第 62 卷　第 2 期　　吉 林 大 学 学 报（理 学 版）　　Vol. 62　No. 2
2024 年 3 月　　Journal of Jilin University (Science Edition)　　Mar　2024

doi：10.13413/j.cnki.jdxblxb.2023343

种内竞争模型的最优控制问题

那　杨，王宏越，杜润梅
（长春工业大学 数学与统计学院，长春 130012）

摘要：考虑一类在 Neumann 边界条件下具有抛物系统种内竞争的最优控制问题. 首先在该系统中讨论种群内部的竞争关系和种群间的交互作用，将目标泛函定义为捕捞得到的利润；其次证明该系统最优控制存在的必要条件，并给出最优控制的表达式.

关键词：抛物系统；最优控制；Neumann 边界条件；竞争模型

中图分类号：O175.26　　**文献标志码：**A　　**文章编号：**1671-5489(2024)02-0243-06

Optimal Control Problem of Intraspecific Competition Model

NA Yang, WANG Hongyue, DU Runmei
(*School of Mathematics and Statistics, Changchun University of Technology, Changchun* 130012, *China*)

Abstract: We considered the optimal control problem for a class of intraspecific competition with parabolic systems under Neumann boundary conditions. Firstly, we discussed the competition relationships within the population and the interactions between the populations in the system, and defined the objective functional as the profit obtained from harvesting. Secondly, we proved the necessary condition for the existence of the optimal control in the system, and gave an expression for the optimal contorl.

Keywords: parabolic system; optimal control; Neumann boundary condition; competition model

0　引　言

考虑如下耦合抛物系统：

$$\frac{\partial w_1}{\partial t}-a_1\Delta w_1=b_1w_1-c_1w_1^2-d_1w_1w_2-u_1w_1,\quad (x,t)\in Q_T,\tag{1}$$

$$\frac{\partial w_2}{\partial t}-a_2\Delta w_2=b_2w_2-c_2w_2^2-d_2w_1w_2-u_2w_2,\quad (x,t)\in Q_T,\tag{2}$$

$$w_1(x,0)=w_{0,1}(x),\quad w_2(x,0)=w_{0,2}(x),\quad x\in\Omega,\tag{3}$$

$$\frac{\partial w_1}{\partial \boldsymbol{n}}(x,t)=\frac{\partial w_2}{\partial \boldsymbol{n}}(x,t)=0,\quad (x,t)\in\Gamma,\tag{4}$$

其中 Ω 是集合 $\mathbb{R}^N$ 中的光滑有界区域，$Q_T=\Omega\times(0,T)$，$\Gamma=\partial\Omega\times(0,T)$，$a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in L^\infty(Q_T)$，$w_{0,1}(x),w_{0,2}(x)\in L^2(\Omega)$，$a_i,b_i,c_i,d_i,u_i,w_{0,i}(i=1,2)$都是非负函数，$\boldsymbol{n}$ 表示单位外法向量.

收稿日期：2023-08-20.

第一作者简介：那　杨(1989—)，男，满族，博士，讲师，从事偏微分方程的研究，E-mail：ny_118@163.com. **通信作者简介：**杜润梅(1985—)，女，汉族，博士，教授，从事应用数学的研究，E-mail：durm_dudu@163.com.

基金项目：国家自然科学基金(批准号：12071067；12161045；12026219)和吉林省科技发展计划项目(批准号：20220101033JC).

种内竞争模型是自然界中物种之间存在的普遍现象，系统(1)～(4)描述了在同一区域内的种群内部竞争模型，函数 w_1 和 w_2 分别表示成年种群密度和年轻种群密度，a_1,a_2 表示严格正的扩散系数，b_1,b_2,c_1,c_2 表示逻辑增长项，d_1,d_2 表示种群间的相互作用系数，u_1,u_2 表示捕获种群的比例.

令 u_1,u_2 表示控制，定义控制集 U 为

$$U=\{(u_1,u_2)\in L^2(Q_T),\ 0\leqslant u_i\leqslant M\}.$$

定义目标泛函为

$$J(u_1,u_2)=\int_0^T\int_\Omega \mathrm{e}^{-\delta t}(K_1u_1w_1+K_2u_2w_2-M_1u_1^2-M_2u_2^2)\mathrm{d}x\mathrm{d}t,$$

其中 $0<\delta<1$，$\mathrm{e}^{-\delta t}$ 是折扣项，$K_1u_1w_1,K_2u_2w_2$ 表示收获收益，$M_1u_1^2,M_2u_2^2$ 表示捕获成本.

关于生物种群系统的最优化问题目前已取得一些研究成果[1-4]. 文献[5]考虑了当 $\delta=0$ 时，抛物型偏微分系统(1)～(4)的最优控制问题. 但通常情况下，捕获物的新鲜度会随时间的增加而降低，因此获利也会随时间的增加而减少；文献[6]考虑了单个扩散种群的最优获利问题，其中表示利润的目标泛函含有 $\mathrm{e}^{-\delta t}$ $(0<\delta<1)$，即考虑了利润与时间的关系. 受上述工作的启发，本文研究当 $\delta>0$ 时，系统(1)～(4)的最优获利问题. 文献[6]研究了在 Dirichlet 边界条件下单方程的最优控制问题，与其不同，本文考虑在 Neumann 边界条件下耦合方程组的最优控制问题，且控制函数的个数为2个. 在生物种群模型中，Dirichlet 边界条件意味着物种不能在边界上生存，Neumann 边界条件意味着物种不向区域外部扩散，因此后者是更有利于物种在栖息地生存的边界条件. 本文首先介绍系统(1)～(4)的适定性，并证明最优控制的存在性；其次，通过最优系统给出最优控制的一阶必要条件和最优控制表达式.

1　最优控制的存在性

首先，给出问题(1)～(4)的适定性.

定义1　对于 $(w_1,w_2)\in(L^\infty(Q_T))\cap(L^2((0,T);H^1(\Omega)))^2$，如果对于任意的 $(\varphi_1,\varphi_2)\in(L^2((0,T);H^1(\Omega)))^2$，$\dfrac{\partial\varphi_1}{\partial t},\dfrac{\partial\varphi_2}{\partial t}\in L^2((0,T);(H^1(\Omega))^*)$，$\dfrac{\partial\varphi_1}{\partial t}=\dfrac{\partial\varphi_2}{\partial t}=0$，$\varphi_2(x,T)=\varphi_2(x,T)=0$，有下列等式成立：

$$-\iint_{Q_T}w_1\frac{\partial\varphi_1}{\partial t}\mathrm{d}x\mathrm{d}t-\int_\Omega w_1(x,0)\varphi_1(x,0)\mathrm{d}x+\iint_{Q_T}a_1\nabla w_1\cdot\nabla\varphi_1\mathrm{d}x\mathrm{d}t=$$

$$\iint_{Q_T}(b_1w_1-c_1w_1^2-d_1w_1w_2-u_1w_1)\varphi_1\mathrm{d}x\mathrm{d}t,$$

$$-\iint_{Q_T}w_2\frac{\partial\varphi_2}{\partial t}\mathrm{d}x\mathrm{d}t-\int_\Omega w_2(x,0)\varphi_2(x,0)\mathrm{d}x+\iint_{Q_T}a_2\nabla w_2\cdot\nabla\varphi_2\mathrm{d}x\mathrm{d}t=$$

$$\iint_{Q_T}(b_2w_2-c_2w_2^2-d_2w_1w_2-u_2w_2)\varphi_2\mathrm{d}x\mathrm{d}t,$$

则称 (w_1,w_2) 是问题(1)～(4)的弱解，其中 $(H^1(\Omega))^*$ 是 $H^1(\Omega)$ 的对偶空间.

引理1[5]　问题(1)～(4)存在唯一弱解 $(w_1,w_2)\in(L^\infty(Q_T))\cap L^2((0,T);H^1(\Omega))^2$，使得对于 $i=1,2$，$(w_i)_t\in L^2((0,T);(H^1(\Omega))^*)$，有

$$\|w_i\|_{L^2((0,T);H^1(\Omega))}+\left\|\frac{\partial w_i}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))^*)}\leqslant C, \tag{5}$$

其中 C 依赖于 Ω,T,M，$\|w_{0,i}\|_{L^2(\Omega)}$，$\|a_i\|_{L^\infty(Q_T)}$，$\|b_i\|_{L^\infty(Q_T)}$，$\|c_i\|_{L^\infty(Q_T)}$，$\|d_i\|_{L^\infty(Q_T)}$，$i=1,2$.

若 $w_{0,i}\in L^\infty(\Omega)$ $(i=1,2)$，则有

$$0\leqslant w_i\leqslant A_1\mathrm{e}^{\mu_1T}, \tag{6}$$

其中 $A_1=\max\{\|w_{0,1}\|_{L^\infty(\Omega)},\|w_{0,2}\|_{L^\infty(\Omega)}\}$，$\mu_1=\max\{\|b_1\|_{L^\infty(Q_T)},\|b_2\|_{L^\infty(Q_T)}\}$. 若 $w_{0,i}\in H^1(\Omega)\cap L^\infty(\Omega)$ $(i=1,2)$，则 $\dfrac{\partial w_i}{\partial t}\in L^2(Q_T)$，且 $\left\|\dfrac{\partial w_i}{\partial t}\right\|_{L^2(Q_T)}\leqslant C$.

类似文献[7]中定理 3.1 的证明，由引理 1 可证明如下最优控制的存在性.

命题 1　存在最优控制(u_1^*,u_2^*)，使得 $J(u_1^*,u_2^*)=\sup\limits_{(u_1,u_2)\in U}J(u_1,u_2)$.

2　最优控制的必要条件

首先，考虑如下问题：

$$\frac{\partial\sigma_1}{\partial t}-a_1\Delta\sigma_1=b_{11}\sigma_1+b_{12}\sigma_2+\tilde{f}_1,\quad (x,t)\in Q_T,\tag{7}$$

$$\frac{\partial\sigma_2}{\partial t}-a_2\Delta\sigma_2=b_{21}\sigma_1+b_{22}\sigma_2+\tilde{f}_2,\quad (x,t)\in Q_T,\tag{8}$$

$$\frac{\partial\sigma_1}{\partial \boldsymbol{n}}=\frac{\partial\sigma_2}{\partial \boldsymbol{n}}=0,\quad (x,t)\in \Gamma,\tag{9}$$

$$\sigma_1(x,0)=\sigma_{0,1}(x),\quad \sigma_2(x,0)=\sigma_{0,2}(x),\quad x\in\Omega,\tag{10}$$

其中 $b_{ij}\in L^\infty(Q_T)$，$\tilde{f}_i\in L^2(Q_T)$，$\sigma_{0,i}\in H^1(\Omega)$，$i,j=1,2$.

引理 2[5]　问题(7)～(10)存在唯一弱解$(\sigma_1,\sigma_2)\in(W_2^{2,1}(Q_T))^2$，使得对于 $i=1,2$，有

$$\|\sigma_i\|_{L^2((0,T);H^1(\Omega))}+\left\|\frac{\partial\sigma_i}{\partial t}\right\|_{L^2(Q_T)}\leqslant C,\tag{11}$$

其中 C 依赖于 $\Omega,T,\|b_{ij}\|_{L^\infty(Q_T)}$，$i,j=1,2$. 此外，如果对于 $i=1,2$，$\sigma_{0,i}\in L^\infty(\Omega)$，$\tilde{f}_i\in L^\infty(Q_T)$，则有

$$\|\sigma_i\|_{L^\infty(Q_T)}\leqslant A_2\mathrm{e}^{\mu_2T},\tag{12}$$

其中 $A_2=\max\limits_{i=1,2}\{\|\sigma_{0,i}\|_{L^\infty(\Omega)},\|\tilde{f}_i\|_{L^\infty(Q_T)}\}$，$\mu_2=\max\limits_{i,j=1,2}\{\|a_{ij}\|_{L^\infty(Q_T)}\}$.

定理 1　假设 $\boldsymbol{u}^*=(u_1^*,u_2^*)$是最优控制，定义$(w_1^*,w_2^*)$是问题(1)～(4)相应于 $u_1=u_1^*$，$u_2=u_2^*$ 的解，记(s_1^*,s_2^*)是如下问题的解：

$$-\frac{\partial s_1^*}{\partial t}-a_1\Delta s_1^*=b_1s_1^*-2c_1w_1^*s_1^*-d_2w_2^*s_2^*-d_1w_2^*s_1^*-$$
$$u_1^*s_1^*-\delta s_1^*+K_1u_1^*,\quad (x,t)\in Q_T,\tag{13}$$

$$-\frac{\partial s_2^*}{\partial t}-a_2\Delta s_2^*=b_2s_2^*-2c_2w_2^*s_2^*-d_2w_1^*s_2^*-d_1w_1^*s_1^*-$$
$$u_2^*s_2^*-\delta s_2^*+K_2u_2^*,\quad (x,t)\in Q_T,\tag{14}$$

$$\frac{\partial s_1^*}{\partial \boldsymbol{n}}(x,t)=\frac{\partial s_2^*}{\partial \boldsymbol{n}}(x,t)=0,\quad (x,t)\in\Gamma,\tag{15}$$

$$s_1^*(x,T)=s_2^*(x,T)=0,\quad x\in\Omega.\tag{16}$$

则对任意的 $\tilde{\boldsymbol{h}}=(\tilde{h}_1,\tilde{h}_2)$，有

$$\iint_{Q_T}\mathrm{e}^{-\delta t}[(\tilde{h}_1-u_1^*)((K_1-s_1^*)w_1^*-2M_1u_1^*)+(\tilde{h}_2-u_2^*)((K_2-s_2^*)w_2^*-2M_2u_2^*)]\mathrm{d}x\mathrm{d}t\leqslant 0.$$

证明：对任意的 $\tilde{\boldsymbol{h}}=(\tilde{h}_1,\tilde{h}_2)\in U$，定义 $\tilde{\boldsymbol{k}}=(\tilde{k}_1,\tilde{k}_2)=(\tilde{h}_1-u_1^*,\tilde{h}_2-u_2^*)$，对任意的 $0\leqslant\varepsilon\leqslant1$，记$(w_1^\varepsilon,w_2^\varepsilon)$是问题(1)～(4)相应于 $u_1=u_1^*+\varepsilon\tilde{k}_1$，$u_2=u_2^*+\varepsilon\tilde{k}_2$ 的解，并记 $\psi_1=w_1^\varepsilon-w_1^*$，$\psi_2=w_2^\varepsilon-w_2^*$，则$(\psi_1,\psi_2)$满足：

$$\frac{\partial\psi_1}{\partial t}-a_1\Delta\psi_1=b_1\psi_1-c_1(w_1^\varepsilon+w_1^*)\psi_1-d_1(w_1^\varepsilon\psi_2+w_2^*\psi_1)-u_1^*\psi_1-\varepsilon\tilde{k}_1w_1^\varepsilon,\quad (x,t)\in Q_T,$$

$$\frac{\partial\psi_2}{\partial t}-a_2\Delta\psi_2=b_2\psi_2-c_2(w_2^\varepsilon+w_2^*)\psi_2-d_2(w_1^\varepsilon\psi_2+w_2^*\psi_1)-u_2^*\psi_2-\varepsilon\tilde{k}_2w_2^\varepsilon,\quad (x,t)\in Q_T,$$

$$\psi_1(x,0)=\psi_2(x,0)=0,\quad x\in\Omega,$$

$$\frac{\partial\psi_1}{\partial \boldsymbol{n}}=\frac{\partial\psi_2}{\partial \boldsymbol{n}}=0,\quad (x,t)\in\Gamma.$$

由引理 1 可知，$w_1^\varepsilon,w_2^\varepsilon,w_1^*,w_2^*$ 在 $L^\infty(Q_T)$上有界，根据引理 2 有$(\psi_1,\psi_2)\in(W_2^{2,1}(Q_T)\cap$

$L^\infty(Q_T))^2$，且

$$\|\psi_1\|_{L^2(Q_T)}+\|\psi_2\|_{L^2(Q_T)}\leqslant\varepsilon C,$$

因此

$$(w_1^\varepsilon,w_2^\varepsilon)\to(w_1^*,w_2^*),\qquad \varepsilon\to0. \tag{17}$$

记 $r_1^\varepsilon=\dfrac{\psi_1}{\varepsilon}$，$r_2^\varepsilon=\dfrac{\psi_2}{\varepsilon}$，则 $(w_1^\varepsilon,w_2^\varepsilon)\in(W_2^{2,1}(Q_T)\cap L^\infty(Q_T))^2$ 是下列问题的解：

$$\frac{\partial r_1^\varepsilon}{\partial t}-a_1\Delta r_1^\varepsilon=b_1r_1^\varepsilon-c_1(w_1^\varepsilon+w_1^*)r_1^\varepsilon-d_1(w_1^\varepsilon r_2^\varepsilon+w_2^*r_1^\varepsilon)-u_1^*r_1^\varepsilon-\tilde{k}_1w_1^\varepsilon,\quad (x,t)\in Q_T, \tag{18}$$

$$\frac{\partial r_2^\varepsilon}{\partial t}-a_2\Delta r_2^\varepsilon=b_2r_2^\varepsilon-c_2(w_2^\varepsilon+w_2^*)r_2^\varepsilon-d_2(w_1^\varepsilon r_2^\varepsilon+w_2^*r_1^\varepsilon)-u_2^*r_2^\varepsilon-\tilde{k}_2w_2^\varepsilon,\quad (x,t)\in Q_T, \tag{19}$$

$$\frac{\partial r_1^\varepsilon}{\partial \boldsymbol{n}}=\frac{\partial r_2^\varepsilon}{\partial \boldsymbol{n}}=0,\quad (x,t)\in\Gamma, \tag{20}$$

$$r_1^\varepsilon(x,0)=r_2^\varepsilon(x,0)=0,\quad x\in\Omega. \tag{21}$$

由引理2可知，存在 $\{(r_1^\varepsilon,r_2^\varepsilon)\}$ 的子序列和 $(r_1,r_2)\in(W_2^{2,1}(Q_T)\cap L^\infty(Q_T))^2$，使得

$$(r_1^\varepsilon,r_2^\varepsilon)\to(r_1,r_2)\ \text{强收敛于}(L^2(Q_T))^2,\qquad \varepsilon\to0, \tag{22}$$

$$(\nabla r_1^\varepsilon,\nabla r_2^\varepsilon)\to(\nabla r_1,\nabla r_2)\ \text{强收敛于}(L^2(Q_T;\mathbb{R}^N))^2,\qquad \varepsilon\to0. \tag{23}$$

因为 $(r_1^\varepsilon,r_2^\varepsilon)$ 是问题(18)～(21)的弱解，因此对任意满足 $\dfrac{\partial\varphi_1}{\partial\boldsymbol{n}}=\dfrac{\partial\varphi_2}{\partial\boldsymbol{n}}=0$，$\varphi_1(x,T)=\varphi_2(x,T)=0$ 的 $\varphi_1,\varphi_2\in C^\infty(\bar{Q}_T)$，有

$$-\iint_{Q_T}r_1^\varepsilon\frac{\partial\varphi_1}{\partial t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}a_1\nabla r_1^\varepsilon\cdot\nabla\varphi_1\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(b_1r_1^\varepsilon-c_1(w_1^\varepsilon+w_1^*)r_1^\varepsilon-d_1(w_1^\varepsilon r_2^\varepsilon+w_2^*r_1^\varepsilon)-u_1^*r_1^\varepsilon-\tilde{k}_1w_1^\varepsilon)\varphi_1\mathrm{d}x\mathrm{d}t,$$

$$-\iint_{Q_T}r_2^\varepsilon\frac{\partial\varphi_2}{\partial t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}a_2\nabla r_2^\varepsilon\cdot\nabla\varphi_2\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(b_2r_2^\varepsilon-c_2(w_2^\varepsilon+w_2^*)r_2^\varepsilon-d_2(w_1^\varepsilon r_2^\varepsilon+w_2^*r_1^\varepsilon)-u_2^*r_2^\varepsilon-\tilde{k}_2w_2^\varepsilon)\varphi_2\mathrm{d}x\mathrm{d}t.$$

当 $\varepsilon\to0$ 时，由式(17),(22),(23)可得，

$$-\iint_{Q_T}r_1\frac{\partial\varphi_1}{\partial t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}a_1\nabla r_1\cdot\nabla\varphi_1\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(b_1r_1-2c_1w_1^*r_1-d_1(w_1^*r_2+w_2^*r_1)-u_1^*r_1-\tilde{k}_1w_1^*)\varphi_1\mathrm{d}x\mathrm{d}t,$$

$$-\iint_{Q_T}r_2\frac{\partial\varphi_2}{\partial t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}a_2\nabla r_2\cdot\nabla\varphi_2\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(b_2r_2-2c_2w_2^*r_2-d_2(w_1^*r_2+w_2^*r_1)-u_2^*r_2-\tilde{k}_2w_2^*)\varphi_2\mathrm{d}x\mathrm{d}t.$$

从而 (r_1,r_2) 是下列问题的解：

$$\frac{\partial r_1}{\partial t}-a_1\Delta r_1=b_1r_1-2c_1w_1^*r_1-d_1(w_1^*r_2+w_2^*r_1)-u_1^*r_1-\tilde{k}_1w_1^*,\quad (x,t)\in Q_T,$$

$$\frac{\partial r_2}{\partial t}-a_2\Delta r_2=b_2r_2-2c_2w_2^*r_2-d_2(w_1^*r_2+w_2^*r_1)-u_2^*r_2-\tilde{k}_2w_2^*,\quad (x,t)\in Q_T,$$

$$r_1(x,0)=r_2(x,0)=0,\quad x\in\Omega,$$

$$\frac{\partial r_1}{\partial\boldsymbol{n}}=\frac{\partial r_2}{\partial\boldsymbol{n}}=0,\quad (x,t)\in\Gamma.$$

通过计算，有

$$J(u^*+\varepsilon\tilde{k})-J(u^*)=\iint_{Q_T}\mathrm{e}^{-\delta t}[K_1w_1^\varepsilon(u_1^*+\varepsilon\tilde{k}_1)-K_1w_1^*u_1^*+K_2w_2^\varepsilon(u_2^*+\varepsilon\tilde{k}_2)-K_2w_2^*u_2^*-$$

$$(M_1(u_1^*+\varepsilon\tilde{k}_1)^2-M_1(u_1^*)^2)-(M_2(u_2^*+\varepsilon\tilde{k}_2)^2-M_2(u_2^*)^2)]\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}\mathrm{e}^{-\delta t}[K_1u_1^*(w_1^\varepsilon-w_1^*)+K_2u_2^*(w_2^\varepsilon-w_2^*)+\varepsilon K_1w_1^\varepsilon\tilde{k}_1+\varepsilon K_2w_2^\varepsilon\tilde{k}_2-$$
$$2\varepsilon M_1u_1^*\tilde{k}_1-2\varepsilon M_2u_2^*\tilde{k}_2-\varepsilon^2M_1\tilde{k}_1^2-\varepsilon^2M_2\tilde{k}_2^2]\mathrm{d}x\mathrm{d}t,$$

故

$$\frac{J(u^*+\varepsilon\tilde{k})-J(u^*)}{\varepsilon}=\iint_{Q_T}\mathrm{e}^{-\delta t}\left[K_1u_1^*\ \frac{w_1^\varepsilon-w_1^*}{\varepsilon}+K_2u_2^*\ \frac{w_2^\varepsilon-w_2^*}{\varepsilon}+K_1w_1^\varepsilon\tilde{k}_1+K_2w_2^\varepsilon\tilde{k}_2-\right.$$
$$\left.2M_1u_1^*\tilde{k}_1-2M_2u_2^*\tilde{k}_2-\varepsilon M_1\tilde{k}_1^2-\varepsilon M_2\tilde{k}_2^2\right]\mathrm{d}x\mathrm{d}t.$$

由于 $r_1^\varepsilon=\dfrac{w_1^\varepsilon-w_1^*}{\varepsilon}$，$r_2^\varepsilon=\dfrac{w_2^\varepsilon-w_2^*}{\varepsilon}$，因此

$$\lim_{\varepsilon\to0}\frac{J(u^*+\varepsilon\tilde{k})-J(u^*)}{\varepsilon}=\lim_{\varepsilon\to0}\iint_{Q_T}\mathrm{e}^{-\delta t}(K_1u_1^*r_1^\varepsilon+K_2u_2^*r_2^\varepsilon+K_1w_1^*\tilde{k}_1+$$
$$K_2w_2^*\tilde{k}_2-2M_1u_1^*\tilde{k}_1-2M_2u_2^*\tilde{k}_2)\mathrm{d}x\mathrm{d}t=$$
$$\lim_{\varepsilon\to0}\iint_{Q_T}\mathrm{e}^{-\delta t}(K_1u_1^*r_1+K_2u_2^*r_2+K_1w_1^*\tilde{k}_1+$$
$$K_2w_2^*\tilde{k}_2-2M_1u_1^*\tilde{k}_1-2M_2u_2^*\tilde{k}_2)\mathrm{d}x\mathrm{d}t. \tag{24}$$

令 (s_1^*,s_2^*) 是如下问题的解：

$$-\frac{\partial s_1^*}{\partial t}-a_1\Delta s_1^*=b_1s_1^*-2c_1w_1^*s_1^*-d_2w_2^*s_2^*-d_1w_2^*s_1^*-u_1^*s_1^*-\delta s_1^*+K_1u_1^*,\quad (x,t)\in Q_T,$$

$$-\frac{\partial s_2^*}{\partial t}-a_2\Delta s_2^*=b_2s_2^*-2c_2^*w_2^*s_2^*-d_2w_1^*s_2^*-d_1w_1^*s_1^*-u_2^*s_2^*-\delta s_2^*+K_2u_2^*,\quad (x,t)\in Q_T,$$

$$\frac{\partial s_1^*}{\partial \boldsymbol{n}}=\frac{\partial s_2^*}{\partial \boldsymbol{n}}=0,\quad (x,t)\in\Gamma,$$

$$s_1^*(x,T)=s_2^*(x,T)=0,\quad x\in\Omega.$$

由 r_1,r_2,s_1^*,s_2^* 满足的方程可知，

$$\iint_{Q_T}\mathrm{e}^{-\delta t}(K_1u_1^*r_1+K_2u_2^*r_2)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\mathrm{e}^{-\delta t}(-s_1^*\tilde{k}_1w_1^*-s_2^*\tilde{k}_2w_2^*)\mathrm{d}x\mathrm{d}t. \tag{25}$$

由式(24),(25)有

$$\lim_{\varepsilon\to0}\frac{J(u^*+\varepsilon\tilde{k})-J(u^*)}{\varepsilon}=\iint_{Q_T}\mathrm{e}^{-\delta t}[\tilde{k}_1((K_1-s_1^*)w_1^*-2M_1u_1^*)+$$
$$\tilde{k}_2((K_2-s_2^*)w_2^*-2M_2u_2^*)]\mathrm{d}x\mathrm{d}t.$$

因为 $\boldsymbol{u}^*=(u_1^*,u_2^*)$ 是最优控制，所以

$$\lim_{\varepsilon\to0}\frac{J(u^*+\varepsilon\tilde{k})-J(u^*)}{\varepsilon}\leqslant0,$$

从而

$$\iint_{Q_T}\mathrm{e}^{-\delta t}[\tilde{k}_1((K_1-s_1^*)w_1^*-2M_1u_1^*)+\tilde{k}_2((K_2-s_2^*)w_2^*-2M_2u_2^*)]\mathrm{d}x\mathrm{d}t\leqslant0.$$

证毕.

注 1　定理 1 给出了最优控制存在的一阶必要条件.

由文献[8]中定理 3.20 的证明，可得如下推论.

推论 1　如果 $\boldsymbol{u}^*=(u_1^*,u_2^*)$ 是最优控制，则

$$u_1^*=\min\left\{M,\max\left\{\frac{1}{2M_1}(K_1-s_1^*)w_1^*,0\right\}\right\},$$

$$u_2^*=\min\left\{M,\max\left\{\frac{1}{2M_2}(K_2-s_2^*)w_2^*,0\right\}\right\},$$

其中$(w_1^*, w_2^*, s_1^*, s_2^*)$是如下最优系统的解：

$$\frac{\partial w_1^*}{\partial t} - a_1 \Delta w_1^* = b_1 w_1^* - c_1 (w_1^*)^2 - d_1 w_1^* w_2^* - u_1^* w_1^*, \quad (x,t) \in Q_T, \tag{26}$$

$$\frac{\partial w_2^*}{\partial t} - a_2 \Delta w_2^* = b_2 w_2^* - c_2 (w_2^*)^2 - d_2 w_1^* w_2^* - u_2^* w_2^*, \quad (x,t) \in Q_T, \tag{27}$$

$$\begin{aligned} -\frac{\partial s_1^*}{\partial t} - a_1 \Delta s_1^* = {} & b_1 s_1^* - 2c_1 w_1^* s_1^* - d_2 w_2^* s_2^* - d_1 w_2^* s_1^* - {} \\ & \delta s_1^* - u_1^* s_1^* + u_1^* K_1, \quad (x,t) \in Q_T, \end{aligned} \tag{28}$$

$$\begin{aligned} -\frac{\partial s_2^*}{\partial t} - a_2 \Delta s_2^* = {} & b_2 s_2^* - 2c_2 w_2^* s_2^* - d_2 w_1^* s_2^* - d_1 w_1^* s_1^* - {} \\ & \delta s_2^* - u_2^* s_2^* + u_2^* K_2, \quad (x,t) \in Q_T, \end{aligned} \tag{29}$$

$$\frac{\partial w_1^*}{\partial \boldsymbol{n}} = \frac{\partial w_2^*}{\partial \boldsymbol{n}} = \frac{\partial s_1^*}{\partial \boldsymbol{n}} = \frac{\partial s_2^*}{\partial \boldsymbol{n}} = 0, \quad (x,t) \in \Gamma, \tag{30}$$

$$w_1^*(x,0) = w_{0,1}^*(x,0), \quad w_2^*(x,0) = w_{0,2}^*(x,0), \quad s_1^*(x,T) = s_2^*(x,T) = 0, \quad x \in \Omega. \tag{31}$$

注2　推论1给出了最优控制的表达式.

参考文献

[1] 梁浩健，李辉来. 含移流项两物种竞争模型关于资源的最优控制[J]. 吉林大学学报(理学版)，2019，57(3)：500-504. (LIANG H J，LI H L. Optimal Control of Resource in Two Competing Species Model with Advection Terms[J]. Journal of Jilin University (Science Edition)，2019，57(3)：500-504.)

[2] 武海辉. 一类带收获和毒素项种群竞争模型的扩散性质[J]. 甘肃科学学报，2022，34(5)：1-4. (WU H H. A Diffusivel Behavior of a Competitive Model with Gain Items and Toxin[J]. Journal of Gansu Sciences，2022，34(5)：1-4.)

[3] 何泽荣，周楠. 具有年龄等级结构的种群竞争系统的最优收获控制[J]. 数学物理学报，2022，42A(1)：228-244. (HE Z R，ZHOU N. Optimal Harvesting in a Competiting System of Hierarchical Age-Structured Populations[J]. Journal of Mathematical Physics，2022，42A(1)：228-244.)

[4] MATHEW S M，DILIP D S. Dynamics of Interspecific *k* Species Competition Model[J]. Journal of Interdisciplinary Mathematics，2022，25(3)：629-638.

[5] LENHART S，MONTERO J A. Optimal Control of Harvesting in a Parabolic System Modeling Two Subpopulations[J]. Mathematical Models & Methods in Applied Sciences，2001，11(7)：1129-1141.

[6] LENHART S，WORKMAN J T. Optimal Control Applied to Biological Models[M]. New York：Chapman and Hau/CRC，2007：216-220.

[7] ZHAO C，WANG M，ZHAO P. Optimal Harvesting Problems for Age-Dependent Interacting Species with Diffusion[J]. Applied Mathematics & Computation，2005，163(1)：117-129.

[8] TRÖLTZSCH F. Optimal Control of Partial Differential Equations：Theory，Methods and Applications[M]. Providence，RI：American Mathematical Society，2010：161-162.

(责任编辑：李　琦)