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$\mathbb{R}^{1+1}$ 中 Maxwell-Chern-Simons-Higgs 模型解的 L^∞ 估计

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摘要： 用能量守恒估计和特征线估计的方法解决(1+1)维 Maxwell-Chern-Simons-Higgs 模型解的 Sobolev 范数增长性估计问题，给出模型有限能量解的一阶导数的 L^∞ 估计，并通过改进指数增长得到解的 H^2 范数的多项式增长.

关键词： Chern-Simons 理论；规范不变性；特征线估计；L^∞ 估计；H^2 范数增长
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L^∞ Estimation for Solution of Maxwell-Chern-Simons-Higgs Model in $\mathbb{R}^{1+1}$

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Abstract： We used the methods of conservative energy estimation and characteristic line estimation to solve the Sobolev norm growth estimation problem of the solution of the (1+1)-dimensional Maxwell-Chern-Simons-Higgs model，gave the L^∞ estimation of the first derivative of the finite energy solution of the model，and obtained the polynomial growth of the H^2 norm of the solution by improving the exponential growth.

Keywords： Chern-Simons theory；gauge invariance；characteristic line estimation；L^∞ estimation；H^2 norm growth

1　引言与预备知识

在规范场理论下研究带有 Chern-Simons 项的偏微分方程对解决实际的物理问题有很大帮助. 在(1+2)维规范场模型中，关于降维模型的研究是一类重要问题. 即可通过假设模型相关的物理场与第二维空间坐标无关，从而将二维空间降为一维空间. 赋予不同的空间维度会展现出方程在其维度所固有的物理特征，一维模型的研究在理论与实际上均有一定价值. 基于一系列降维模型[1-6]的研究，本文考虑在(1+1)维 Minkowski 空间上的 Maxwell-Chern-Simons-Higgs(MCSH)模型[7]. 该模型由一个规范场(电磁场)$\boldsymbol{A}=(\boldsymbol{A}_0,\boldsymbol{A}_1,\boldsymbol{A}_2)$、一个复标量场(Higgs 场)$\varphi$ 和一个中性实标量场 N 组成，它是通过(1+2)维 MCSH 模型[8]降维得到的. 首先，通过变分法求得(1+1)维 MCSH 模型对应的 Euler-

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Lagrange 方程:

$$\partial_1 \boldsymbol{F}_{01} + \kappa \partial_1 B + 2e\mathrm{Im}(\varphi \overline{D_0 \varphi}) = 0, \tag{1}$$

$$\partial_0 \boldsymbol{F}_{01} + \kappa \partial_0 B + 2e\mathrm{Im}(\varphi \overline{D_1 \varphi}) = 0, \tag{2}$$

$$\partial_0 \partial_0 B - \partial_1 \partial_1 B - \kappa \boldsymbol{F}_{01} + V_B(|\varphi|^2, N, B) = 0, \tag{3}$$

$$\partial_0 \partial_0 N - \partial_1 \partial_1 N + V_N(|\varphi|^2, N, B) = 0, \tag{4}$$

$$D_0 D_0 \varphi - D_1 D_1 \varphi + V_{\bar{\varphi}}(|\varphi|^2, N, B) = 0; \tag{5}$$

其次,构造模型的守恒能量函数

$$E(t) = \sum_{\mu=0,1} \int_{\mathbb{R}} \frac{1}{2}(\boldsymbol{F}_{01}^2 + |\partial_\mu B|^2 + |\partial_\mu N|^2) + |D_\mu \varphi|^2 + V(|\varphi|^2, N, B)\mathrm{d}x = E(0), \tag{6}$$

其中

$$V(|\varphi|^2, N, B) = \frac{1}{2}(e|\varphi|^2 + \kappa N - ev^2)^2 + e^2(N^2 + B)^2 |\varphi|^2,$$

e 是电子的电荷,$\kappa(\kappa>0)$是 Chern-Simons 耦合常数,v 是非零常数,$\partial_0=\partial_t$,$\partial_1=\partial_x$,$V_{\bar{\varphi}}, V_N, V_B$ 分别表示 $V(|\varphi|^2, N, B)$关于变量 $\bar{\varphi}, N, B$ 求偏导数. 本文讨论非拓扑边界条件情形($(\varphi, N, \boldsymbol{A}_\mu, B) \to (0, ev^2/\kappa, 0, 0)$, $|x| \to \infty$). 为得到模型(1)～(5)在标准 Sobolev 空间中的解($(\varphi, N, \boldsymbol{A}_\mu, B) \to (0,0,0,0)$, $|x| \to \infty$),可通过式 $\widetilde{N} = N - ev^2/\kappa$ 替换原模型中的 N,为记号简单仍用 N 表示 $\widetilde{N}$. 定义通常意义下的规范场强和协变导数为

$$\boldsymbol{F}_{\mu\nu} \triangleq \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu, \qquad D_\mu \triangleq \partial_\mu - \mathrm{i}e\boldsymbol{A}_\mu.$$

模型(1)～(5)在如下的规范变换下保持不变:

$$\varphi \to \varphi' = \mathrm{e}^{-\mathrm{i}e\chi}\varphi, \quad \boldsymbol{A}_\mu \to \boldsymbol{A}'_\mu = \boldsymbol{A}_\mu + \partial_\mu \chi, \quad D_\mu \to D'_\mu = \partial_\mu - \mathrm{i}e\boldsymbol{A}'_\mu,$$

其中 χ: $\mathbb{R}^{1+1} \to \mathbb{R}^{1+1}$是光滑函数. 赋予(1+1)维 MCSH 模型 Lorenz 规范条件$\partial_0 \boldsymbol{A}_0 - \partial_1 \boldsymbol{A}_1 = 0$,并引入记号$\square = \partial_\mu \partial^\mu$. 模型(1)～(5)的 Cauchy 问题为

$$\begin{aligned} &\square \boldsymbol{A}_0 = -\kappa \partial_1 B - 2e\mathrm{Im}(\varphi \overline{D_0 \varphi}), \\ &\square \boldsymbol{A}_1 = -\kappa \partial_0 B - 2e\mathrm{Im}(\varphi \overline{D_1 \varphi}), \\ &\square B = \kappa \boldsymbol{F}_{01} - 2e^2 B|\varphi|^2, \\ &\square N = -V_N(|\varphi|^2, N, B), \\ &\square \varphi = 2\mathrm{i}e\boldsymbol{A}_\mu \partial^\mu \varphi + e^2 \boldsymbol{A}_\mu \boldsymbol{A}^\mu \varphi - V_{\bar{\varphi}}(|\varphi|^2, N, B), \end{aligned} \tag{7}$$

对应的初始数据为

$$\begin{aligned} &\varphi(0,\cdot) = \varphi_0, \quad \partial_t \varphi(0,\cdot) = \varphi_1, \quad \boldsymbol{A}_\mu(0,\cdot) = a_{0\mu}, \quad \partial_t \boldsymbol{A}_\mu(0,\cdot) = a_{1\mu}, \\ &B(0,\cdot) = b_0, \quad \partial_t B(0,\cdot) = b_1, \quad N(0,\cdot) = n_0, \quad \partial_t N(0,\cdot) = n_1, \end{aligned} \tag{8}$$

对应的约束方程为

$$\begin{aligned} &a_{10} - \partial_1 a_{01} = 0, \\ &\partial_1 \partial_1 a_{00} - \partial_1 a_{11} - \kappa \partial_1 b_0 - 2e\mathrm{Im}(\varphi_0 \bar{\varphi}_1 + \mathrm{i}ea_{00}\varphi_0^2) = 0. \end{aligned} \tag{9}$$

2 主要结果

记 $I = \max\{\|\boldsymbol{F}_{10}(0,\cdot)\|_{L^\infty}, \|B(0,\cdot)\|_{L^\infty}, \|N(0,\cdot)\|_{L^\infty}, \|\varphi(0,\cdot)\|_{L^\infty}, \|\partial_\mu B(0,\cdot)\|_{L^\infty}, \|\partial_\mu N(0,\cdot)\|_{L^\infty}, \|D_\mu \varphi(0,\cdot)\|_{L^\infty}\}$, $E(0)=E_0$. 用 $A \lesssim B$ 表示估计 $A \leqslant CB$,其中C 为常量.

定理 1 设$(\varphi, N, \boldsymbol{A}_\mu, B) \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); H^1(\mathbb{R}))$是(1+1)维 MCSH 系统(1)～(5)在初始条件$(\varphi_0, n_0, a_{0\mu}, b_0) \in H^2(\mathbb{R})$, $(\varphi_1, n_1, a_{1\mu}, b_1) \in H^1(\mathbb{R})$下的整体解,则

$$\|\boldsymbol{F}_{10}(t,\cdot)\|_{L^\infty} \lesssim (1 + I + E_0^{1/2})^2 (1 + t^{1/2})^2, \qquad \|\partial_\mu B(t,\cdot)\|_{L^\infty} \lesssim (1 + I + E_0^{1/2})^2 (1 + t^{1/2})^2,$$

$$\|\partial_\mu N(t,\cdot)\|_{L^\infty} \lesssim (1 + I + E_0^{1/2})^2 (1 + t^{1/2})^2, \qquad \|D_\mu \varphi(t,\cdot)\|_{L^\infty} \lesssim (1 + I + E_0^{1/2})^3 (1 + t^{1/2})^5.$$

证明:为方便,对$\forall \varphi, \psi \in \mathbb{C}$引入记号 $\varphi\bar{\psi} = \langle \varphi, \psi \rangle$. 于是可得如下计算性质:

$$\langle\psi,\varphi\rangle=\overline{\langle\varphi,\psi\rangle},\quad \partial_\mu\langle\varphi,\psi\rangle=\langle D_\mu\varphi,\psi\rangle+\langle\varphi,D_\mu\psi\rangle,\quad D_\mu D_\nu\varphi=D_\nu D_\mu\varphi-\mathrm{i}e\boldsymbol{F}_{\mu\nu}\varphi. \tag{10}$$

根据式(2)～(5)可得如下等式：

$$\partial_0\left[\sum_{\mu=0,1}\frac{1}{2}(|\boldsymbol{F}_{01}|^2+|\partial_\mu B|^2+|\partial_\mu N|^2)+|D_\mu\varphi|^2+V(|\varphi|^2,N,B)\right]-\partial_1[2\mathrm{Re}\langle D_1\varphi,\overline{D_0\varphi}\rangle+\partial_0 B\partial_1 B+\partial_0 N\partial_1 N]=0. \tag{11}$$

对 $t_0>0$，$x_0\in\mathbb{R}$，定义三角形区域 $\Delta(x_0,t_0)$如下：

$$\Delta(x_0,t_0)\triangleq\{(x,t)\,|\,0<t<t_0,\ x_0-t_0+t<x<x_0+t_0-t\},$$

在区域 $\Delta(x_0,t_0)$上使用格林公式可得

$$\begin{aligned}\iint_{\Delta(x_0,t_0)}\partial_t F(x,t)-\partial_x G(x,t)\mathrm{d}t\mathrm{d}x=\oint_{\partial\Delta(x_0,t_0)}G(x,t)\mathrm{d}t+F(x,t)\mathrm{d}x=\\ \int_{x_0-t_0}^{x_0+t_0}F(s,0)\mathrm{d}s-\int_0^{t_0}F(x_0+t_0-s,s)\mathrm{d}s-\int_0^{t_0}F(x_0-t_0+s,s)\mathrm{d}s+\\ \int_0^{t_0}G(x_0+t_0-s,s)\mathrm{d}s-\int_0^{t_0}G(x_0-t_0+s,s)\mathrm{d}s.\end{aligned} \tag{12}$$

根据式(12)在区域 $\Delta(x_0,t_0)$上对式(11)进行积分，可得如下估计：

$$\begin{aligned}&\int_0^{t_0}\left(|D_0\varphi-D_1\varphi|^2+\frac{1}{2}|\partial_0 N-\partial_1 N|^2+\frac{1}{2}|\partial_0 B-\partial_1 B|^2+\frac{1}{2}|\boldsymbol{F}_{01}|^2+V\right)(x_0+t_0-s,s)\mathrm{d}s+\\ &\int_0^{t_0}\left(|D_0\varphi+D_1\varphi|^2+\frac{1}{2}|\partial_0 N+\partial_1 N|^2+\frac{1}{2}|\partial_0 B+\partial_1 B|^2+\frac{1}{2}|\boldsymbol{F}_{01}|^2+V\right)(x_0-t_0+s,s)\mathrm{d}s=\\ &\int_0^{t_0}\left(|D_0\varphi-D_1\varphi|^2+\frac{1}{2}|\partial_0 N-\partial_1 N|^2+\frac{1}{2}|\partial_0 B-\partial_1 B|^2\right)(x_0+t_0-s,s)\mathrm{d}s+\\ &\int_0^{t_0}\left(|D_0\varphi+D_1\varphi|^2+\frac{1}{2}|\partial_0 N+\partial_1 N|^2+\frac{1}{2}|\partial_0 B+\partial_1 B|^2\right)(x_0-t_0+s,s)\mathrm{d}s+\\ &\int_0^{t_0}\left[\frac{1}{2}|\boldsymbol{F}_{01}|^2+\frac{1}{2}(e|\varphi|^2+\kappa N)^2+e^2B^2|\varphi|^2+e^2\left(N+\frac{ev^2}{\kappa}\right)^2|\varphi|^2\right](x_0+t_0-s,s)\mathrm{d}s+\\ &\int_0^{t_0}\left[\frac{1}{2}|\boldsymbol{F}_{01}|^2+\frac{1}{2}(e|\varphi|^2+\kappa N)^2+e^2B^2|\varphi|^2+e^2\left(N+\frac{ev^2}{\kappa}\right)^2|\varphi|^2\right](x_0-t_0+s,s)\mathrm{d}s=\\ &\int_{x_0-t_0}^{x_0+t_0}E(s,0)\mathrm{d}s\leqslant\int_{\mathbb{R}}E(s,0)\mathrm{d}s=E_0,\end{aligned} \tag{13}$$

1) 关于 $\boldsymbol{F}_{10}$. 首先根据式(1)和(2)可得关于 $\boldsymbol{F}_{10}$的输运方程：

$$\begin{aligned}(\partial_0+\partial_1)\boldsymbol{F}_{10}+\kappa(\partial_0+\partial_1)B+2e\mathrm{Im}\langle\varphi,D_0\varphi+D_1\varphi\rangle=0,\\ (\partial_0-\partial_1)\boldsymbol{F}_{10}+\kappa(\partial_0-\partial_1)B+2e\mathrm{Im}\langle\varphi,D_1\varphi-D_0\varphi\rangle=0,\end{aligned} \tag{14}$$

沿特征线对式(14)进行积分可得

$$\begin{aligned}\boldsymbol{F}_{10}(x,t)=\boldsymbol{F}_{10}(x-t,0)+\int_0^t[\kappa(\partial_0 B+\partial_1 B)+2e\mathrm{Im}\langle\varphi,D_0\varphi+D_1\varphi\rangle](x-t+s,s)\mathrm{d}s=0,\\ \boldsymbol{F}_{10}(x,t)=\boldsymbol{F}_{10}(x+t,0)+\int_0^t[\kappa(\partial_0 B-\partial_1 B)+2e\mathrm{Im}\langle\varphi,D_1\varphi-D_0\varphi\rangle](x+t-s,s)\mathrm{d}s=0.\end{aligned} \tag{15}$$

应用 Hölder 不等式，考虑关于 φ 的 L^2 范数估计，有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{L^2}^2=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\varphi(t,x)|^2\mathrm{d}x=\mathrm{Re}\int_{\mathbb{R}^2}\varphi\overline{D_0\varphi}(t,x)\mathrm{d}x\leqslant\|\varphi\|_{L^2}\|D_0\varphi\|_{L^2}\leqslant E_0^{1/2}\|\varphi\|_{L^2},$$

此处使用了能量守恒估计 $\|D_0\varphi\|_{L^2}\leqslant E_0^{1/2}$. 于是得到估计 $\|\varphi\|_{L^2}\leqslant\|\varphi_0\|_{L^2}+E_0^{1/2}t$. 同理还可得 $\|B\|_{L^2}\lesssim\|b_0\|_{L^2}+E_0^{1/2}t$ 和 $\|N\|_{L^2}\lesssim\|n_0\|_{L^2}+E_0^{1/2}t$. 应用协变 Gagliardo-Nirenberg 不等式[9]可得关于 φ 的 L^∞范数估计：

$$\|\varphi(\cdot,t)\|_{L^\infty}\leqslant C\|\varphi(\cdot,t)\|_{L^2}^{1/2}\|D_1\varphi(\cdot,t)\|_{L^2}^{1/2}\lesssim(I+E_0^{1/2})(1+t^{1/2}). \tag{16}$$

根据估计(13)和(16)并应用 Hölder 不等式，由式(15)可得关于 $\boldsymbol{F}_{10}$的估计如下：

$$|\boldsymbol{F}_{10}(x,t)|\leqslant|\boldsymbol{F}_{10}(x-t,0)|+\kappa\left[\int_0^t|\partial_0 B+\partial_1 B|^2(x-t+s,s)\mathrm{d}s\right]^{1/2}\left(\int_0^t 1\mathrm{d}s\right)^{1/2}+$$

$$2e\left[\int_0^t |D_0\varphi+D_1\varphi|^2(x-t+s,s)\mathrm{d}s\right]^{1/2}\left(\int_0^t 1\mathrm{d}s\right)^{1/2}\|\varphi\|_{L^\infty}\lesssim$$
$$(1+I+E_0^{1/2})^2(1+t^{1/2})^2.$$

2) 关于$\partial_\mu B$ 和$\partial_\mu N$. 可将式(3)按如下两种方式表示:

$$\begin{aligned}(\partial_0+\partial_1)(\partial_0 B-\partial_1 B)&=\kappa\boldsymbol{F}_{01}-2e^2B|\varphi|^2,\\(\partial_0-\partial_1)(\partial_0 B+\partial_1 B)&=\kappa\boldsymbol{F}_{01}-2e^2B|\varphi|^2.\end{aligned}\tag{17}$$

沿特征线对式(17)积分, 并应用估计(13)和 Hölder 不等式可得

$$|(\partial_0 B-\partial_1 B)(x,t)|\lesssim|(\partial_0 B-\partial_1 B)(x-t,0)|+(1+I+E_0^{1/2})^2(1+t^{1/2})^2,$$
$$|(\partial_0 B+\partial_1 B)(x,t)|\lesssim|(\partial_0 B+\partial_1 B)(x+t,0)|+(1+I+E_0^{1/2})^2(1+t^{1/2})^2,$$

从而可得关于$\partial_\mu B$ 的估计为

$$|\partial_\mu B|\leqslant\frac{1}{2}|(\partial_0 B-\partial_1 B)(x,t)|+\frac{1}{2}|(\partial_0 B+\partial_1 B)(x,t)|\lesssim(1+I+E_0^{1/2})^2(1+t^{1/2})^2.$$

类似地, 通过式(4)可得关于$\partial_\mu N$ 的估计为

$$|\partial_\mu N|\lesssim(1+I+E_0^{1/2})^2(1+t^{1/2})^2.$$

3) 关于 $D_\mu\varphi$. 定义符号 $\Phi_+\triangleq D_0\varphi+D_1\varphi$, $\Phi_-\triangleq D_0\varphi-D_1\varphi$. 由式(15)和性质(10)可得

$$\begin{aligned}(D_0+D_1)\Phi_-&=-V_{\bar\varphi}-\mathrm{i}e\boldsymbol{F}_{10}\varphi,\\(D_0-D_1)\Phi_+&=-V_{\bar\varphi}+\mathrm{i}e\boldsymbol{F}_{10}\varphi,\end{aligned}\tag{18}$$

分别对式(18)中第一个等式两边同乘 $\overline{\Phi}_-$、第二个等式两边同乘 $\overline{\Phi}_+$, 并在两边分别取实部可得

$$(\partial_0+\partial_1)\frac{1}{2}|\Phi_-|^2=e\boldsymbol{F}_{10}\mathrm{Im}\langle\varphi,\Phi_-\rangle-\mathrm{Re}\langle V_{\bar\varphi},\Phi_-\rangle,$$
$$(\partial_0-\partial_1)\frac{1}{2}|\Phi_+|^2=-e\boldsymbol{F}_{10}\mathrm{Im}\langle\varphi,\Phi_+\rangle-\mathrm{Re}\langle V_{\bar\varphi},\Phi_+\rangle.$$

从而得到估计

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}|\Phi_-(x-t+s,s)|&\leqslant(e|\boldsymbol{F}_{10}||\varphi|+|V_{\bar\varphi}|)(x-t+s,s),\\\frac{\mathrm{d}}{\mathrm{d}s}|\Phi_+(x+t-s,s)|&\leqslant(e|\boldsymbol{F}_{10}||\varphi|+|V_{\bar\varphi}|)(x+t-s,s),\end{aligned}\tag{19}$$

沿特征线对式(19)进行积分可得

$$\begin{aligned}|\Phi_-(x,t)|&\leqslant|\Phi_-(x-t,0)|+e\int_0^t|\boldsymbol{F}_{10}||\varphi|(x-t+s,s)\mathrm{d}s+\int_0^t|V_{\bar\varphi}(x-t+s,s)|\mathrm{d}s,\\|\Phi_+(x,t)|&\leqslant|\Phi_+(x+t,0)|+e\int_0^t|\boldsymbol{F}_{10}||\varphi|(x+t-s,s)\mathrm{d}s+\int_0^t|V_{\bar\varphi}(x+t-s,s)|\mathrm{d}s.\end{aligned}\tag{20}$$

下面对式(20)第一个不等式右侧项进行估计:

$$e\int_0^t|\boldsymbol{F}_{10}||\varphi|(x-t+s,s)\mathrm{d}s\leqslant e\left[\int_0^t|\boldsymbol{F}_{10}(x-t+s,s)|^2\mathrm{d}s\right]^{1/2}\left(\int_0^t 1\mathrm{d}s\right)^{1/2}\|\varphi\|_{L^\infty}\lesssim$$
$$(I+E_0^{1/2})^2(1+t^{1/2})^2,$$
$$\int_0^t|V_{\bar\varphi}(x-t+s,s)|\mathrm{d}s\leqslant\int_0^t[e|\varphi||e|\varphi|^2+\kappa N|+e^2B^2|\varphi|+e^2(N+ev^2/\kappa)^2|\varphi|]\times$$
$$(x-t+s,s)\mathrm{d}s\leqslant I_1+I_2+I_3,$$

其中

$$I_1=e\left[\int_0^t(e|\varphi|^2+\kappa N)^2(x-t+s,s)\mathrm{d}s\right]^{1/2}\left(\int_0^t 1\mathrm{d}s\right)^{1/2}\|\varphi\|_{L^\infty}\lesssim(I+E_0^{1/2})^2(1+t^{1/2})^2,$$
$$I_2=e^2\|B\|_{L^\infty}^2\|\varphi\|_{L^\infty}\int_0^t 1\mathrm{d}s\lesssim(I+E_0^{1/2})^3(1+t^{1/2})^5,$$
$$I_3=e^2\left[\int_0^t(N+ev^2/\kappa)^2|\varphi|^2(x-t+s,s)\mathrm{d}s\right]^{1/2}\left[\int_0^t(N+ev^2/\kappa)^2(x-t+s,s)\mathrm{d}s\right]^{1/2}\lesssim$$
$$(I+E_0^{1/2})^2(1+t^{1/2}),$$

这里利用了

$$\|B\|_{L^\infty}^2 \leqslant \|\partial_1 B\|_{L^2}\|B\|_{L^2} \lesssim (I+E_0^{1/2})^2(1+t^{1/2})^2,$$
$$\int_0^t (N+ev^2/\kappa)^2(x-t+s,s)\mathrm{d}s \lesssim (I+E_0^{1/2})^2(1+t^{1/2})^2.$$

类似可对式(20)第二个不等式的右侧项进行估计. 由式(20)可得

$$\begin{aligned}|\Phi_-(x,t)| &\lesssim |\Phi_-(x-t,0)| + (1+I+E_0^{1/2})^3(1+t^{1/2})^5,\\ |\Phi_+(x,t)| &\leqslant |\Phi_+(x+t,0)| + (1+I+E_0^{1/2})^3(1+t^{1/2})^5.\end{aligned} \tag{21}$$

根据式(21)可得关于 $D_\mu\varphi$ 的估计为

$$\begin{aligned}|D_\mu\varphi(x,t)| &\leqslant \frac{1}{2}|\Phi_+(x,t)| + \frac{1}{2}|\Phi_-(x,t)| \lesssim \\ &\frac{1}{2}|\Phi_-(x-t,0)| + \frac{1}{2}|\Phi_+(x+t,0)| + (1+I+E_0^{1/2})^3(1+t^{1/2})^5 \lesssim \\ &(1+I+E_0^{1/2})^3(1+t^{1/2})^5.\end{aligned}$$

证毕.

注 1　定理 1 对(1+1)维 MCSH 模型的解进行了一阶导数 L^∞ 估计.

定理 2　设 $(\varphi, N, \boldsymbol{A}_\mu, B) \in C([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); H^1(\mathbb{R}))$ 是 (1+1) 维 MCSH 系统(1)～(5)在初始条件 $(\varphi_0, n_0, a_{0\mu}, b_0) \in H^2(\mathbb{R}), (\varphi_1, n_1, a_{1\mu}, b_1) \in H^1(\mathbb{R})$ 下的整体解，则

$$\sum_{\mu,\nu=0,1} \|D_\mu D_\nu\varphi(t,\cdot)\|_{L^2} \lesssim \|DD\varphi(0,\cdot)\|_{L^2} + (1+I+E_0^{1/2})^4(1+t^{1/2})^7.$$

证明：为方便，记 $|DD\varphi| = \sum_{\mu,\nu=0,1}|D_\mu D_\nu\varphi|$，$|\partial f| = \sum_{\mu=0,1}|\partial_\mu f|$，$|D\psi| = \sum_{\mu=0,1}|D_\mu\psi|$. 用算子 D_ν 作用于式(5)可得

$$\begin{aligned}D_\nu(D_0D_0\varphi - D_1D_1\varphi + V_{\bar\varphi}) = {}& D_0D_0D_\nu\varphi - D_1D_1D_\nu\varphi + \mathrm{i}e\varphi\partial_0\boldsymbol{F}_{0v} + 2\mathrm{i}e\boldsymbol{F}_{0v}D_0\varphi - \mathrm{i}e\varphi\partial_1\boldsymbol{F}_{1v} - \\ &2\mathrm{i}e\boldsymbol{F}_{1v}D_1\varphi + e\varphi\partial_\nu(e|\varphi|^2+\kappa N) + e(e|\varphi|^2+\kappa N)D_\nu\varphi + \\ &e^2[B^2+(N+ev^2/\kappa)^2]D_\nu\varphi + e^2\varphi\partial_\nu[B^2+(N+ev^2/\kappa)^2] = 0.\end{aligned} \tag{22}$$

对式(22)等式两端分别乘 $\overline{D_0D_\nu\varphi}$ 后取实部，并在区域 $\mathbb{R}$ 上对 x 积分可得

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}}\langle(D_0D_0 - D_1D_1)D_\nu\varphi, D_0D_\nu\varphi\rangle\mathrm{d}x = {}& -\mathrm{Re}\int_{\mathbb{R}}\{e\partial_\nu(e|\varphi|^2+\kappa N)\langle\varphi, D_0D_\nu\varphi\rangle + \\ &e(e|\varphi|^2+\kappa N)\langle D_\nu\varphi, D_0D_\nu\varphi\rangle + e^2[B^2+(N+ev^2/\kappa)^2]\langle D_\nu\varphi, D_0D_\nu\varphi\rangle + \\ &e^2\partial_\nu[B^2+(N+ev^2/\kappa)^2]\langle\varphi, D_0D_\nu\varphi\rangle\}\mathrm{d}x + e\mathrm{Im}\int_{\mathbb{R}}(\partial_0\boldsymbol{F}_{0v}\langle\varphi, D_0D_\nu\varphi\rangle + \\ &2\boldsymbol{F}_{0v}\langle D_0\varphi, D_0D_\nu\varphi\rangle - \partial_1\boldsymbol{F}_{1v}\langle\varphi, D_0D_\nu\varphi\rangle - 2\boldsymbol{F}_{1v}\langle D_1\varphi, D_0D_\nu\varphi\rangle)\mathrm{d}x.\end{aligned} \tag{23}$$

由式(23)可得

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mu,\nu=0,1}\|D_\mu D_\nu\varphi(\cdot,t)\|_{L^2}^2 = {}& \mathrm{Re}\int_{\mathbb{R}}[\langle(D_0D_0 - D_1D_1)D_\nu\varphi, D_0D_\nu\varphi\rangle]\mathrm{d}x + \\ &e\boldsymbol{F}_{01}\int_{\mathbb{R}}\mathrm{Im}\langle D_\nu\varphi, D_1D_\nu\varphi\rangle\mathrm{d}x,\end{aligned}$$

于是得到如下估计：

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mu,\nu=0,1}\|D_\mu D_\nu\varphi(\cdot,t)\|_{L^2}^2 = {}& -\mathrm{Re}\int_{\mathbb{R}}\{(e^2\partial_\nu|\varphi|^2 + e\kappa\partial_\nu N)\langle\varphi, D_0D_\nu\varphi\rangle + \\ &e(e|\varphi|^2+\kappa N)\langle D_\nu\varphi, D_0D_\nu\varphi\rangle + e^2[B^2+(N+ev^2/\kappa)^2]\langle D_\nu\varphi, D_0D_\nu\varphi\rangle \\ &+ e^2\partial_\nu[B^2+(N+ev^2/\kappa)^2]\langle\varphi, D_0D_\nu\varphi\rangle\}\mathrm{d}x + e\mathrm{Im}\int_{\mathbb{R}}(\partial_0\boldsymbol{F}_{0v}\langle\varphi, D_0D_\nu\varphi\rangle + \\ &2\boldsymbol{F}_{0v}\langle D_0\varphi, D_0D_\nu\varphi\rangle - \partial_1\boldsymbol{F}_{1v}\langle\varphi, D_0D_\nu\varphi\rangle - \\ &2\boldsymbol{F}_{1v}\langle D_1\varphi, D_0D_\nu\varphi\rangle + \boldsymbol{F}_{01}\langle D_\nu\varphi, D_1D_\nu\varphi\rangle)\mathrm{d}x =: J_1 + J_2,\end{aligned}$$

其中

$$J_1=-\mathrm{Re}\int_{\mathbb{R}}\{(e^2\partial_\nu|\varphi|^2+e\kappa\partial_\nu N)\langle\varphi,D_0D_\nu\varphi\rangle+e(e|\varphi|^2+\kappa N)\langle D_\nu\varphi,D_0D_\nu\varphi\rangle+$$
$$e^2[B^2+(N+ev^2/\kappa)^2]\langle D_\nu\varphi,D_0D_\nu\varphi\rangle+e^2\partial_\nu[B^2+(N+ev^2/\kappa)^2]\langle\varphi,D_0D_\nu\varphi\rangle\}\mathrm{d}x\lesssim$$
$$(E_0^{1/2}\|\varphi\|_{L^\infty}^2+\|\partial N\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|D\varphi\|_{L^\infty}+E_0^{1/2}\|B\|_{L^\infty}^2+E_0^{1/2}\|N\|_{L^\infty}^2+$$
$$E_0^{1/2}+\|B\|_{L^\infty}\|\partial B\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|\partial N\|_{L^\infty})\|DD\varphi\|_{L^2},$$

$$J_2=e\mathrm{Im}\int_{\mathbb{R}}(\partial_0\boldsymbol{F}_{0v}\langle\varphi,D_0D_\nu\varphi\rangle+2\boldsymbol{F}_{0v}\langle D_0\varphi,D_0D_\nu\varphi\rangle-\partial_1\boldsymbol{F}_{1v}\langle\varphi,D_0D_\nu\varphi\rangle-$$
$$2\boldsymbol{F}_{1v}\langle D_1\varphi,D_0D_\nu\varphi\rangle+\boldsymbol{F}_{01}\langle D_\nu\varphi,D_1D_\nu\varphi\rangle)\mathrm{d}x\lesssim$$
$$(\|\partial B\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|\varphi\|_{L^\infty}^2+E_0^{1/2}\|\boldsymbol{F}_{10}\|_{L^\infty})\|DD\varphi\|_{L^2}.$$

这里根据式(1)和(2)并应用了估计$\partial_\nu|\varphi|^2=2\mathrm{Re}(\overline{\varphi}\partial_\nu\varphi)=2\mathrm{Re}(\overline{\varphi}D_\nu\varphi)$. 利用 Gagliardo-Nirenberg 不等式[10]可得关于N的估计$\|N\|_{L^\infty}^2\leqslant\|\partial_1N\|_{L^2}\|N\|_{L^2}\lesssim(I+E_0^{1/2})^2(1+t^{1/2})^2$，从而可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|DD\varphi\|_{L^2}^2\lesssim(E_0^{1/2}\|\varphi\|_{L^\infty}^2+\|\partial N\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|D\varphi\|_{L^\infty}+$$
$$E_0^{1/2}\|B\|_{L^\infty}^2+E_0^{1/2}\|N\|_{L^\infty}^2+E_0^{1/2}+\|B\|_{L^\infty}\|\partial B\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|\partial N\|_{L^\infty}+$$
$$\|\partial B\|_{L^\infty}\|\varphi\|_{L^2}+E_0^{1/2}\|\varphi\|_{L^\infty}^2+E_0^{1/2}\|\boldsymbol{F}_{10}\|_{L^\infty})\|DD\varphi\|_{L^2}\lesssim$$
$$(1+I+E_0^{1/2})^4(1+t^{1/2})^5\|DD\varphi\|_{L^2},$$

这里利用了定理1. 于是，得到关于$DD\varphi$的L^2范数估计：

$$\|DD\varphi\|_{L^2}\lesssim\|DD\varphi(\cdot,0)\|_{L^2}+(1+I+E_0^{1/2})^4(1+t^{1/2})^5t\lesssim$$
$$\|DD\varphi(\cdot,0)\|_{L^2}+(1+I+E_0^{1/2})^4(1+t^{1/2})^7.$$

证毕.

注2 定理2对(1+1)维 MCSH 模型的解进行了 H^2 范数的多项式增长估计.

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