

一类带次线性中立项和分布时滞的 三阶阻尼微分方程的振动性

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摘要: 利用处理次线性中立项的技术、广义 Riccati 变换和积分平均技巧, 首先, 给出一种估计 Riccati 变换不等式的解析方法; 其次, 考虑一类具有次线性中立项和分布时滞的三阶阻尼微分方程, 给出解振动或收敛于零的一些充分条件; 最后, 通过实例验证所得结果.

关键词: 振动性; 分布时滞; 次线性中立项

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Oscillation of a Class of Third-Order Damped Differential Equations with Sublinear Neutral Terms and Distributed Delays

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Abstract: By using the techniques of dealing with sublinear neutral terms, generalized Riccati transformation and integral averaging techniques, firstly, the author gave an analytical method for estimating Riccati transformation inequality. Secondly, the author considered a class of third-order damped differential equations with sublinear neutral terms and distributed delays, and obtained some sufficient conditions for the solution to oscillate or converge to zero. Finally, the results were verified by some examples.

Keywords: oscillation; distributed delay; sublinear neutral term

0 引言

考虑具有分布时滞和阻尼项的三阶中立型微分方程:

$$\left[p(x) \left(z(x) \left[\left(y(x) + \int_a^\beta r(x,t) y''(\tau(x,t)) dt \right)' \right]^\kappa \right)' \right]' + q(x) \left(z(x) \left[\left(y(x) + \int_a^\beta r(x,t) y''(\tau(x,t)) dt \right)' \right]^\kappa \right)' + \int_a^b G(x, \xi, y(\sigma(x, \xi))) d\xi = 0 \quad (1)$$

的振动性, 且满足下列条件:

(H₁) $0 < \mu \leq 1$, $\kappa \geq 1$, μ, κ 均为两个正奇数之商;

(H₂) $p(x) \in C^1([x_0, \infty), (0, \infty))$, $q(x) \in C([x_0, \infty), (0, \infty))$, $\int_{x_0}^\infty \frac{1}{p(x)} \exp\left\{-\int_{x_0}^x \frac{q(s)}{p(s)} ds\right\} dx =$

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∞;

$$(H_3) z(x) \in C^1([x_0, \infty), (0, \infty)), \int_{x_0}^{\infty} \frac{1}{z^{1/\kappa}(x)} dx = \infty;$$

$$(H_4) r(x, t) \in C([x_0, \infty) \times [\alpha, \beta], (-\infty, +\infty)), 0 \leq r(x) = \int_a^\beta r(x, t) dt \leq r < 1, \lim_{x \rightarrow \infty} r(x) = 0;$$

$$(H_5) \tau(x, t) \in C([x_0, \infty) \times [\alpha, \beta], (-\infty, +\infty)), \frac{\partial \tau(x, t)}{\partial t} \geq 0, \tau(x, t) \leq x, \lim_{x \rightarrow \infty} \min_{t \in [\alpha, \beta]} \tau(x, t) =$$

+∞;

$$(H_6) G(x, \xi, v) \in C([x_0, \infty) \times [a, b] \times (0, \infty), (0, \infty)), m(x, \xi) \in C([x_0, \infty) \times [a, b], (0, \infty)), G(x, \xi, v)/v^\kappa \geq m(x, \xi);$$

$$(H_7) \sigma(x, \xi) \in C([x_0, \infty) \times [a, b], (-\infty, +\infty)), \frac{\partial \sigma(x, \xi)}{\partial \xi} \geq 0, \sigma(x, \xi) \leq x, \lim_{x \rightarrow \infty} \min_{\xi \in [a, b]} \sigma(x, \xi) =$$

+∞.

令

$$w(x) = y(x) + \int_a^\beta r(x, t) y^\mu(\tau(x, t)) dt. \tag{2}$$

若函数 $y(x)$ 满足方程(1)及 $y(x) \in C^1([T_y, \infty), (-\infty, +\infty))$, $p(x)[z(x)(w'(x))^\kappa]' \in C^1([T_y, \infty), T_y \geq t_0)$, 则称 $y(x)$ 是方程(1)的一个解. 若 $y(x)$ 既非最终为正, 也非最终为负, 则称 $y(x)$ 在 $[T_y, \infty)$ 上振动; 否则, 称 $y(x)$ 是非振动的.

由于中立型微分方程广泛应用于自然科学和应用技术等领域, 因此关于三阶中立型微分方程解的振动性和渐近性研究备受关注^[1-12]. 文献[3]研究了三阶方程

$$(a(x)(b(x)(y(x) + py(x - \tau))')')' + q(x)f(y(x - \sigma)) = 0$$

的振动性; 文献[7]研究了具连续分布滞量的三阶微分方程

$$\left[r(x) \left[y(x) + \int_a^\beta p(x, s)y(\tau(x, s)) ds \right]'' \right]' + \int_a^b q(x, \xi)f(y(\sigma(x, \xi))) d\xi = 0$$

的振动性; 文献[8]研究了方程(1)当 $\mu = \kappa = 1$ 及 $Z(x)$ 恒为 1 时特例的振动性; 文献[10]研究了三阶阻尼微分方程

$$[r(t)([x(t) + p(t)x(\tau(t))]^\mu)']^a + m(t)([x(t) + p(t)x(\tau(t))]^\mu)^\alpha + q(t)f(x(\sigma(t)))g(x'(t)) = 0$$

的振动性; 文献[11]研究了方程(1)当 $\mu = \kappa = 1$ 时特例的振动性, 得到了若干解的振动性定理.

本文受文献[7-10]的启发, 考虑中立型方程(1)当 $0 < \mu \leq 1$ 和 $\kappa \geq 1$ 时更广泛的情形, 给出该方程的振动性条件及实例.

1 引 理

引理 1 若 $y(x)$ 是方程(1)的正解, 则当 $x \geq x_1 \geq x_0$ 时有下列两种可能:

- 1) $w(x) > 0, w'(x) > 0, (z(x)[w'(x)]^\kappa)' > 0;$
- 2) $w(x) > 0, w'(x) < 0, (z(x)[w'(x)]^\kappa)' > 0.$

证明: 首先, 设 $y(x)$ 是方程(1)在 $[x_0, \infty)$ 上的一个正解, 由条件(H₄)和(H₅)知, 存在 $x_1 > x_0$, 使得当 $x \geq x_1$ 时, 有 $y(\tau(x, t)) > 0, y(\sigma(x, \xi)) > 0$. 于是由式(2)和条件(H₃)有 $w(x) > y(x) > 0$.

其次, 由方程(1)和条件(H₆)有

$$(p(x)(z(x)[w'(x)]^\kappa)')' + q(x)(z(x)[w'(x)]^\kappa)' = - \int_a^b G(x, \xi, y(\sigma(x, \xi))) d\xi \leq - \int_a^b m(x, \xi)y(\sigma(x, \xi)) d\xi < 0,$$

因此可得

$$\left[\exp \left\{ \int_{x_1}^x \frac{q(s)}{p(s)} ds \right\} p(x)(z(x)[w'(x)]^\kappa)' \right]' \leq 0,$$

从而 $\exp\left\{\int_{x_1}^x \frac{q(s)}{p(s)} ds\right\} p(x)(z(x)[w'(x)]^\kappa)'$ 递减且最终定号. 由条件(H₁)知, 当 $x \geq x_2 \geq x_1$ 时, 有

$$(z(x)[w'(x)]^\kappa)' < 0 \text{ 或 } (z(x)[w'(x)]^\kappa)' > 0.$$

下面证明 $(z(x)[w'(x)]^\kappa)' > 0$. 假设 $(z(x)[w'(x)]^\kappa)' \leq 0$, 由 $\exp\left\{\int_{x_1}^x \frac{q(s)}{p(s)} ds\right\} p(x) \times (z(x)[w'(x)]^\kappa)'$ 的单调性知, 存在常数 $T > 0$, 使得

$$\exp\left\{\int_{x_1}^x \frac{q(s)}{p(s)} ds\right\} p(x)(z(x)[w'(x)]^\kappa)' \leq -T, \tag{3}$$

将式(3)从 x_2 到 x 积分得

$$z(x)[w'(x)]^\kappa \leq z(x_2)[w'(x_2)]^\kappa - T \int_{x_2}^x \frac{1}{p(s)} \exp\left\{-\int_{x_1}^s \frac{q(v)}{p(v)} dv\right\} ds.$$

令 $x \rightarrow \infty$, 由条件(H₁)得 $z(x)[w'(x)]^\kappa \rightarrow -\infty$. 因此由 $(z(t)[w'(t)]^\kappa)' \leq 0$, 可知当 $x \geq x_3 \geq x_2$ 时,

$$z(x)[w'(x)]^\kappa \leq z(x_3)[w'(x_3)]^\kappa < 0,$$

从而有

$$w'(x) \leq z^{1/\kappa}(x_3)w'(x_3) \frac{1}{z^{1/\kappa}(x)}. \tag{4}$$

对式(4)从 x_3 到 x 积分, 有

$$w(x) - w(x_3) \leq z^{1/\kappa}(x_3)w'(x_3) \int_{x_3}^x \frac{1}{z^{1/\kappa}(s)} ds.$$

令 $x \rightarrow \infty$, 由条件(H₂)可得 $w(x) \rightarrow -\infty$, 与 $w(x) > 0$ 矛盾, 进而有 $(z(x)[w'(x)]^\kappa)' > 0$. 证毕.

引理 2 设 $y(x)$ 是方程(1)的最终正解, $w(x)$ 满足引理 1 中结论 2), 若

$$\int_{x_0}^\infty \left(\frac{1}{z(v)} \int_v^\infty \frac{1}{p(u)} \int_u^\infty m(s) ds du \right)^{1/\kappa} dv = \infty, \tag{5}$$

则 $\lim_{x \rightarrow \infty} y(x) = 0$, 其中 $m(x) = \int_a^b m(x, \xi) d\xi$.

证明: 由于 $w(t)$ 满足引理 1 的结论 2), 即 $w(x) > 0, w'(x) < 0$, 又由单调有界定理可知 $\lim_{x \rightarrow \infty} w(x)$ 存在, 记 $\lim_{x \rightarrow \infty} w(x) = j$, 则 $j \geq 0$. 假设 $j > 0$, 由 $\lim_{x \rightarrow \infty} r(x) = 0$, 则 $\forall 0 < \epsilon < \min\{j, 2^{-\mu} j^{1-\mu}\}$, 存在 $j < w(x) < j + \epsilon, 0 \leq r(x) < \epsilon$, 使得

$$\begin{aligned} y(x) &= w(x) - \int_a^\beta r(x, t) y^\mu(\tau(x, t)) dt \geq j - \int_a^\beta r(x, t) w^\mu(\tau(x, t)) dt > \\ &j - (j + \epsilon)^\mu \epsilon \geq j - (2j)^\mu \epsilon = K(j + \epsilon) > Kw(x), \end{aligned} \tag{6}$$

其中 $K = \frac{j - (2j)^\mu \epsilon}{j + \epsilon} > 0$, 利用条件(H₆), (H₇)及引理 1 中结论 2)和式(6), 可得

$$\begin{aligned} (p(x)(z(x)[w'(x)]^\kappa)')' + q(x)(z(x)[w'(x)]^\kappa)' &\leq - \int_a^b m(x, \xi) y^\kappa(\sigma(x, \xi)) d\xi \leq \\ &- K^\kappa w^\kappa(\sigma(x, b))m(x). \end{aligned}$$

令 $\alpha(x) = \exp\left\{\int_{x_1}^x \frac{q(s)}{p(s)} ds\right\}$, 则有

$$(\alpha(x)p(x)(z(x)[w'(x)]^\kappa)')' \leq -K^\kappa \alpha(x)w^\kappa(\sigma(x, b))m(x). \tag{7}$$

对式(7)在 $(x, +\infty)$ 上积分, 有

$$-\alpha(x)p(x)(z(x)[w'(x)]^\kappa)' + K^\kappa \int_x^\infty \alpha(s)w^\kappa(\sigma(s, b))m(s) ds \leq 0.$$

再注意到 $w(\sigma(x, b)) > j$ 和 $w'(x) > 0$, 有

$$-(z(x)[w'(x)]^\kappa)' + \frac{K^\kappa j^\kappa}{p(x)} \int_x^\infty m(s) ds < 0,$$

从而

$$z(x)[w'(x)]^\kappa + K^\kappa j^\kappa \int_x^\infty \frac{1}{p(u)} \int_u^\infty m(s) ds du < 0,$$

进而有

$$-w'(x) \geq K_j \left(\frac{1}{z(x)} \int_x^\infty \frac{1}{p(u)} \int_u^\infty m(s) ds du \right)^{1/\kappa}. \tag{8}$$

对式(8)在 (x_1, ∞) 上积分, 得

$$\int_{x_1}^\infty \left(\frac{1}{z(v)} \int_v^\infty \frac{1}{p(u)} \int_u^\infty m(s) ds du \right)^{1/\kappa} dv < \frac{w(x_1)}{K_j},$$

与条件(5)矛盾. 因此 $j=0$. 又因为 $w(x) > y(x) > 0$, 所以有 $\lim_{x \rightarrow \infty} y(x) = 0$. 证毕.

引理 3^[13] 设 $0 < \lambda \leq 1$, 则:

- 1) $X^\lambda + Y^\lambda \leq 2^{1-\lambda} (X+Y)^\lambda$, X, Y 为非负实数;
- 2) $(1+X)^\lambda \leq 1 + \lambda X$, 其中 $1+X > 0$.

2 主要结果

下面利用 Riccati 变换和文献[14]中的一些估计, 证明方程(1)的一些新的振动性结果. 记

$$E = \{(x, s) \mid x \geq s \geq x_0\}, \quad E_0 = \{(x, s) \mid x > s \geq x_0\}.$$

如果函数 $J(x, s) \in C(E, R)$ 具有下列性质, 则称 $J(x, s)$ 属于 Y 类, 记作 $J \in Y$:

- (i) $J(x, x) = 0, x \geq x_0, J(x, s) > 0, (x, s) \in E_0$;
- (ii) $\frac{\partial J(x, s)}{\partial s} \leq 0, (x, s) \in E$;
- (iii) 存在函数 $f(x, s) \in C(E, R)$, 使得

$$\frac{\partial J(x, s)}{\partial s} = -f(x, s) \sqrt{J(x, s)}, \quad (x, s) \in E.$$

定理 1 设式(5)成立, 且存在函数 $J \in Y$ 和 $\psi \in C^1([x_0, \infty), (-\infty, +\infty))$, 使得

$$0 < \inf_{s \geq x_0} \left[\liminf_{x \rightarrow \infty} \frac{J(x, s)}{J(x, x_0)} \right] \leq \infty, \tag{9}$$

$$\int_{x_0}^\infty \frac{\phi_+^2(x)}{g(x)p(x)} dx = \infty, \tag{10}$$

$$\limsup_{x \rightarrow \infty} \frac{1}{J(x, X)} \int_X^x \left[J(x, s)\beta(s) - \frac{1}{4} \eta g(s)p(s)f^2(x, s) \right] ds \geq \psi(X), \tag{11}$$

其中

$$\begin{aligned} \varphi_+(s) &= \max\{\varphi(s), 0\}, \\ g(x) &= \exp \int_{x_0}^x \frac{q(s)}{p(s)} ds, \\ \beta(x) &= g(x) \left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^\kappa \frac{\gamma m(x)}{z(x)}, \quad \eta > 1, \quad \gamma > 0. \end{aligned} \tag{12}$$

则方程(1)的解 $y(x)$ 振动或 $\lim_{x \rightarrow \infty} y(x) = 0$.

证明: 设 $y(x)$ 是方程(1)的一个非振动解, 不失一般性, 设 $y(x) > 0, x \in [x_1, \infty)$. 由条件 $(H_4), (H_5)$ 有

$$y(\tau(x, t)) > 0, \quad (x, t) \in [x_1, \infty) \times [\alpha, \beta], \quad y(\sigma(x, \xi)) > 0, \quad (x, \xi) \in [x_1, \infty) \times [a, b].$$

当 $w(x)$ 满足引理 1 中结论 1) 时, 由条件 $(H_4), (H_5)$, 有

$$\begin{aligned} y(x) &\geq w(x) - (1 + w^\mu(x))r + r \geq w(x) - 2^{1-\mu}(1 + w(x))^\mu r + r \geq \\ &\quad (1 - \mu 2^{1-\mu} r)w(x) - (2^{1-\mu} - 1)r. \end{aligned} \tag{13}$$

由引理 1 中结论 1) 知, $w(x) > 0, w'(x) > 0$, 所以

$$w(\sigma(x, a)) \geq w(\sigma(x_1, a)) = k_1, \quad x \geq x_1. \tag{14}$$

从而由条件 $(H_6), (H_7)$ 和上述不等式, 有

$$(p(x)(z(x)[w'(x)]^\kappa)')' + q(x)(z(x)[w'(x)]^\kappa)' \leq - \int_a^b m(x, \xi) y^\kappa(\sigma(x, \xi)) d\xi \leq$$

$$\begin{aligned}
& - \int_a^b m(x, \xi) [(1 - \mu 2^{1-\mu} r) w(\sigma(x, a)) - (2^{1-\mu} - 1)r]^{\kappa} d\xi \leq \\
& - w^{\kappa}(\sigma(x, a)) \left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^{\kappa} m(x).
\end{aligned}$$

令 $H(x) = g(x) \frac{p(x)(z(x)[w'(x)]^{\kappa})' }{z(x)[w'(x)]^{\kappa}}$, $x \geq x_1$, 其中 $g(x)$ 由式(12)定义, 于是

$$\begin{aligned}
H'(x) & \leq \frac{g'(x)}{g(x)} H(x) + g(x) \frac{(p(x)(z(x)[w'(x)]^{\kappa})')'}{z(x)[w'(x)]^{\kappa}} - \frac{H^2(x)}{g(x)p(x)} \leq \\
& \frac{g'(x)}{g(x)} H(x) - g(x) \left[\left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^{\kappa} \frac{w^{\kappa}(\sigma(x, a))m(x)}{z(x)[w'(x)]^{\kappa}} + \right. \\
& \left. \frac{q(x)(z(x)[w'(x)]^{\kappa})' }{z(x)[w'(x)]^{\kappa}} - \frac{H^2(x)}{g(x)p(x)} \right] = \\
& - g(x) \left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^{\kappa} \frac{w^{\kappa}(\sigma(x, a))m(x)}{z(x)[w'(x)]^{\kappa}} - \frac{H^2(x)}{g(x)p(x)}. \tag{15}
\end{aligned}$$

由引理 1 中结论 1) 知, 存在极限 $\lim_{x \rightarrow \infty} \frac{1}{[w'(x)]^{\kappa}}$, 记为 $\lim_{x \rightarrow \infty} \frac{1}{[w'(x)]^{\kappa}} = \theta$. 选取 $\varepsilon = \frac{\theta}{2}$, 则

$\frac{1}{[w'(x)]^{\kappa}} > \frac{\theta}{2}$, $x \geq x_2 \geq x_1$. 再取 $\gamma = \frac{w^{\kappa}(\lambda)\theta}{2}$, 则由条件(H₆)有 $\sigma(x, a) \geq \lambda$, 从而

$$H'(x) = -\beta(x) - \frac{H^2(x)}{g(x)p(x)}. \tag{16}$$

将式(16)两边乘以 $J(x, s)$, 并在 $[x_2, x]$ 上积分, 得

$$\begin{aligned}
\int_{x_2}^x J(x, s)\beta(s)ds & \leq J(x, x_2)H(x_2) - \int_{x_2}^x f(x, s) \sqrt{J(x, s)} H(s)ds - \int_{x_2}^x \frac{J(x, s)H^2(s)}{g(s)p(s)}ds = \\
& J(x, x_2)H(x_2) - \int_{x_2}^x \left(f(x, s) \frac{\sqrt{\eta g(s)p(s)}}{2} + H(s) \sqrt{\frac{J(x, s)}{\eta g(s)p(s)}} \right)^2 ds + \\
& \int_{x_2}^x \frac{\eta g(s)p(s)f^2(x, s)}{4}ds - \int_{x_2}^x \frac{(\eta - 1)J(x, s)H^2(s)}{\eta g(s)p(s)}ds,
\end{aligned}$$

从而

$$\begin{aligned}
\limsup_{x \rightarrow \infty} \frac{1}{J(x, x_2)} \int_{x_2}^x \left(J(x, s)\beta(s) - \frac{\eta g(s)p(s)f^2(x, s)}{4} \right) ds & \leq \\
H(x_2) - \liminf_{x \rightarrow \infty} \frac{1}{J(x, x_2)} \int_{x_2}^x \frac{(\eta - 1)J(x, s)H^2(s)}{\eta g(s)p(s)} ds, \tag{17}
\end{aligned}$$

于是

$$\liminf_{x \rightarrow \infty} \frac{1}{J(x, x_2)} \int_{x_2}^x \frac{(\eta - 1)J(x, s)H^2(s)}{\eta g(s)p(s)} ds \leq H(x_2) - \psi(x_2) < \infty, \quad x \geq x_2. \tag{18}$$

断言 $\int_{x_2}^{\infty} \frac{H^2(x)}{g(x)p(x)} dx < \infty$. 若 $\int_{x_2}^{\infty} \frac{H^2(x)}{g(x)p(x)} dx = \infty$, 则由式(9), 对某个 $\delta > 0$ 有

$$\inf_{s \geq x_0} \left[\liminf_{x \rightarrow \infty} \frac{J(x, s)}{J(x, x_0)} \right] > \delta.$$

表明 $\frac{J(x, x_3)}{J(x, x_0)} > \delta$, $x \geq x_3 \geq x_2$, 从而对任意 $N > 0$, 有

$$\int_{x_3}^x \frac{H^2(x)}{g(x)p(x)} dx \geq \frac{N}{\delta},$$

进而

$$\begin{aligned}
\frac{1}{J(x, x_0)} \int_{x_3}^x \frac{J(x, s)H^2(s)}{g(s)p(s)} ds & = \frac{1}{J(x, x_0)} \int_{x_3}^x \left(-\frac{\partial J(x, s)}{\partial s} \right) \int_{x_3}^s \frac{H^2(v)}{g(v)p(v)} dv ds \geq \\
& \frac{1}{J(x, x_0)} \frac{N}{\delta} \int_{x_3}^x \left(-\frac{\partial J(x, s)}{\partial s} \right) ds = \frac{N}{\delta} \frac{J(x, x_3)}{J(x, x_0)} \geq N.
\end{aligned}$$

于是

$$\liminf_{x \rightarrow \infty} \frac{1}{J(x, x_0)} \int_{x_3}^x \frac{J(x, s) H^2(s)}{g(s) p(s)} ds = \infty,$$

与式(18)矛盾. 因此有 $\int_{x_2}^{\infty} \frac{H^2(x)}{g(x) p(x)} dx < \infty$, 与条件(10)矛盾.

当 $w(x)$ 满足引理 1 中结论 2) 时, 由式(5)和引理 2, 有 $\lim_{x \rightarrow \infty} w(x) = \lim_{x \rightarrow \infty} y(x) = 0$. 证毕.

定理 2 设式(5)成立, 若存在函数 $J \in Y$ 和 $B(x) \in C([x_0, \infty), (-\infty, +\infty))$, 使得对 $x \geq X \geq x_0$, 满足

$$\limsup_{x \rightarrow \infty} \frac{1}{J(x, X)} \int_x^X \left(J(x, s) A(s) - \frac{\eta g(s) p(s) f^2(x, s)}{4(\eta - 1)} \right) ds = \infty, \quad (19)$$

则方程(1)的解 $y(x)$ 振动或 $\lim_{x \rightarrow \infty} y(x) = 0$, 其中 $\eta > 1, \gamma > 0$,

$$A(x) = M(x) - \eta g(x) p(x) B^2(x), \quad g(x) = \exp \int_{x_0}^x \frac{q(s)}{p(s)} ds, \quad (20)$$

$$M(x) = g(x) \left[p(x) B^2(x) - q(x) B(x) - (p(x) B(x))' + \left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^\kappa \frac{\gamma m(x)}{z(x)} \right]. \quad (21)$$

证明: 设 $y(x)$ 是方程(1)的一个非振动解, 不失一般性, 设 $y(x) > 0, x \in [x_1, \infty)$. 由条件 $(H_4), (H_5)$ 有

$$y(\tau(x, t)) > 0, \quad (x, t) \in [x_1, \infty) \times [\alpha, \beta], \quad y(\sigma(x, \xi)) > 0, \quad (x, \xi) \in [x_1, \infty) \times [a, b].$$

当 $w(x)$ 满足引理 1 中结论 1) 时, 令

$$H(x) = g(x) \left[\frac{p(x)(z(x)[w'(x)]^\kappa)'}{z(x)[w'(x)]^\kappa} + p(x) B(x) \right], \quad x \geq x_1,$$

于是

$$H'(x) \leq \frac{g'(x)}{g(x)} H(x) + g(x) (p(x) B(x))' - g(x) \left[\left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^\kappa \frac{w^\kappa(\sigma(x, a)) m(x)}{z(x)[w'(x)]^\kappa} + \frac{q(x)(z(x)[w'(x)]^\kappa)'}{z(x)[w'(x)]^\kappa} + p(x) \left(\frac{z(x)[w'(x)]^\kappa}{z(x)[w'(x)]^\kappa} \right)^2 \right].$$

类似式(15), 取 $\lim_{x \rightarrow \infty} \frac{1}{[w'(x)]^\kappa} = \theta, \varepsilon = \frac{\eta}{2}, \gamma = \frac{w^\kappa(\lambda) \theta}{2}$, 则有

$$H'(x) \leq \frac{g'(x)}{g(x)} H(x) + g(x) (p(x) B(x))' - g(x) \left[\left\{ 1 - \left[\mu 2^{1-\mu} + \frac{(2^{1-\mu} - 1)}{k_1} \right] r \right\}^\kappa \frac{\gamma m(x)}{z(x)} + \frac{q(x) H(x)}{g(x) p(x)} - q(x) B(x) + p(x) \left(\frac{H(x)}{g(x) p(x)} - B(x) \right)^2 \right] = -M(x) + 2B(x) H(x) - \frac{H^2(x)}{\eta g(x) p(x)} - \frac{(\eta - 1) H^2(x)}{\eta g(x) p(x)},$$

其中 $M(x)$ 由式(21)定义. 利用不等式 $Du - Cu^2 \leq \frac{D^2}{4C}, C > 0, u \in \mathbb{R}$, 得

$$H'(x) \leq -A(x) - \frac{(\eta - 1) H^2(x)}{\eta g(x) p(x)}, \quad (22)$$

其中 $A(x)$ 来自式(20). 将式(22)两边乘以 $J(x, s)$, 并在 $[X, x]$ 上积分得

$$\begin{aligned} \int_X^x J(x, s) A(s) ds &\leq J(x, X) H(X) - \int_{x_2}^x \left(f(x, s) \sqrt{J(x, s)} H(s) + \frac{(\eta - 1) J(x, s) H^2(s)}{\eta g(s) p(s)} \right) ds = \\ &J(x, X) H(X) - \int_X^x \left(f(x, s) \sqrt{\frac{\eta g(s) p(s)}{4(\eta - 1)}} + H(s) \sqrt{\frac{(\eta - 1) J(x, s)}{\eta g(s) p(s)}} \right)^2 ds + \\ &\int_X^x \frac{\eta g(s) p(s) f^2(x, s)}{4(\eta - 1)} ds, \end{aligned}$$

从而有

$$\frac{1}{J(x, X)} \int_x^X \left(J(x, s)A(s) - \frac{\eta g(s)p(s)f^2(x, s)}{4(\eta - 1)} \right) ds \leq H(X),$$

与式(16)矛盾.

当 $w(x)$ 满足引理 1 中结论 2) 时, 由式(5)和引理 2 有 $\lim_{x \rightarrow \infty} w(x) = \lim_{x \rightarrow \infty} y(x) = 0$. 证毕.

注 1 若方程(1)中取 $\mu = \kappa = 1$, 则定理 1 和定理 2 即为文献[10]的振动结果, 进而改进并推广了文献[7-9]的相应结果.

3 实 例

例 1 考虑三阶阻尼微分方程

$$\left[\frac{1}{x} \left(e^{-x} \left[\left(y(x) + \int_1^2 \frac{t}{6x} y^{1/3}(x-t) dt \right)' \right]^2 \right)' \right]' + \frac{2}{x^2} \left(e^{-x} \left[\left(y(x) + \int_1^2 \frac{t}{6x} y^{1/3}(x-t) dt \right)' \right]^2 \right)' + \int_{1/2}^1 2x\xi y^2((x-3\sqrt{x})\xi) d\xi = 0. \tag{23}$$

记 $p(x) = \frac{1}{x}$, $q(x) = \frac{1}{x^2}$, $z(x) = e^{-x}$, $r(x, t) = \frac{t}{6x}$, $\kappa = 2$, $\alpha = 1$, $\beta = 2$, $a = \frac{1}{2}$, $b = 1$, $\tau(x, t) = x - t$, $\sigma(x, \xi) = (x - 3\sqrt{x})\xi$, $\mu = \frac{1}{3}$, 则

$$r(x) = \int_a^\beta r(x, t) dt = \int_1^2 \frac{t}{6x} dt = \frac{1}{4x} \leq \frac{1}{2}, \quad \lim_{x \rightarrow \infty} r(x) = \lim_{x \rightarrow \infty} \frac{1}{4x} = 0,$$

于是满足条件 $(H_1) \sim (H_7)$. 进一步, 令

$$J(x, s) = (x - s)^2, \quad \eta = 2, \quad \gamma = 2, \quad r = \frac{1}{2}, \quad m(x, \xi) = 2x\xi, \quad \psi(x) = x, \quad x_0 = 1,$$

利用定理 1, 有

$$f(x, s) = 2, \quad g(x) = x^2 - 1, \quad m(x) = \frac{3}{8}x, \quad \beta(x) = \frac{3}{8}x(x^2 - 1)e^x,$$

$$\limsup_{x \rightarrow \infty} \frac{1}{J(x, X)} \int_x^X \left[J(x, s)\beta(s) - \frac{1}{4}\eta g(s)p(s)f^2(x, s) \right] ds \geq X = \psi(X).$$

因此满足定理 1 的所有条件. 故由定理 1 知, 方程(23)的任意解振动或趋于 0.

例 2 考虑三阶阻尼微分方程

$$\left[\frac{2}{x} \left(\sqrt{x} \left[\left(y(x) + \int_0^1 e^{-x} t^2 y^{1/3} \left(\frac{1}{3}xt \right) dt \right)' \right]^2 \right)' \right]' + \frac{1}{x^2} \left(\sqrt{x} \left[\left(y(x) + \int_0^1 e^{-x} t^2 y^{1/3} \left(\frac{1}{3}xt \right) dt \right)' \right]^2 \right)' + \int_{1/2}^1 \frac{\xi}{\sqrt{x}} y^2(x\xi) d\xi = 0. \tag{24}$$

记 $p(x) = \frac{2}{x}$, $q(x) = \frac{1}{x^2}$, $z(x) = \sqrt{x}$, $r(x, t) = e^{-x}t^2$, $\kappa = 2$, $\alpha = 0$, $\beta = 1$, $a = \frac{1}{2}$, $b = 1$, $\tau(x, t) = \frac{1}{3}xt$, $\mu = \frac{1}{3}$, $\sigma(x, \xi) = x\xi$, 则满足条件 $(H_1) \sim (H_7)$. 进一步, 令 $J(x, s) = (x - s)^2$, $\eta = 2$, $\gamma = 2$,

$r = 0.9$, $m(x, \xi) = \frac{\xi}{\sqrt{x}}$, $B(x) = \frac{1}{x}$, $x_0 = 1$, 利用定理 2, 有

$$f(x, s) = 2, \quad g(x) = \sqrt{x} - 1, \quad m(x) = \frac{3}{8\sqrt{x}}, \quad A(x) = (\sqrt{x} - 1) \left(\frac{3}{16x} + \frac{1}{x^3} \right),$$

$$\limsup_{x \rightarrow \infty} \frac{1}{J(x, X)} \int_x^X \left(J(x, s)A(s) - \frac{\eta g(s)p(s)f^2(x, s)}{4(\eta - 1)} \right) ds = \infty.$$

因此满足定理 2 的所有条件. 故由定理 2 知, 方程(24)的任意解振动或趋于 0.

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