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一类双曲守恒律方程组退化 Goursat 问题整体光滑解的存在性

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摘要：针对一类双曲守恒律方程组退化 Goursat 问题，研究其整体光滑解的存在性. 首先，引入特征角 α,β，建立 α,β 和压力 P 的特征分解；其次，利用 α,β 的特征分解得到不变区域，进而得到特征角的最大模估计；最后，通过压力 P 的特征分解以及连续性方法建立解的梯度估计，从而证明退化 Goursat 问题解的存在性.

关键词：双曲守恒律方程组；特征分解；退化 Goursat 问题；平面稀疏波
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Existence of Global Smooth Solutions for Degenerate Goursat Problem of a Class of Hyperbolic Conservation Law Systems

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Abstract: We studied the existence of the global smooth solutions for degenerate Gourset problem of a class of hyperbolic conversation law systems. Firstly, we introduced characteristic angles α,β, and established characteristic decompositions for α,β and pressure P. Secondly, the characteristic decompositions of α,β were used to obtain the invariant region, and then the maximum norm estimate of the characteristic angles were obtained. Finally, the gradient estimates of the solution were established by the characteristic decomposition of pressure P and continuity method, which proved the existence of the solutions to the degenerate Gourset problem.

Keywords: hyperbolic conservation law system; characteristic decomposition; degenerate Goursat problem; planar rarefaction wave

1　引言与预备知识

非线性双曲守恒律方程组来源于物理和力学等中的许多自然现象. 其中二维 Riemann 问题的研究在理论和实际中均具有重要意义. 对于 Euler 方程的二维 Riemann 问题，文献[1]基于广义特征分析法和数值实验提出了一系列猜想. 但跨音速结构的存在以及小尺度结构[2-3]使得该类问题的解更复杂. 文献[4-5]根据对称性避免了这些结构，得到了一些有意义的结果. 对于 Goursat 问题，Barthwal 等[6]

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为得到对应二维 Riemann 问题直到真空边界的全局解的存在性，考虑 Euler 方程组 Goursat 型边值问题，并在文献[7]中通过特征分解和自举法的思想得到了简单波和平面稀疏波相互作用相应的二维 Riemann 问题在真空边界处解的整体存在性；Dai 等[8]考虑气体动力学压差方程的 Goursat 问题，证明了退化真空 Goursat 问题整体光滑解的存在性；Hu 等[9]利用文献[8]的方法将非线性波动方程局部解延拓为整体解.

Song 等[10]为得到 $f(P)=P$ 条件下退化到声速线上解的存在性，在二维压力梯度系统的研究中构造了一种新型波，这种新型波称为解的半双曲片. 半双曲片作为上述结构中的一种，经常出现在 Euler 方程及相关模型的二维 Riemann 问题中. Li 等[11]通过引入特征角研究了二维可压缩 Euler 方程半双曲片结构；Hu 等[12]对等温 Euler 系统的半双曲片解进行了研究. 该类型解在翼型的跨音速流和 Guderley 反射中应用广泛. Feng 等[13]证明了一类压强定律的二维可压缩 Euler 方程半双曲片解的存在性；Chen 等[14]研究了二维可压缩 Euler 系统中非理想气体在相应大小尖角处的膨胀问题，用推广局部解并结合估计和双曲性得到了非理想气体与真空交界处全局解的存在性. 本文与文献[10]不同，主要利用特征角建立 C^1 估计得到整体光滑解.

考虑下列双曲守恒律方程组的退化 Goursat 问题：

$$\begin{cases} u_t + P_x = 0, \\ v_t + P_y = 0, \\ P_t + f(P)(u_x + v_y) = 0, \end{cases} \tag{1}$$

其中 P 为压力，(u,v) 为速度，$f(P)\in C^2$ 满足

$$f>0,\quad f'>0,\quad \frac{1}{2}f'^2+ff''>0. \tag{2}$$

令 $v_4=0$，c_1,c_4,v_1 为 3 个实数且 $c_1>c_4>0$，定义 $\eta_i \triangleq \sqrt{f(P_i)}$，$i=1,4$. 考虑平面稀疏波 $R_{14}(\eta)$：

$$\begin{cases} \eta=\sqrt{f(P)}, & \eta_4 \leqslant \eta \leqslant \eta_1, \\ v=\displaystyle\int_{P_4}^{P}\frac{1}{\sqrt{f(s)}}\mathrm{d}s, & 0<P_4\leqslant P\leqslant P_1, \\ u=u_1=u_4=0. \end{cases} \tag{3}$$

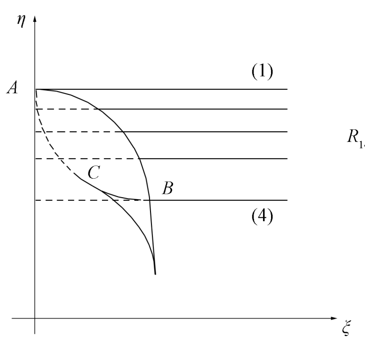

图 1 退化双曲区域

Fig. 1 Degenerate hyperbolic regions

考虑在稀疏波 $R_{14}(\eta)$ 中的正特征线 $\widehat{AB}$，其中点 A 为声速点. 令 $\widehat{BC}$ 为严格凸的负特征线，且点 C 为声速点. 在区域退化双曲区域 ABC 内求解，其中 $\widehat{AC}$ 为声速线，如图 1 所示.

令 $(\xi,\eta)=\left(\dfrac{x}{t},\dfrac{y}{t}\right)$，则式(1)为

$$\begin{cases} -\xi u_\xi - \eta u_\eta + P_\xi = 0, \\ -\xi v_\xi - \eta v_\eta + P_\eta = 0, \\ -\xi P_\xi - \eta P_\eta + f(P)(u_\xi + v_\eta) = 0, \end{cases} \tag{4}$$

其特征值为

$$\Lambda_\pm = \frac{\xi\eta \pm \sqrt{f(P)(\xi^2+\eta^2-f(P))}}{\xi^2 - f(P)}\left(=\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right). \tag{5}$$

将式(4)化简，得

$$(f(P)-\xi^2)P_{\xi\xi} + (f(P)-\eta^2)P_{\eta\eta} - 2\xi\eta P_{\xi\eta} - 2(\xi P_\xi + \eta P_\eta) + \frac{f'(P)}{f(P)}(\xi P_\xi + \eta P_\eta)^2 = 0. \tag{6}$$

引入特征角：$\tan(\alpha)=\Lambda_+$，$\tan(\beta)=\Lambda_-$；马赫角：$\omega=\dfrac{\alpha-\beta}{2}$；流角：$\tau=\dfrac{\alpha+\beta}{2}$. 方向导数定义为 $\partial^\pm=\partial_\xi+\Lambda_\pm\partial_\eta$. 从而

$$\begin{cases} \xi=\sqrt{f}\ \dfrac{\cos(\tau)}{\sin(\omega)}, \\ \eta=\sqrt{f}\ \dfrac{\sin(\tau)}{\sin(\omega)}. \end{cases} \tag{7}$$

2　P 的特征分解

根据文献[15]，首先限制 $\alpha\in\left[\dfrac{\pi}{2},\pi\right]$，$\beta\in\left[-\dfrac{\pi}{2},0\right]$. 设

$$\bar{\partial}^{+}=\cos(\alpha)\partial_{\xi}+\sin(\alpha)\partial_{\eta},\qquad \bar{\partial}^{-}=\cos(\beta)\partial_{\xi}+\sin(\beta)\partial_{\eta}. \tag{8}$$

用$\bar{\partial}^{\pm}P$ 表示特征角 α，β 的一阶方向导数.

性质 1　一阶偏导关系为

$$\begin{cases} \bar{\partial}^{+}\alpha=\dfrac{f'}{2f}\tan(\omega)\bar{\partial}^{+}P, \\ \bar{\partial}^{-}\alpha=-\dfrac{2\sin^2(\omega)}{\sqrt{f}}+\dfrac{f'}{2f}\tan(\omega)\bar{\partial}^{-}P, \\ \bar{\partial}^{+}\beta=\dfrac{2\sin^2(\omega)}{\sqrt{f}}-\dfrac{f'}{2f}\tan(\omega)\bar{\partial}^{+}P, \\ \bar{\partial}^{-}\beta=-\dfrac{f'}{2f}\tan(\omega)\bar{\partial}^{-}P, \\ \bar{\partial}^{\pm}\tau=\pm\dfrac{\sin^2(\omega)}{\sqrt{f}}, \\ \bar{\partial}^{\pm}\omega=-\dfrac{\sin^2(\omega)}{\sqrt{f}}+\dfrac{f'}{2f}\tan(\omega)\bar{\partial}^{\pm}P. \end{cases} \tag{9}$$

证明：由式(7)，(8)得

$$\begin{aligned} \bar{\partial}^{+}\xi=\cos(\alpha)=&\frac{f'}{2\sqrt{f}}\bar{\partial}^{+}P\ \frac{\cos(\tau)}{\sin(\omega)}- \\ &\sqrt{f}\ \frac{[\sin(\tau)\sin(\omega)+\cos(\tau)\cos(\omega)]\bar{\partial}^{+}\alpha+[\sin(\tau)\sin(\omega)-\cos(\tau)\cos(\omega)]\bar{\partial}^{+}\beta}{2\sin^2(\omega)}, \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{\partial}^{+}\eta=\sin(\alpha)=&\frac{f'}{2\sqrt{f}}\bar{\partial}^{+}P\ \frac{\sin(\tau)}{\sin(\omega)}+ \\ &\sqrt{f}\ \frac{[\cos(\tau)\sin(\omega)-\sin(\tau)\cos(\omega)]\bar{\partial}^{+}\alpha+[\cos(\tau)\sin(\omega)+\sin(\tau)\cos(\omega)]\bar{\partial}^{+}\beta}{2\sin^2(\omega)}. \end{aligned} \tag{11}$$

结合式(10)及式(11)得

$$\bar{\partial}^{+}\alpha=\frac{2\sin^2(\omega)}{\sqrt{f}}-\bar{\partial}^{+}\beta, \tag{12}$$

同理

$$\bar{\partial}^{-}\beta=-\frac{2\sin^2(\omega)}{\sqrt{f}}-\bar{\partial}^{-}\alpha. \tag{13}$$

将式(12)代入式(10)得

$$\bar{\partial}^{+}\beta=\frac{2\sin^2(\omega)}{\sqrt{f}}-\frac{f'}{2f}\tan(\omega)\bar{\partial}^{+}P, \tag{14}$$

同理

$$\bar{\partial}^{-}\alpha=-\frac{2\sin^2(\omega)}{\sqrt{f}}+\frac{f'}{2f}\tan(\omega)\bar{\partial}^{-}P. \tag{15}$$

由式(12)，(14)得

$$\bar{\partial}^{+}\alpha=\frac{f'}{2f}\tan(\omega)\bar{\partial}^{+}P, \tag{16}$$

同理

$$\bar{\partial}^{-}\beta=-\frac{f'}{2f}\tan(\omega)\bar{\partial}^{-}P. \tag{17}$$

根据τ,ω与特征角α,β的关系得

$$\bar{\partial}^{\pm}\tau=\pm\frac{\sin^{2}(\omega)}{\sqrt{f}}, \tag{18}$$

$$\bar{\partial}^{\pm}\omega=-\frac{\sin^{2}(\omega)}{\sqrt{f}}+\frac{f'}{2f}\tan(\omega)\bar{\partial}^{\pm}P. \tag{19}$$

定理1(**$\bar{\partial}^{\pm}$的交换子关系**) 对任意的C^{2}-光滑函数$I(\xi,\eta)$，有

$$\bar{\partial}^{+}\bar{\partial}^{-}I-\bar{\partial}^{-}\bar{\partial}^{+}I=\frac{1}{\sin(2\omega)}[(\bar{\partial}^{+}\beta-\cos(2\omega)\bar{\partial}^{-}\alpha)\bar{\partial}^{+}I-(\cos(2\omega)\bar{\partial}^{+}\beta-\bar{\partial}^{-}\alpha)\bar{\partial}^{-}I]. \tag{20}$$

定理1证明略.

定理2 压力P的特征分解为

$$\begin{cases}\bar{\partial}^{-}\bar{\partial}^{+}P=\bar{\partial}^{+}P\left[-\dfrac{\sin(2\omega)}{\sqrt{f}}+\dfrac{f'}{f}\dfrac{1}{4\cos^{2}(\omega)}\bar{\partial}^{+}P+\left(\dfrac{f'}{2f}+\dfrac{f'}{f}\dfrac{1}{4\cos^{2}(\omega)}\right)\bar{\partial}^{-}P\right],\\ \bar{\partial}^{+}\bar{\partial}^{-}P=\bar{\partial}^{-}P\left[-\dfrac{\sin(2\omega)}{\sqrt{f}}+\dfrac{f'}{f}\dfrac{1}{4\cos^{2}(\omega)}\bar{\partial}^{-}P+\left(\dfrac{f'}{2f}+\dfrac{f'}{f}\dfrac{1}{4\cos^{2}(\omega)}\right)\bar{\partial}^{+}P\right].\end{cases} \tag{21}$$

证明：

$$\begin{aligned}\bar{\partial}^{-}\bar{\partial}^{+}P=&\bar{\partial}^{-}[\cos(\alpha)P_{\xi}+\sin(\alpha)P_{\eta}]=\\&[-\sin(\alpha)P_{\xi}+\cos(\alpha)P_{\eta}]\bar{\partial}^{-}\alpha+\cos(\alpha)\bar{\partial}^{-}P_{\xi}+\sin(\alpha)\bar{\partial}^{-}P_{\eta}.\end{aligned} \tag{22}$$

令

$$\text{Ⅰ}=[-\sin(\alpha)P_{\xi}+\cos(\alpha)P_{\eta}]\bar{\partial}^{-}\alpha,\qquad \text{Ⅱ}=\cos(\alpha)\bar{\partial}^{-}P_{\xi}+\sin(\alpha)\bar{\partial}^{-}P_{\eta}.$$

在Ⅰ中，将P代入式(8)，可得

$$\begin{cases}P_{\xi}=-\dfrac{\sin(\beta)\bar{\partial}^{+}P-\sin(\alpha)\bar{\partial}^{-}P}{\sin(2\omega)},\\ P_{\eta}=-\dfrac{\cos(\beta)\bar{\partial}^{+}P-\cos(\alpha)\bar{\partial}^{-}P}{\sin(2\omega)}.\end{cases} \tag{23}$$

由式(15)及式(23)，有

$$\begin{aligned}\text{Ⅰ}=&-\frac{2\tan(\omega)\cos^{2}(\omega)}{\sqrt{f}}\bar{\partial}^{+}P+\frac{\tan(\omega)}{\sqrt{f}}\bar{\partial}^{+}P+\frac{\tan(\omega)}{\sqrt{f}}\bar{\partial}^{-}P+\\&\frac{f'}{2f}\bar{\partial}^{+}P\bar{\partial}^{-}P-\frac{f'}{4f}\frac{1}{\cos^{2}(\omega)}\bar{\partial}^{+}P\bar{\partial}^{-}P-\frac{f'}{4f}\frac{1}{\cos^{2}(\omega)}(\bar{\partial}^{-}P)^{2}.\end{aligned} \tag{24}$$

在Ⅱ中，有

$$\text{Ⅱ}=\frac{\cos(\alpha)\cos(\beta)}{f-\xi^{2}}\left[(f-\xi^{2})P_{\xi\xi}+\frac{\tan(\alpha)+\tan(\beta)}{(f-\xi^{2})}P_{\xi\eta}+\frac{\tan(\alpha)\tan(\beta)}{(f-\xi^{2})}P_{\eta\eta}\right]. \tag{25}$$

根据特征角的定义，可得

$$\tan(\alpha)+\tan(\beta)=\frac{2\xi\eta}{\xi^{2}-f},\qquad \tan(\alpha)\tan(\beta)=\frac{\eta^{2}-f}{\xi^{2}-f}. \tag{26}$$

将式(26)代入式(25)，并结合式(6)有

$$\text{Ⅱ}=\frac{\cos(\alpha)\cos(\beta)}{f-\xi^{2}}\left[2(\xi P_{\xi}+\eta P_{\eta})-\frac{f'}{f}(\xi P_{\xi}+\eta P_{\eta})^{2}\right], \tag{27}$$

根据式(7)及式(23)可得

$$\xi P_{\xi}+\eta P_{\eta}=\sqrt{f}\frac{1}{\sin(\omega)}\frac{\bar{\partial}^{+}P+\bar{\partial}^{-}P}{2\cos(\omega)}, \tag{28}$$

$$(\xi P_{\xi}+\eta P_{\eta})^{2}=\frac{f}{4\sin^{2}(\omega)\cos^{2}(\omega)}(\bar{\partial}^{+}P+\bar{\partial}^{-}P)^{2}. \tag{29}$$

根据式(24)可得

$$\frac{\cos(\alpha)\cos(\beta)}{f-\xi^2}=-\frac{\sin^2(\omega)}{f}. \tag{30}$$

再结合式(28)～(30)，有

$$\mathrm{II}=-\frac{\tan(\omega)}{\sqrt{f}}(\bar{\partial}^+P+\bar{\partial}^-P)+\frac{f'}{f}\frac{1}{4\cos^2(\omega)}(\bar{\partial}^+P+\bar{\partial}^-P)^2. \tag{31}$$

于是

$$\bar{\partial}^-\bar{\partial}^+P=\bar{\partial}^+P\left[-\frac{\sin(2\omega)}{\sqrt{f}}+\frac{f'}{f}\frac{1}{4\cos^2(\omega)}\bar{\partial}^+P+\left(\frac{f'}{2f}+\frac{f'}{f}\frac{1}{4\cos^2(\omega)}\right)\bar{\partial}^-P\right], \tag{32}$$

同理

$$\bar{\partial}^+\bar{\partial}^-P=\bar{\partial}^-P\left[-\frac{\sin(2\omega)}{\sqrt{f}}+\frac{f'}{f}\frac{1}{4\cos^2(\omega)}\bar{\partial}^-P+\left(\frac{f'}{2f}+\frac{f'}{f}\frac{1}{4\cos^2(\omega)}\right)\bar{\partial}^+P\right]. \tag{33}$$

推论 1　压力 P 的二阶偏导齐次形式为

$$\begin{cases}\bar{\partial}^+\left(-\dfrac{\bar{\partial}^-P}{\sin^2(\omega)}\right)=-\dfrac{f'}{4f}\tan^2(\omega)\left(-\dfrac{\bar{\partial}^-P}{\sin^2(\omega)}\right)^2-\dfrac{f'}{2f}\left(-\dfrac{\bar{\partial}^-P}{\sin^2(\omega)}\right)\bar{\partial}^+P+\\ \qquad\dfrac{f'}{4f\cos^2(\omega)}(\bar{\partial}^+P)\left(-\dfrac{\bar{\partial}^-P}{\sin^2(\omega)}\right),\\ -\bar{\partial}^-\left(\dfrac{\bar{\partial}^+P}{\sin^2(\omega)}\right)=-\dfrac{f'}{4f}\tan^2(\omega)\left(\dfrac{\bar{\partial}^+P}{\sin^2(\omega)}\right)^2-\dfrac{f'}{2f}\left(\dfrac{\bar{\partial}^+P}{\sin^2(\omega)}\right)(-\bar{\partial}^-P)+\\ \qquad\dfrac{f'}{4f\cos^2(\omega)}(-\bar{\partial}^-P)\left(\dfrac{\bar{\partial}^+P}{\sin^2(\omega)}\right).\end{cases} \tag{34}$$

3　边界值估计和局部存在性

首先给出$\widehat{AB}$,$\widehat{BC}$的边界条件. 假设点 B 的位置靠近点 A，圆弧$\widehat{AB}$不超过圆的$\frac{1}{4}$. 对于凸负特征线$\widehat{BC}$在点 C 处的倾斜角 β_C，设$-\frac{\pi}{2}<\beta_C<0$. 局部区域如图 2 所示，给出方向导数$\bar{\partial}^+$和$-\bar{\partial}^-$在区域 ABC 内的指向.

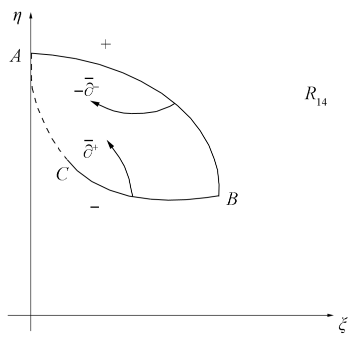

图 2　局部区域

Fig. 2　Local areas

边界值条件：已知 α,β 在边界$\widehat{AB}$,$\widehat{BC}$上的值

$$\begin{aligned}&\beta|_{\widehat{AB}}=0,\\ &\frac{\pi}{2}\leqslant\alpha|_{\widehat{BC}}\leqslant\pi+\beta|_C,\\ &\beta_C\in\left(-\frac{\pi}{2},0\right).\end{aligned} \tag{35}$$

引理 1(边界值估计)　α,β 在边界$\widehat{AB}$,$\widehat{BC}$上，下式成立：

$$\frac{\pi}{2}\leqslant\alpha|_{\widehat{AB}}\leqslant\pi,\quad \beta|_C\leqslant\beta|_{\widehat{BC}}\leqslant0,\quad \frac{\pi}{4}\leqslant\omega|_{\widehat{AB},\widehat{BC}}\leqslant\frac{\pi}{2}. \tag{36}$$

引理 2(局部解的存在性)　对于边值问题(1)-(2)-(35)，存在 $\delta>0$，使得在区域 D_δ 上存在唯一 C^1 解，且 δ 只依赖于在边界$\widehat{AB}$,$\widehat{BC}$上 α,β,P 的 C^1 范数.

引理 1 和引理 2 的证明参见文献[16].

引理 3(不变三角形区域)　对任意局部 C^2 解，有

$$\alpha\geqslant\frac{\pi}{2},\quad \beta\leqslant0,\quad \frac{\pi}{2}\leqslant\alpha-\beta\leqslant\pi,\quad \pm\bar{\partial}^{\pm}\alpha>0,\quad \pm\bar{\partial}^{\pm}\beta<0. \tag{37}$$

证明：由局部解的存在性可知，存在 $\omega_0\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$，使得在 D_{ω_0} 内存在解. 当 $\varepsilon\to\frac{\pi}{4}$时，在 D_ε 内有

$\bar{\partial}^+\alpha>0$，$\bar{\partial}^-\beta>0$.

根据式(14)及式(17)有

$$\bar{\partial}^+P\,|_{\widehat{AB}}>0,\qquad -\bar{\partial}^-P\,|_{\widehat{BC}}>0.\tag{38}$$

结合式(34)知，在 D_{ω_0} 内有$\bar{\partial}^+P>0$，$-\bar{\partial}^-P>0$. 利用式(15)～(17)，在 D_{ω_0} 内有$\bar{\partial}^+\alpha>0$，$-\bar{\partial}^-\alpha>0$，$-\bar{\partial}^-\beta<0$. 由式(9)及式(32)，有

$$-\bar{\partial}^-\,\bar{\partial}^+\beta-M_2\bar{\partial}^+\beta=\frac{2\tan(\omega)}{\sqrt{f}}\left[2\cos^2(\omega)\left(1-\frac{ff''}{f'^2}\right)-\frac{3}{2}\right]\bar{\partial}^-\beta,\tag{39}$$

其中

$$M_2=\frac{\tan(\omega)}{2\sin^2(\omega)}\bar{\partial}^+\beta-\left(\frac{1}{\tan(\omega)}-\frac{3}{\sin(2\omega)}-\frac{2ff''}{f'^2\tan(\omega)}\right)\bar{\partial}^-\beta+\frac{\tan(\omega)\cos(2\omega)}{\sqrt{f}}.$$

由式(2)及式(39)，有

$$\bar{\partial}^+\beta\,|_{\widehat{AB}}=0,\tag{40}$$

从而得$\bar{\partial}^+\beta<0$. 因此$\bar{\partial}^+\omega=2\bar{\partial}^+(\alpha-\beta)>0$. 于是，在 D_{ω_0} 内有$\pm\bar{\partial}^\pm\alpha>0$，$\pm\bar{\partial}^\pm\beta<0$，进而得到不变区域

$$\alpha\geqslant\frac{\pi}{2},\quad \beta\leqslant 0,\quad \frac{\pi}{2}\leqslant\alpha-\beta\leqslant\pi.\tag{41}$$

4 梯度估计

引理 4($\bar{\partial}^\pm P$ 的一致有界) 对于在退化双曲区域 ABC 内的光滑解，$\bar{\partial}^\pm P$ 在 ABC 内的最大值是一致有界的：

$$\bar{\partial}^+P>0,\qquad -\bar{\partial}^-P>0,$$
$$\max_{ABC}\{\bar{\partial}^+P,-\bar{\partial}^-P\}\leqslant 2\max_{\widehat{AB},\widehat{BC}}\{\bar{\partial}^+P,-\bar{\partial}^-P\}.\tag{42}$$

证明：在引理3证明中有$\bar{\partial}^+P>0$，$-\bar{\partial}^-P>0$. 根据式(34)可证式(42)成立，即证

$$\max_{ABC}\left\{\frac{\bar{\partial}^+P}{\sin^2(\omega)},\frac{-\bar{\partial}^-P}{\sin^2(\omega)}\right\}\leqslant 2\max_{\widehat{AB},\widehat{BC}}\left\{\frac{\bar{\partial}^+P}{\sin^2(\omega)},\frac{-\bar{\partial}^-P}{\sin^2(\omega)}\right\}\triangleq M.$$

因此，只需证明对于$\forall\mu>0$，$\max\limits_{ABC}\left\{\frac{\bar{\partial}^+P}{\sin^2(\omega)},\frac{-\bar{\partial}^-P}{\sin^2(\omega)}\right\}<M+\mu$. 显然，当 $\varepsilon\to\omega_B$ 时结论成立，即在点 B 附近区域 D_ε 满足结论. 假设结论不成立，则必存在第一个点 P_1 在 $\omega=\omega_1$ 上，有

$$\frac{\bar{\partial}^+P}{\sin^2(\omega)}=M+\mu\text{ 或 }\frac{-\bar{\partial}^-P}{\sin^2(\omega)}=M+\mu,$$

且在 D_{ω_1} 内有

$$\frac{\bar{\partial}^+P}{\sin^2(\omega)}<M+\mu,\qquad \frac{-\bar{\partial}^-P}{\sin^2(\omega)}<M+\mu.$$

不妨设在 $\omega=\omega_1$ 处，$\frac{\bar{\partial}^+P}{\sin^2(\omega)}=M+\mu$，$\frac{-\bar{\partial}^-P}{\sin^2(\omega)}<M+\mu$.

过点 P_1 作负特征线，由于

$$-\bar{\partial}^-\left(\frac{\bar{\partial}^+P}{\sin^2(\omega)}\right)\Bigg|_{P_1}=\frac{f'}{4f}\frac{1}{\cos^2(\omega)}(-\bar{\partial}^-P-\bar{\partial}^+P)\frac{\bar{\partial}^+P}{\sin^2(\omega)}-\frac{f'}{2f}\frac{\bar{\partial}^+P}{\sin^2(\omega)}(-\bar{\partial}^-P)<0,\tag{43}$$

因此由连续性可知，存在 P_1 的邻域$\cup(P_1)$，使得$-\bar{\partial}^-\left(\frac{\bar{\partial}^+P}{\sin^2\omega}\right)\Big|_{\cup(P_1)}<0$. 则在邻域$\cup(P_1)$内存在点 $P_1'\in D_{\omega_1}\cap\cup(P_1)$，使得$\frac{\bar{\partial}^+P}{\sin^2(\omega)}\Big|_{P_1}<\frac{\bar{\partial}^+P}{\sin^2(\omega)}\Big|_{P_1'}$，与假设矛盾. 由 μ 的任意性知结论成立.

引理 5(α,β 的一阶梯度估计) 假设在 D_ε 内有 C^1 连续解$\left(\varepsilon<\frac{\pi}{2}\right)$，则存在与 ε 无关的常数 C，使得

$$\|(\alpha,\beta)\|_{C^1(D_\varepsilon)}\leqslant C\tan^2(\varepsilon).\tag{44}$$

证明：根据$\bar{\partial}^{\pm}P$ 的一致有界，有

$$|\bar{\partial}^{+}\alpha|=\left|\frac{f'}{2f}\tan(\omega)\bar{\partial}^{+}P\right|\leqslant C\tan(\varepsilon),\tag{45}$$

同理

$$|\bar{\partial}^{-}\alpha,\bar{\partial}^{\pm}\beta|\leqslant C\tan(\varepsilon).\tag{46}$$

由式(8)有$\partial_{\xi}=\dfrac{-\sin(\beta)\bar{\partial}^{+}+\sin(\alpha)\bar{\partial}^{-}}{\sin(\alpha-\beta)}$成立，则

$$|\partial_{\xi}\alpha|\leqslant\frac{-\sin(\beta)|\bar{\partial}^{+}\alpha|+\sin(\alpha)|\bar{\partial}^{-}\alpha|}{\sin(\alpha-\beta)}\leqslant C\tan^{2}(\varepsilon),\tag{47}$$

同理可得$|\partial_{\eta}\alpha,\partial_{\xi}\beta,\partial_{\eta}\beta|\leqslant C\tan^{2}(\varepsilon)$.

5　全局解

引理 6(C^2 估计)　假设在 D_{ε} 内存在 C^2 解$\left(\varepsilon<\dfrac{\pi}{2}\right)$，则存在与 ε 无关的常数 C，使得

$$\|(\alpha,\beta)\|_{C^{2}(D_{\varepsilon})}\leqslant C\tan^{5}(\varepsilon).\tag{48}$$

证明：由式(15),(16)有

$$\bar{\partial}^{-}\bar{\partial}^{+}\alpha=A_{1}\bar{\partial}^{-}\bar{\partial}^{+}P+\text{低阶项},\qquad \bar{\partial}^{+}\bar{\partial}^{-}\alpha=B_{1}\bar{\partial}^{+}\bar{\partial}^{-}P+\text{低阶项},\tag{49}$$

因此$\bar{\partial}^{-}\bar{\partial}^{+}\alpha,\bar{\partial}^{+}\bar{\partial}^{-}\alpha$ 有界. 结合式(20)，对于$\bar{\partial}^{+}\bar{\partial}^{+}\alpha$ 有

$$\bar{\partial}^{-}(\bar{\partial}^{+}\bar{\partial}^{+}\alpha)+D_{1}\bar{\partial}^{+}\bar{\partial}^{+}\alpha=\text{低阶项}.\tag{50}$$

因此$\bar{\partial}^{+}\bar{\partial}^{+}\alpha$ 有界，同理可得$\bar{\partial}^{-}\bar{\partial}^{-}\alpha$ 有界.

综上，得到了$\bar{\partial}^{+}\bar{\partial}^{+}\alpha,\bar{\partial}^{+}\bar{\partial}^{-}\alpha,\bar{\partial}^{-}\bar{\partial}^{+}\alpha,\bar{\partial}^{-}\bar{\partial}^{-}\alpha$ 的有界性，同理可证明$\bar{\partial}^{+}\bar{\partial}^{+}\beta,\bar{\partial}^{+}\bar{\partial}^{-}\beta,\bar{\partial}^{-}\bar{\partial}^{+}\beta,\bar{\partial}^{-}\bar{\partial}^{-}\beta$ 的有界性.

由式(19)，得

$$|\bar{\partial}^{\pm}\omega|\leqslant C\tan(\varepsilon),\tag{51}$$

由式(15),(16)，得

$$|\bar{\partial}^{-}\bar{\partial}^{+}\alpha|\leqslant C\tan^{3}(\varepsilon),\qquad |\bar{\partial}^{+}\bar{\partial}^{-}\alpha|\leqslant C\tan^{3}(\varepsilon),\tag{52}$$

同理

$$|\bar{\partial}^{+}\bar{\partial}^{+}\alpha|\leqslant C\tan^{3}(\varepsilon),\qquad |\bar{\partial}^{-}\bar{\partial}^{-}\alpha|\leqslant C\tan^{3}(\varepsilon).\tag{53}$$

则

$$\begin{aligned}|\partial_{\xi\xi}\alpha|=&\left|\left(\frac{-\sin(\beta)\bar{\partial}^{+}\alpha+\sin(\alpha)\bar{\partial}^{-}\alpha}{\sin(\alpha-\beta)}\right)_{\xi}\right|=\\&\left|\frac{-\sin(\beta)\bar{\partial}^{+}\left(\dfrac{-\sin(\beta)\bar{\partial}^{+}\alpha+\sin(\alpha)\bar{\partial}^{-}\alpha}{\sin(\alpha-\beta)}\right)+\sin(\alpha)\bar{\partial}^{-}\left(\dfrac{-\sin(\beta)\bar{\partial}^{+}\alpha+\sin(\alpha)\bar{\partial}^{-}\alpha}{\sin(\alpha-\beta)}\right)}{\sin(\alpha-\beta)}\right|\leqslant\\&C\tan^{5}(\varepsilon),\end{aligned}\tag{54}$$

同理

$$|\partial_{\xi\eta}\alpha,\partial_{\eta\eta}\alpha,\partial_{\eta\xi}\alpha|\leqslant C\tan^{5}(\varepsilon).\tag{55}$$

再同理 β 成立.

由局部存在性定理及先验估计，并结合式(6),(7)和式(1)可得：

定理 3　退化 Goursat 问题(1)-(2)-(35)在区域 ABC 内存在整体光滑解.

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