|V\|_{L^\infty}^2)\|\nabla d\|_{\boldsymbol{H}^n}^2+$$
$$C(1+\|\nabla d\|_{\boldsymbol{H}^n}^2)^2(1+\|V\|_{H^{n-1}}+\|f\|_{H^{n-1}}+\|u\|_{H^{n-1}}+$$
$$\|\nabla d\|_{\boldsymbol{H}^{n-1}})^{n+2}\leqslant\frac{\mu}{2}\|\nabla u\|_{H^n}^2+C\|V\|_{H^{n+1}}^2+$$
$$C(1+\|V\|_{L^\infty}^2)(1+\|u\|_{H^n}^2+\|\nabla d\|_{\boldsymbol{H}^n}^2)^2(1+\|V\|_{H^{n-1}}+$$
$$\|f\|_{H^{n-1}}+\|u\|_{H^{n-1}}+\|\nabla d\|_{\boldsymbol{H}^{n-1}})^{n+2}.$$

From Eq. (29), we may assume that for any $2\leqslant l\leqslant n-1$, there exists $K_l>0$ such that

$$\|u\|_{H^l}^2+\|\nabla d\|_{\boldsymbol{H}^l}^2+\mu\int_0^t\|\nabla u\|_{H^l}^2\mathrm{d}s+$$
$$\left(\frac{\beta}{2}+2\varepsilon\right)\int_0^T\|\nabla d\|_{\boldsymbol{H}^{l+1}}^2\mathrm{d}s\leqslant K_l,t\in[0,T].$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^n}^2+\|\nabla d\|_{\boldsymbol{H}^n}^2)+\mu\|\nabla u\|_{H^n}^2+\left(\frac{\beta}{2}+2\varepsilon\right)\|\nabla d\|_{\boldsymbol{H}^{n+1}}^2\leqslant$$
$$\tilde{C}(1+\|u\|_{H^n}^2+\|\nabla d\|_{\boldsymbol{H}^n}^2)^2+C\|V\|_{H^{n+1}}^2.\tag{30}$$

Using the comparison lemma, we obtain that there exists a time $T^*(\mathbb{S}^2,\|u_0\|_{H^2(M)},\|\nabla d_0\|_{\boldsymbol{H}^2(M)})$, and $K_n(T,\|u_0\|_{H^n(M)},\|\nabla d_0\|_{\boldsymbol{H}^n(M)})>0$ such that for any $T<T^*$, we have

$$\sup_{0\leqslant t\leqslant T}(\|u\|_{H^n}^2+\|\nabla d\|_{\boldsymbol{H}^n}^2)+\mu\int_0^T\|\nabla u\|_{H^n}^2\mathrm{d}s+$$
$$\left(\frac{\beta}{2}+2\varepsilon\right)\int_0^T\|\nabla d\|_{\boldsymbol{H}^{n+1}}^2\mathrm{d}s\leqslant K_n.\tag{31}$$

It is easy to find that the solution to Eq. (2) with $\varepsilon\in(0,1)$ must exist on the time interval $[0,T]$. Thus, we always extend the time interval of existence to cover $[0,T]$, i.e., for $T_\varepsilon\geqslant T$ always holds. Thus, we complete the proof of Lemma 6.

Next, we use Lemma 6 to prove the local existence of 1.

Proof of Theorem 1 Since $u_0\in H^k(\mathbb{R}^m,\mathbb{R}^m)$, $d_0\in H^{k+1}(\mathbb{R}^m,\mathbb{S}^2)$ for $k\geqslant 2$, by the density theorem of Sobolev spaces and Lemma 5, there exists a sequence $\{(u_0^i,d_0^i)\}\in H^k(\mathbb{R}^m,\mathbb{R}^m)\times H_Q^{k+1}(\mathbb{R}^m,\mathbb{S}^2)$, satisfying $u_0^i\in C_0^\infty(\mathbb{R}^m,\mathbb{R}^m)$ and $d_0^i-Q\in C_0^\infty(\mathbb{R}^m,\mathbb{R}^3)$ such that

$$(u_0^i,d_0^i)\to(u_0,d_0)\in H^k(\mathbb{R}^m,\mathbb{R}^m)\times H_Q^{k+1}(\mathbb{R}^m,\mathbb{S}^2),$$
as $i\to\infty$. (32)

For a section F of $d^*T\mathbb{S}^2$, we have the relation between $\nabla_\alpha F$ and $D_\alpha F$ as follow:

$$\nabla_\alpha F=D_\alpha F+A(d)(Dd,F),$$

where A is the second fundamental form of $\mathbb{S}^2$ in $\mathbb{R}^3$. Then, we have multi-linear vector valued functions B_i on $\mathbb{R}^3$ such that

$$D_a d=\nabla_a d+\sum B_{\sigma(a)}(d)(\nabla_{a_1}d,\cdots,\nabla_{a_s}d),\tag{33}$$

where the sum is over all multi-indices $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_s$ such that $|\boldsymbol{a}_j|\geqslant 1$ for $j=1,\cdots,s$ and

$$\sigma(\boldsymbol{a})=(\boldsymbol{a}_1,\cdots,\boldsymbol{a}_s)$$

is a permutation of $\boldsymbol{a}$ and $|\boldsymbol{a}|\geqslant 2$. Thus, by Eqs. (32) and (33), we may obtain

$$\|Dd_0^i\|_{\boldsymbol{H}^k}\to\|Dd_0\|_{\boldsymbol{H}^k},\text{ as } i\to\infty.$$

Next, let Ω_i be the support of $(u_{i0},d_{i0}-Q)$ and there exists R_i sufficiently large such that $\Omega_i\subset\subset\tilde{\Omega}_i\equiv\overbrace{[-R_i,R_i]\times\cdots\times[-R_i,R_i]}^{2m}$. Then, (u_{i0},d_{i0}) may be regarded as a function defined on a flat torus $\mathbb{T}_i^m\times\mathbb{T}_i^m=\mathbb{R}^m/(2R_i\cdot\mathbb{Z})^m\times\mathbb{R}^m/(2R_i\cdot\mathbb{Z})^m$, and thus, we shall consider the following Cauchy problem:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \mu\Delta u - \lambda\nabla\cdot(\nabla d\odot\nabla d), \\ \quad \text{on } \mathbb{T}_i^m\times(0,T], \\ d_t + u\cdot\nabla d + \alpha d\times(\Delta d + V\cdot\nabla d + \nabla\times d) + \\ \quad \beta d\times(d\times(\Delta d + V\cdot\nabla d + \nabla\times d)) = 0, \\ \quad \text{on } \mathbb{T}_i^m\times(0,T], \\ \nabla\cdot u = 0, \\ (u,d)|_{t=0} = (u_0,d_0), \quad \text{on } \mathbb{T}_i^m\times\mathbb{T}_i^m\to\mathbb{R}^m\times\mathbb{S}^2. \end{cases} \tag{34}$$

By Lemma 1 and Lemma 6, we can obtain that there exists $T>0$, which does not depend on i, such that Eq. (34) admits a smooth solution (u_i,d_i) on $\mathbb{T}_i^d\times\mathbb{T}_i^d\times[0,T]$. Furthermore, the following bound holds uniformly with respect to i:

$$\sup_{0\leqslant t\leqslant T}(\|u_i\|^2_{H^n(\mathbb{T}_i^m)} + \lambda\|\nabla d_i\|^2_{\boldsymbol{H}^n(\mathbb{T}_i^m)}) + \mu\int_0^T\|\nabla u_i\|^2_{H^n(\mathbb{T}_i^m)}\mathrm{d}s + \left(\frac{\beta}{2}+2\varepsilon\right)\int_0^T\|\nabla d_i\|^2_{\boldsymbol{H}^{n+1}(\mathbb{T}_i^m)}\mathrm{d}s \leqslant C(T,\|u_0\|_{H^n(\mathbb{T}_i^m)},\|\nabla d_0\|_{\boldsymbol{H}^n(\mathbb{T}_i^m)}).$$

Combining this and by Lemma 4, we can further obtain

$$\sup_{0\leqslant t\leqslant T}(\|u_i\|^2_{H^n(\mathbb{T}_i^m)} + \lambda\|\nabla d_i\|^2_{\boldsymbol{H}^n(\mathbb{T}_i^m)}) + \mu\int_0^T\|\nabla u_i\|^2_{H^n(\mathbb{T}_i^m)}\mathrm{d}s + \left(\frac{\beta}{2}+2\varepsilon\right)\int_0^T\|\nabla d_i\|^2_{H_Q^{n+1}(\mathbb{T}_i^m)}\mathrm{d}s \leqslant C(T,\|u_0\|_{H^n(\mathbb{T}_i^m)},\|\nabla d_0\|_{H_Q^n(\mathbb{T}_i^m)}).$$

We regard each (u_i,d_i) as a function from $[-R_i,R_i]^m\times[-R_i,R_i]^m\times[0,T]$ into $\mathbb{R}^m\times\mathbb{S}^2$, then there exists a $(u,d)\in L^\infty([0,T];H^n(\mathbb{R}^m)\times H_Q^{n+1}(\mathbb{R}^m))$ and a subsequence $\{(u_i,d_i)\}$ such that for any compact domain $O_1,O_2\subset\mathbb{R}^m$

$$(u_i,d_i)\to(u,d)\,[\textit{weakly}^*]$$
$$\text{in } L^\infty([0,T];H^n(O_1))\cap L^2([0,T];$$
$$H^{n+1}(O_1))\times L^\infty([0,T];$$
$$H_Q^{n+1}(O_2))\cap L^2([0,T];H_Q^{n+2}(O_2)),$$

and thus, it is easy to obtain that (u,d) is a strong solution to the Cauchy problem. This completes the proof of local existence.

4 Uniqueness

In this section, we research the uniqueness of the solution by using the ideas of McGahagan[23] and Song, et al.[24].

Assume that $(u_i,d_i)\in H^k\times H_Q^{k+1}, i=1,2$ are two solutions for the system (1) with the same initial map $(u_0,d_0)\in H^k\times H_Q^{k+1}$.

Since $\mathbb{S}^2\subset\mathbb{R}^3$ and by Eq. (1), we have for $\lambda=1,2$,

$$\|d_\lambda(x,t)-d_0(x)\|_{L^2}\leqslant\left\|\int_0^t\partial_s d_\lambda(x,s)\mathrm{d}s\right\|_{L^2}\leqslant Ct\|u_\lambda\cdot\nabla d_\lambda+\alpha d_\lambda\times(\Delta d_\lambda+V\cdot\nabla d_\lambda+\nabla\times d_\lambda)+\beta d_\lambda\times(d_\lambda\times(\Delta d_\lambda+V\cdot\nabla d_\lambda+\nabla\times d_\lambda))\|_{L^2}\leqslant Ct.$$

This implies

$$\|d_\lambda-d_0\|_{L^\infty}\leqslant C\|d_\lambda-d_0\|_{L^2}^{1-\frac{m}{4}}\|\Delta(d_\lambda-d_0)\|_{L^2}^{\frac{m}{4}}\leqslant Ct^{1-\frac{m}{4}}.$$

From this, for any $\varepsilon>0$ sufficiently small, there exists $T'>0$ such that $|d_1-d_2|<\varepsilon$ for any $(x,t)\in\mathbb{R}^m\times[0,T']$. And this implies there exists a unique minimizing geodesic $\gamma_{(x,t)}(s):[0,l(x,t)]\to\mathbb{S}^2$ between the points $d_1(x,t)$ and $d_2(x,t)$, where $\gamma_{(x,t)}(0)=d_1(x,t)$, $\gamma_{(x,t)}(l)=d_2(x,t)$, and l is the length of the geodesic γ. Let (x,t) vary, the family of geodesics gives rise to a map $U:[0,1]\times[0,T^*]\times\mathbb{R}^m\to\mathbb{S}^2$ connecting d_1 and d_2, where $U(s,t,x)=\gamma_{(t,x)}(s)$. Therefore, we can define a global bundle morphism $X(s,0):d_1^*T\mathbb{S}^2=\gamma(0)^*T\mathbb{S}^2\to\gamma(s)^*T\mathbb{S}^2$ for any $s\in[0,l(x,t)]$ by the parallel transportation along each geodesic.

Using the similar argument to Ref. [23], we have the following lemma.

Lemma 9 We have the following inequalities for derivatives of the geodesics γ and their lengths l:

$$|\partial_k l|\leqslant|\nabla d_2 - X\nabla d_1|,$$
$$|\partial_k\gamma|\lesssim|\nabla d_1|+|\nabla d_2|,$$
$$|\partial_t\gamma|\lesssim|D_k\partial_k d_1|+|D_k\partial_k d_2|+|u_1\cdot\nabla d_1|+|u_2\cdot\nabla d_2|+(1+|V|)(|\nabla d_1|+|\nabla d_2|),$$
$$|D_j\partial_k\gamma|\lesssim|D_j\partial_k d_1|+|D_j\partial_k d_2|+(|\partial_j d_1|+|\partial_j d_2|)(|\partial_k d_1|+|\partial_k d_2|).$$

Next, in order to get uniqueness, we need to prove

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_2-u_1\|^2_{L^2}+\|d_2-d_1\|^2_{L^2}+\|\nabla d_2-X\nabla d_1\|^2_{L^2})\leqslant C(\|u_2-u_1\|^2_{L^2}+\|d_2-d_1\|^2_{L^2}+\|\nabla d_2-X\nabla d_1\|^2_{L^2}), \tag{35}$$

where the constant C depends on $\|u_\lambda\|_{H^k}$ $(k=2,3)$ and $\|\nabla d_\lambda\|_{H^2}$ for $\lambda=1,2$.

First, we refer to Ref. [23] and the d - equation of Eq. (1), subtracting the equation for the parallel transport of d_1 from the equation for d_2, we have the identity

$$\begin{aligned}&\partial_t d_2+u_2\cdot\nabla d_2+\alpha J(d_2)D_k\partial_k d_2+\\&\quad\alpha J(d_2)(V\cdot\nabla d_2+\nabla\times d_2)-\\&\quad\beta D_k\partial_k d_2-\beta(V\cdot\nabla d_2+\nabla\times d_2)-J(d_2)\times f-\\&\quad X(l,0)[\partial_t d_1+u_1\cdot\nabla d_1+\alpha J(d_1)D_k\partial_k d_1+\\&\quad\alpha J(d_1)(V\cdot\nabla d_1+\nabla\times d_1)-\beta D_k\partial_k d_1-\\&\quad\beta(V\cdot\nabla d_1+\nabla\times d_1)-J(d_1)\times f]=0.\end{aligned}$$

Next, we use the fact (cf. Ref. [23]) that for every $F\in T_{d_1}\mathbb{S}^2$, we have

$$X(s,0)J(\gamma(0))F=J(\gamma(s))X(s,0)F.$$

Then, one gets

$$\begin{aligned}&\partial_t d_2-X(l,0)\partial_t d_1+u_2\cdot\nabla d_2-u_1\cdot X(l,0)\nabla d_1+\\&\quad\alpha J(d_2)(D_k\partial_k d_2-X(l,0)D_k\partial_k d_1)+\\&\quad\alpha J(d_2)V\cdot(\nabla d_2-X(l,0)\nabla d_1)+\\&\quad\alpha J(d_2)(\nabla\times d_2-X(l,0)\nabla\times d_1)-\\&\quad\beta(D_k\partial_k d_2-X(l,0)D_k\partial_k d_1)-\\&\quad\beta V\cdot(\nabla d_2-X(l,0)\nabla d_1)=0,\end{aligned}$$

Taking the inner product of the above identity with $D_k\partial_k d_2-X(l,0)D_k\partial_k d_1$ and integrating in space, one gets

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d_2-X\nabla d_1\|_{L^2}^2\leqslant\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}\{\|\nabla u_2-\nabla u_1\|_{L^2}+\\&\quad\|u_2-u_1\|_{L^4}\|D\partial d_2\|_{L^4}+\|\nabla d_2-X\nabla d_1\|_{L^2}+\\&\quad\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}+\|[D_t,X]\partial_k d_1\|_{L^2}+\\&\quad\|[D_k,X]\partial_t d_1\|_{L^2}+\|D[D_k,X]\partial_k d_1\|_{L^2}+\\&\quad\|[D,X]\nabla d_1\|_{L^2}\}-\beta\|[D_k,X]\partial_k d_1\|_{L^2}^2-\\&\quad\beta\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2\leqslant\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}\{\|\nabla u_2-\nabla u_1\|_{L^2}+\\&\quad\|u_2-u_1\|_{L^4}\|D\partial d_2\|_{L^4}+\|\nabla d_2-X\nabla d_1\|_{L^2}+\\&\quad\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}+\|l\sup_{s\in[0,l]}|\partial_t\gamma||\nabla d_1|\|_{L^2}+\\&\quad\|l\sup_{s\in[0,l]}|\partial\gamma||\partial_t d_1|\|_{L^2}+\|\nabla l\|_{L^2}\|\nabla d_1\|_{L^\infty}\|\nabla d_2\|_{L^\infty}+\\&\quad\|l\sup_{s\in[0,l]}|DR(\partial\gamma,\partial_s\gamma)X(s)\nabla d_1|\|_{L^2}+\\&\quad\|l^2\sup_{s\in[0,l]}|\partial\gamma||\nabla d_1|^2\|_{L^2}+\|l\sup_{s\in[0,l]}|\partial\gamma||\nabla d_1|\|_{L^2}\}-\\&\quad\beta\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2.\end{aligned}\tag{36}$$

For $m=3$, we may estimate each term with a factor of l by taking l in $L^{\frac{2m}{m-2}}$ and the rest of the term in L^m. Then, by Sobolev embedding and Lemma 9, Eq. (36) becomes

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d_2-X\nabla d_1\|_{L^2}^2\leqslant\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}\{\|\nabla u_2-\nabla u_1\|_{L^2}+\\&\quad\|u_2-u_1\|_{L^2}^{1-\frac{m}{4}}\|\nabla u_2-\nabla u_1\|_{L^2}^{\frac{m}{4}}+\|\nabla d_2-X\nabla d_1\|_{L^2}+\\&\quad\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}+\|\nabla l\|_{L^2}C(u_\lambda,d_\lambda)\}-\\&\quad\beta\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+\\&\quad C\|u_2-u_1\|_{L^2}^2+C\|\nabla d_2-X\nabla d_1\|_{L^2}^2+\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}^2C(u_\lambda,d_\lambda)-\\&\quad(\beta-\varepsilon_1)\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+\\&\quad C(\|u_2-u_1\|_{L^2}^2+\|\nabla d_2-X\nabla d_1\|_{L^2}^2)+\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}^2C(u_\lambda,d_\lambda),\end{aligned}$$

where we choose positive constant ε_1 small enough such that

$$\beta-\varepsilon_1\geqslant0,$$

and

$$\begin{aligned}C(u_\lambda,d_\lambda)=&\|(|D_k\partial_k d_2|+|D_k\partial_k d_1|+|u_2\cdot\nabla d_2|+\\&|u_1\cdot\nabla d_1|+(1+|V|)(|\nabla d_1|+|\nabla d_2|))|\nabla d_1|\|_{L^m}+\\&\|(|\nabla d_2|+|\nabla d_1|)(|u_1\cdot\nabla d_1|+|D\nabla d_1|+\\&(|V|+1)|\nabla d_1|+|f|)\|_{L^m}+\|\nabla d_1\|_{L^\infty}\|\nabla d_2\|_{L^\infty}+\\&\|(|D\partial d_2|+|D\partial d|_1)|\nabla d_1|(|\nabla d_2|+|\nabla d_1|)\|_{L^m}+\\&\|(|\nabla d_2|+|\nabla d_1|)|\nabla d_1|\|_{L^m}+\\&\|l(|\nabla d_2|+|\nabla d_1|)|\nabla d_1|^2\|_{L^m}+\\&\|(|\nabla d_2|+|\nabla d_1|)^2|\nabla d_1|(1+|\nabla d_2|+|\nabla d_1|)\|_{L^m}.\end{aligned}$$

For $m=2$, we must bound l in L^∞. In this borderline case, we apply a theorem due to Brezis, et al.[26]:

$$\begin{aligned}\|l\|_{L^\infty}\approx&\|d_1-d_2\|_{L^\infty}\lesssim\\&\|d_1-d_2\|_{H^1}(1+\log^{\frac{1}{2}}(1+\|\partial^2(d_1-d_2)\|_{L^2}))\leqslant\\&\|d_1-d_2\|_{H^1}.\end{aligned}$$

Then, Eq. (36) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d_2-X\nabla d_1\|_{L^2}^2\leqslant\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}\{\|\nabla u_2-\nabla u_1\|_{L^2}+\\&\quad\|u_2-u_1\|_{L^2}^{1-\frac{m}{4}}\|\nabla u_2-\nabla u_1\|_{L^2}^{\frac{m}{4}}+\|\nabla d_2-X\nabla d_1\|_{L^2}+\\&\quad\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}+(\|l\|_{L^\infty}+\|\nabla l\|_{L^2})C(u_\lambda,d_\lambda)\}-\\&\quad\beta\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+\\&\quad C\|u_2-u_1\|_{L^2}^2+C\|\nabla d_2-X\nabla d_1\|_{L^2}^2-\\&\quad(\beta-\varepsilon_2)\|D_k\partial_k d_2-XD_k\partial_k d_1\|_{L^2}^2+\\&\quad C\|\nabla d_2-X\nabla d_1\|_{L^2}(\|\nabla d_2-X\nabla d_1\|_{L^2}+\\&\quad\|d_2-d_1\|_{H^1})C(u_\lambda,d_\lambda)\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+\\&\quad C(\|u_2-u_1\|_{L^2}^2+\|\nabla d_2-X\nabla d_1\|_{L^2}^2)+\\&\quad C(\|\nabla d_2-X\nabla d_1\|_{L^2}^2+\|d_2-d_1\|_{L^2}^2)C(u_\lambda,d_\lambda),\end{aligned}$$

where we choose positive constant ε_2 small enough such that

$$\beta-\varepsilon_2\geqslant 0.$$

By Sobolev embedding, we can bound $C(u_\lambda,d_\lambda)$ by

$$C(u_\lambda,d_\lambda)\leqslant C(1+\|\nabla d_1\|_{H^2}+\|\nabla d_2\|_{H^2})^4(1+\|u_1\|_{H^2}+\|u_2\|_{H^2}).$$

Then, we obtain

$$\frac{d}{dt}\|\nabla d_2-X\nabla d_1\|_{L^2}^2\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}+C(\|u_2-u_1\|_{L^2}^2+\|d_2-d_1\|_{L^2}^2+\|\nabla d_2-X\nabla d_1\|_{L^2}^2).\quad(37)$$

Second, by the u-equation of (1), we have the identity

$$\partial_t(u_2-u_1)+(u_2-u_1)\cdot\nabla u_2+u_1(\nabla u_2-\nabla u_1)=\mu\Delta(u_2-u_1)-\lambda\nabla\cdot(\nabla d_2\odot\nabla d_2)+\lambda\nabla\cdot(\nabla d_1\odot\nabla d_1).$$

Making the inner product of the above identity with u_2-u_1 and integrating in space, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2-u_1\|_{L^2}^2+\mu\|\nabla u_2-\nabla u_1\|_{L^2}^2\leqslant C\|\nabla u_2-\nabla u_1\|_{L^2}(\|u_2-u_1\|_{L^2}+\|\nabla d_2-\nabla d_1\|_{L^2})\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+C(\|u_2-u_1\|_{L^2}^2+\|\nabla d_2-\nabla d_1\|_{L^2}^2).\quad(38)$$

Using properties of the parallel transport, we have

$$[\nabla d_2-\nabla d_1]_{L^2}\lesssim\|\nabla d_2-X\nabla d_1\|_{L^2}+\|l\|_{L^2}\lesssim\|\nabla d_2-X\nabla d_1\|_{L^2}+\|d_2-d_1\|_{L^2}.\quad(39)$$

Then, Eq. (38) becomes

$$\frac{1}{2}\frac{d}{dt}\|u_2-u_1\|_{L^2}^2+\mu\|\nabla u_2-\nabla u_1\|_{L^2}^2\leqslant\frac{\mu}{4}\|\nabla u_2-\nabla u_1\|_{L^2}^2+C(\|u_2-u_1\|_{L^2}^2+\|d_2-d_1\|_{L^2}^2+\|\nabla d_2-X\nabla d_1\|_{L^2}^2).\quad(40)$$

Finally, by d-equation, we have the identity

$$\partial_t(d_2-d_1)+(u_2-u_1)\cdot\nabla d_2+u_1(\nabla d_2-\nabla d_1)+\alpha(d_2-d_1)\times(\Delta d_2+V\cdot\nabla d_2+\nabla\times d_2)+d_1\times(\Delta(d_2-d_1)+V\cdot\nabla(d_2-d_1)+\nabla\times(d_2-d_1))+\beta(d_2-d_1)\times d_2\times(\Delta d_2+V\cdot\nabla d_2+\nabla\times d_2)+\beta d_1\times(d_2-d_1)\times(\Delta d_2+V\cdot\nabla d_2+\nabla\times d_2)+\beta d_1\times d_1\times(V\cdot\nabla(d_2-d_1)+\nabla\times(d_2-d_1))-\beta\Delta(d_2-d_1)-|\nabla d_2|^2(d_2-d_1)-\beta(|\nabla d_2|^2-|\nabla d_1|^2)d_1=(d_2-d_1)\times f.\quad(41)$$

Making the inner product of the above identity with d_2-d_1 and integrating in space, we get

$$\frac{1}{2}\frac{d}{dt}\|d_2-d_1\|_{L^2}^2\leqslant-\beta\|\nabla d_2-\nabla d_1\|_{L^2}^2+\|u_2-u_1\|_{L^2}\|d_2-d_1\|_{L^2}\|\nabla d_2\|_{H^2}+C\|d_2-d_1\|_{L^2}\|\nabla d_2-\nabla d_1\|_{L^2}\{1+\|u_1\|_{H^2}+\|V\|_{L^\infty}+\|\nabla d_1\|_{H^2}+\|\nabla d_2\|_{H^2}\}+\beta\|\nabla d_2\|_{H^2}^2\|d_2-d_1\|_{L^2}^2\leqslant C(\|u_2-u_1\|_{L^2}^2+\|d_2-d_1\|_{L^2}^2+\|\nabla d_2-\nabla d_1\|_{L^2}^2).$$

Then, by Eq. (39) and (41) become

$$\frac{1}{2}\frac{d}{dt}\|d_2-d_1\|_{L^2}^2\leqslant C(\|u_2-u_1\|_{L^2}^2+\|d_2-d_1\|_{L^2}^2+\|\nabla d_2-X\nabla d_1\|_{L^2}^2).\quad(42)$$

Hence, from Eqs. (37), (40) and (42), the bound Eq. (35) is acceptable. And since

$$\|u_2-u_1\|_{L^2}=\|d_2-d_1\|_{L^2}=\|\nabla d_2-X\nabla d_1\|_{L^2}=0$$

at initial time, then by Eq. (35) and using Gronwall's inequality, we can easily obtain

$$\sup_{0\leqslant t\leqslant T'}\|u_2-u_1\|_{L^2}^2+\sup_{0\leqslant t\leqslant T'}\|d_2-d_1\|_{L^2}^2+\sup_{0\leqslant t\leqslant T'}\|\nabla d_2-X\nabla d_1\|_{L^2}^2\leqslant 0.$$

Hence, we obtain $(u_1,d_1)=(u_2,d_2)$ on $[0,T']$. By repeating the above argument starting at time T', we can prove $(u_1,d_1)=(u_2,d_2)$ for time $t\in[0,T]$. This completes the proof of the uniqueness.

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