doi：10.13413/j.cnki.jdxblxb.2023527

分数阶 Boussinesq-Coriolis 方程在变指数 Fourier-Besov 空间中解的整体适定性和正则性

李风娟，孙小春，吴育联
（西北师范大学 数学与统计学院，兰州 730070）

摘要：基于变指数 Fourier-Besov 函数空间理论，利用 Littlewood-Paley 分解工具、Fourier 局部化方法和 Banach 压缩映射原理，通过建立线性项与非线性项的估计，证明分数阶 Boussinesq-Coriolis 方程在临界变指数空间 $\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}(\mathbb{R}^3)$ 中解的整体适定性和 Gevrey 类正则性.

关键词：Boussinesq-Coriolis 方程；变指数 Fourier-Besov 空间；整体适定性；Gevrey 类正则性
中图分类号：O174.2　　**文献标志码**：A　　**文章编号**：1671-5489(2024)05-1043-09

Global Well-Posedness and Regularity of Solutions to Fractional Boussinesq-Coriolis Equations in Variable Exponent Fourier-Besov Spaces

LI Fengjuan, SUN Xiaochun, WU Yulian
(*College of Mathematics and Statistics, Northwest Normal University, Lanzhou* 730070, *China*)

Abstract: Based on the theory of variable exponent Fourier-Besov function spaces, we used Littlewood-Paley decomposition tools, Fourier localization methods and Banach contraction mapping principle. By establishing estimations for both linear and nonlinear terms, we proved the global well-posedness and the Gevrey class regularity of the solutions to the fractional Boussinesq-Coriolis equations in critical variable exponent Fourier-Besov spaces $\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}(\mathbb{R}^3)$.

Keywords: Boussinesq-Coriolis equation; variable exponent Fourier-Besov space; global well-posedness; Gevrey class regularity

0 引　言

考虑三维分数阶不可压缩 Boussinesq-Coriolis 方程解的初值问题：

$$\begin{cases} \partial_t \boldsymbol{u} + \nu(-\Delta)^\alpha \boldsymbol{u} + \Omega \boldsymbol{e}_3 \times \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = g\theta \boldsymbol{e}_3, & \text{在 } \mathbb{R}^3 \times (0,\infty) \text{ 内}, \\ \partial_t \theta + \mu(-\Delta)^\alpha \theta + (\boldsymbol{u}\cdot\nabla)\theta = 0, & \text{在 } \mathbb{R}^3 \times (0,\infty) \text{ 内}, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{在 } \mathbb{R}^3 \times (0,\infty) \text{ 内}, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0, \quad \theta(\boldsymbol{x},0) = \theta_0, & \text{在 } \mathbb{R}^3 \text{ 内}, \end{cases} \tag{1}$$

其中：$\boldsymbol{u}=(u_1(\boldsymbol{x},t),u_2(\boldsymbol{x},t),u_3(\boldsymbol{x},t))$，$p=p(\boldsymbol{x},t)$，$\theta=\theta(\boldsymbol{x},t)$分别为流体在点$(\boldsymbol{x},t)\in\mathbb{R}^3\times(0,\infty)$的

收稿日期：2023-12-13.　**网络首发日期**：2024-07-15.
第一作者简介：李风娟(1999—)，女，汉族，硕士研究生，从事调和分析的研究，E-mail：lfj15268976471@163.com. **通信作者简介**：孙小春(1981—)，男，汉族，博士，教授，从事调和分析的研究，E-mail：sunxiaochun@nwnu.edu.cn.
基金项目：国家自然科学基金(批准号：11601434).
网络首发地址：https://link.cnki.net/urlid/22.1340.o.20240710.1055.001.

未知速度、未知压力和温度；正常数 ν,μ 和 g 分别为黏度系数、热扩散系数和重力加速度；$\Omega \boldsymbol{e}_3\times\boldsymbol{u}$ 为 Coriolis 力，$\Omega\in\mathbb{R}$ 为流体绕垂直单位矢量 $\boldsymbol{e}_3=(0,0,1)$ 的旋转速度；$g\theta\boldsymbol{e}_3$ 表示浮力；$\Delta=\sum\limits_{j=1}^{3}\dfrac{\partial^2}{\partial x_j^2}$ 表示 Laplace 算子. 文献[1-3]给出了问题(1)的物理意义. Wang[4] 证明了三维分数阶磁流体方程在临界变指数 Fourier-Besov 空间中解的整体适定性和解析性；Abidin 等[5] 证明了当初值 $\boldsymbol{u}_0$ 属于变指数 Fourier-Besov-Morrey 空间 $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),h(\cdot),q}$ 时，分数阶 Navier-Stokes 方程解的整体适定性；Abidin 等[6] 证明了 Prandtl 常数为 1 时三维地球物理原始方程在变指数 Fourier-Besov 空间中小初值意义下解的整体适定性和解析性；当方程(1)中 $\alpha=1$ 时，Sun 等[7] 证明了当旋转速度 $|\Omega|$ 充分大时，方程(1)在 Besov 空间中整体温和解的存在唯一性；Sun 等[8] 证明了当初值 $(\boldsymbol{u}_0,\theta_0)$ 属于变指数 Fourier-Besov 空间 $\mathscr{F}\dot{\mathscr{B}}^{2-3/p(\cdot)}_{p(\cdot),q}$ 时，方程(1)解的整体适定性. 文献[9-16]给出了不可压缩 Navier-Stokes 方程、微极流体方程、磁流体方程、多孔介质方程和地球物理原始方程在变指数函数空间中解的适定性结果. 本文主要研究 $\nu=\mu=1$ 的情形.

二阶椭圆微分算子的原型是 Laplace 算子 $\Delta_n=\sum\limits_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}$，其 Fourier 变换的象征为 $-|\boldsymbol{\xi}|^2$. Laplace 算子与 Brown 运动中的 Markov 运动有关，即一个给定的微分算子构造一个 Markov 过程. 例如与 L_p 次 Markov 半群相关的 Hunt 过程就是利用微分算子构造的. Demuth 和 van Casteren 发展的随机谱分析理论用一般对称的 Feller 过程代替了 Laplace 运动或 Brown 运动[17]. 分数阶 Laplace 算子 $(-\Delta)^\alpha$ 是椭圆微分算子的推广，其象征为 $|\boldsymbol{\xi}|^{2\alpha}$. Riesz 在处理分数阶 Laplace 算子 $(-\Delta)^\alpha$ 时首先注意到了该算子的非局部特征，该特征不仅与物理学中的扩散过程、材料学中的热传导等问题密不可分，而且在图像处理、信号处理和金融学等领域应用广泛[17].

与 Fourier-Besov 空间相比，一般的变指数 Fourier-Besov 空间中不再具有平移不变性质，从而使一些经典理论如 Young 不等式和乘积定理均不再成立. 因此，在变指数函数空间中研究方程的适定性有一定困难. 本文研究分数阶 Boussinesq-Coriolis 方程在变指数 Fourier-Besov 空间中解的整体适定性和 Gevrey 类正则性，所得结果推广了文献[8]的相应结果.

1　预备知识

设 $\mathscr{S}(\mathbb{R}^3)$ 是 $\mathbb{R}^3$ 上的光滑速降函数空间，$\mathscr{S}'(\mathbb{R}^3)$ 表示 $\mathscr{S}(\mathbb{R}^3)$ 的拓扑对偶空间，也称为缓增分布空间. 设非负光滑函数 $\chi,\varphi\in\mathscr{S}(\mathbb{R}^3)$ 满足

$$\operatorname{supp}\chi\subset\left\{\boldsymbol{\xi}\in\mathbb{R}^3:|\boldsymbol{\xi}|\leqslant\frac{4}{3}\right\},\qquad \operatorname{supp}\varphi\subset\left\{\boldsymbol{\xi}\in\mathbb{R}^3:\frac{4}{3}\leqslant|\boldsymbol{\xi}|\leqslant\frac{8}{3}\right\},$$

$$\chi(\boldsymbol{\xi})+\sum_{j\geqslant0}\varphi(2^{-j}\boldsymbol{\xi})=1,\quad \boldsymbol{\xi}\in\mathbb{R}^3,\qquad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\boldsymbol{\xi})=1,\quad \forall\,\boldsymbol{\xi}\in\mathbb{R}^3\backslash\{0\},$$

其中 $\varphi_j(\boldsymbol{\xi})=\varphi(2^{-j}\boldsymbol{\xi})$. 定义局部化算子

$$\Delta_j f=\mathscr{F}^{-1}\varphi_j\mathscr{F}f=\int_{\mathbb{R}^3}f_j(\boldsymbol{y})f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y},\qquad \forall\,j\in\mathbb{Z},$$

$$S_j f=\sum_{i\leqslant j-1}\Delta_i f,$$

其中 $f_j=\mathscr{F}^{-1}\varphi_j$. 由上述定义可知，当 $|j-i|\geqslant2$ 时，$\Delta_j\Delta_i$ 恒为 0；当 $|j-i|\geqslant5$ 时，$\Delta_j(\Delta_{i-1}-\Delta_i)$ 恒为 0.

对可测函数 $p(\cdot)$，记

$$\mathscr{P}_0(\mathbb{R}^3):=\{p(\cdot):\mathbb{R}^3\to(0,\infty]\,|\,0<p_-=\operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathbb{R}^3}p(\boldsymbol{x}),\ p_+=\operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^3}p(\boldsymbol{x})<\infty\}.$$

变指数 Lebesgue 函数空间定义为 $L^{p(\cdot)}=\{f:\mathbb{R}^3\to\mathbb{R},\ \|f\|_{L^{p(\cdot)}}<\infty\}$，$f$ 的 $L^{p(\cdot)}$ 范数定义为

$$\|f\|_{L^{p(\cdot)}}:=\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left(\frac{|f(\boldsymbol{x})|}{\lambda}\right)^{p(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\leqslant1\right\}.$$

由于 $L^{p(\cdot)}$ 中不具有平移不变性，为保证 Hardy-Littlewood 极大算子在 $L^{p(\cdot)}(\mathbb{R}^3)$ 上有界，因此假设 $p(\cdot)$ 满足 log-Hölder 连续条件.

若存在正常数 $C_{\log}(p)$，使得对任意的 $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$ 且 $\boldsymbol{x}\neq\boldsymbol{y}$，$p(\cdot)$满足

$$|p(\boldsymbol{x})-p(\boldsymbol{y})|\leqslant\frac{C_{\log}(p)}{\log(\mathrm{e}+|\boldsymbol{x}-\boldsymbol{y}|^{-1})},\tag{2}$$

或若存在正常数 $C_{\log}(p)$和 p_∞，使得对任意的 $\boldsymbol{x}\in\mathbb{R}^3$，$p(\cdot)$满足

$$|p(\boldsymbol{x})-p_\infty|\leqslant\frac{C_{\log}(p)}{\log(\mathrm{e}+|\boldsymbol{x}|)},\tag{3}$$

则 $p(\cdot)$分别称为局部 log-Hölder 连续的和整体 log-Hölder 连续的. 本文所考虑的 $p(\cdot)$属于既局部 log-Hölder 连续又整体 log-Hölder 连续的函数空间，记为 $C_{\log}(\mathbb{R}^3)$.

设 $p(\cdot),q(\cdot)\in\mathscr{P}_0(\mathbb{R}^3)$，$l^{q(\cdot)}(L^{p(\cdot)})$表示由$\mathbb{R}^3$ 上所有可测函数序列$\{f_j\}_{j\in\mathbb{Z}}$组成的函数空间，记

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(L^{p(\cdot)})}:=\inf\left\{\mu>0,\ \rho_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j\in\mathbb{Z}}\right)\leqslant 1\right\},$$

其中

$$\rho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}})=\sum_{j\in\mathbb{Z}}\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left(\frac{|f_j(\boldsymbol{x})|}{\lambda^{1/q(\boldsymbol{x})}}\right)^{p(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\leqslant 1\right\}.$$

假设 $q_+<\infty$，$\rho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}})=\sum_{j\in\mathbb{Z}}\||f_j|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}}$ 成立.

定义 1　设 $p(\cdot),q(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$且 $s(\cdot)\in C_{\log}(\mathbb{R}^3)$，则齐次变指数 Besov 空间定义为

$$\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}=\{f\in\dot{\mathscr{S}}'(\mathbb{R}^3):\ \|f\|_{\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}}<\infty\},$$

f 的 $\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$范数定义为

$$\|f\|_{\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}}:=\|\{2^{j(\cdot)}\dot{\Delta}_jf\}_{j\in\mathbb{Z}}\|_{\rho_{l^{q(\cdot)}(L^{p(\cdot)})}},$$

其中 $\dot{\mathscr{S}}'(\mathbb{R}^3)$表示 $\dot{\mathscr{S}}(\mathbb{R}^3)=\{f\in\mathscr{S}(\mathbb{R}^3):(D^\alpha\hat{f})(0)=0,\ \forall\alpha\in\mathbb{N}^+\}$的对偶空间.

设 $T>0$，$\rho\in[1,\infty]$，$L^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$空间中缓增分布函数 f 的范数定义为

$$\|f\|_{L^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})}:=\left\|\Big(\sum_{j\in\mathbb{Z}}\|2^{js(\cdot)}\dot{\Delta}_jf\|^r_{L^{p(\cdot)}}\Big)^{1/r}\right\|_{L^\rho_T},$$

$\widetilde{L}^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$空间中缓增分布函数 f 的范数定义为

$$\|f\|_{\widetilde{L}^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})}:=\Big(\sum_{j\in\mathbb{Z}}\|2^{js(\cdot)}\dot{\Delta}_jf\|^r_{L^\rho_TL^{p(\cdot)}}\Big)^{1/r}.$$

记 $L^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}:=L^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$，$\widetilde{L}^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}:=\widetilde{L}^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$. 根据 Minkowski 不等式，有

$$\|f\|_{\widetilde{L}^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})}\leqslant\|f\|_{L^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})},\qquad\rho\leqslant r,$$

$$\|f\|_{L^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})}\leqslant\|f\|_{\widetilde{L}^\rho(0,T;\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})},\qquad r\leqslant\rho.$$

定义 2(**齐次变指数 Fourier-Besov 空间**)　设 $p(\cdot),q(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$且 $s(\cdot)\in C_{\log}(\mathbb{R}^3)$，则齐次变指数 Fourier-Besov 空间定义为

$$\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}=\{f\in\dot{\mathscr{S}}'(\mathbb{R}^3):\ \|f\|_{\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}}<\infty\},$$

f 的 $\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$范数定义为

$$\|f\|_{\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}}:=\|\{2^{js(\cdot)}\varphi_j\hat{f}\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(L^{p(\cdot)})}.$$

类似可定义 $L^\rho(0,T;\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$，$\widetilde{L}^\rho(0,T;\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$空间中缓增分布函数 f 的范数.

引理 1(**Hölder 不等式**)[6]　给定一个可测集 A 和可测函数 $r(\cdot),q(\cdot),p(\cdot)\in\mathscr{P}_0(A)$，$\dfrac{1}{p(\cdot)}=\dfrac{1}{q(\cdot)}+\dfrac{1}{r(\cdot)}$，则存在一个常数 C，使得当 $f\in L^{q(\cdot)}(A)$，$g\in L^{r(\cdot)}(A)$且 $fg\in L^{p(\cdot)}(A)$时，有

$$\|fg\|_{p(\cdot)}\leqslant C\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

引理 2[8]　设 $s>0$，$1\leqslant p,r,\rho\leqslant\infty$，$p_1(\cdot),p_2(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$，$\dfrac{1}{p}=\dfrac{1}{p_1(\cdot)}+\dfrac{1}{p_2(\cdot)}$，$\dfrac{1}{\rho}=\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}$且 $u,v\in\widetilde{L}^{\rho_1}_T\dot{\mathscr{B}}^0_{p_1(\cdot),r}\cap\widetilde{L}^{\rho_2}_T\dot{\mathscr{B}}^s_{p_2(\cdot),r}$，则有

$$\|uv\|_{\tilde{L}^{\rho}_T\dot{\mathscr{B}}^{s}_{p,r}} \lesssim \|u\|_{\tilde{L}^{\rho_1}_T\dot{\mathscr{B}}^{0}_{p_1(\cdot),r}}\|v\|_{\tilde{L}^{\rho_2}_T\dot{\mathscr{B}}^{s}_{p_2(\cdot),r}} + \|v\|_{\tilde{L}^{\rho_1}_T\dot{\mathscr{B}}^{0}_{p_1(\cdot),r}}\|u\|_{\tilde{L}^{\rho_2}_T\dot{\mathscr{B}}^{s}_{p_2(\cdot),r}}.$$

引理 3(**Sobolev 不等式**)[18] 设 $p_0(\cdot),p_1(\cdot)\in\mathscr{P}_0(\mathbb{R}^3)$ 且 $s_0(\cdot),s_1(\cdot)\in L^\infty(\mathbb{R}^3)\cap C_{\log}(\mathbb{R}^3)$ $(s_0(\cdot)>s_1(\cdot))$.

1) 若 $q(\cdot)\in\mathscr{P}_0(\mathbb{R}^3)$, $\dfrac{1}{q(\cdot)}$ 和 $s_0(\cdot)-\dfrac{n}{p_0(\cdot)}=s_1(\cdot)-\dfrac{n}{p_1(\cdot)}$ 是局部 log-Hölder 连续的, 则 $\dot{\mathscr{B}}^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}\hookrightarrow\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}$;

2) 若 $q_0(\cdot),q_1(\cdot)\in\mathscr{P}_0(\mathbb{R}^3)$, $\dfrac{1}{q_0(\cdot)},\dfrac{1}{q_1(\cdot)}$ 和 $s_0(\cdot)-\dfrac{n}{p_0(\cdot)}=s_1(\cdot)-\dfrac{n}{p_1(\cdot)}+\varepsilon(\boldsymbol{x})$ 是局部 log-Hölder 连续的, 则 $\dot{\mathscr{B}}^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}\hookrightarrow\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}$, 其中 $\operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathbb{R}^3}\varepsilon(\boldsymbol{x})>0$.

引理 4(**Mollification 不等式**)[19] 设 $p(\cdot)\in C_{\log}(\mathbb{R}^3)$, $f\in L^{p(\cdot)}(\mathbb{R}^3)$, $\psi\in L^1(\mathbb{R}^3)$, $\Psi(\boldsymbol{x})=\sup\limits_{\boldsymbol{y}\notin B(0,|\boldsymbol{x}|)}|\psi(\boldsymbol{y})|$ 可积, 则有

$$\|f*\psi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^3)}\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^3)}\|\Psi\|_{L^1(\mathbb{R}^3)},$$

其中 $\psi_\varepsilon=\dfrac{1}{\varepsilon^n}\psi\left(\dfrac{1}{\varepsilon}\right)$ 且 C 仅依赖于 n.

引理 5[10] 设 $0\leqslant\alpha\leqslant2$ 且 $0<s\leqslant t<\infty$, 则对任意的 $\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^3$, 有

$$t|\boldsymbol{y}|^{\alpha/2}-\frac{1}{2}(t^2-s^2)|\boldsymbol{y}|^{\alpha}-s|\boldsymbol{y}-\boldsymbol{z}|^{\alpha/2}-s|\boldsymbol{z}|^{\alpha/2}\leqslant\frac{1}{2}.$$

2 线性项和非线性项的估计

考虑下列方程:

$$\begin{cases}\partial_t\boldsymbol{u}+(-\Delta)^\alpha\boldsymbol{u}+\Omega\boldsymbol{e}_3\times\boldsymbol{u}+\nabla p=0, & \text{在 }\mathbb{R}^3\times(0,\infty)\text{ 内},\\ \operatorname{div}\boldsymbol{u}=0, & \text{在 }\mathbb{R}^3\times(0,\infty)\text{ 内},\\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0, & \text{在 }\mathbb{R}^3\text{ 内}.\end{cases}\tag{4}$$

方程(4)的解可由 Stokes-Coriolis 半群 $T_{\Omega,\alpha}(t)$ 表示为

$$T_{\Omega,\alpha}(t)\boldsymbol{u}=\mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\boldsymbol{\xi}|}t\right)\mathrm{e}^{-t|\boldsymbol{\xi}|^{2\alpha}}\boldsymbol{I}+\sin\left(\Omega\frac{\xi_3}{|\boldsymbol{\xi}|}t\right)\mathrm{e}^{-t|\boldsymbol{\xi}|^{2\alpha}}\boldsymbol{R}(\boldsymbol{\xi})\right]*\boldsymbol{u}=$$
$$\mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\boldsymbol{\xi}|}t\right)\boldsymbol{I}+\sin\left(\Omega\frac{\xi_3}{|\boldsymbol{\xi}|}t\right)\boldsymbol{R}(\boldsymbol{\xi})\right]*(\mathrm{e}^{-t(-\Delta)^\alpha}\boldsymbol{u}),$$

其中 $\boldsymbol{u}\in\dot{\mathscr{S}}'(\mathbb{R}^3)$, $\boldsymbol{I}$ 是 $\boldsymbol{M}_{3\times3}(\mathbb{R})$ 中的单位矩阵, $\boldsymbol{R}(\boldsymbol{\xi})$ 是斜对称矩阵:

$$\boldsymbol{R}(\boldsymbol{\xi}):=\frac{1}{|\boldsymbol{\xi}|}\begin{pmatrix}0&\xi_3&-\xi_2\\-\xi_3&0&\xi_1\\\xi_2&-\xi_1&0\end{pmatrix},\qquad\boldsymbol{\xi}\in\mathbb{R}^3\backslash\{0\}.$$

根据 Duhamel 原则, 方程(1)的解可写为

$$\begin{cases}\boldsymbol{u}(t)=T_{\Omega,\alpha}(t)\boldsymbol{u}_0-\displaystyle\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau+\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}g\theta\boldsymbol{e}_3\mathrm{d}\tau,\\ \theta(t)=\mathrm{e}^{-t(-\Delta)^\alpha}\theta_0-\displaystyle\int_0^t\mathrm{e}^{-(t-\tau)(-\Delta)^\alpha}[(\boldsymbol{u}\cdot\nabla)\theta]\mathrm{d}\tau,\end{cases}\tag{5}$$

对 $T_{\Omega,\alpha}(t)$ 的推导可参考文献[20].

下面给出关于半群 $\{T_{\Omega,\alpha}(t)\}_{t>0}$ 的估计.

引理 6 设 $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$, $2\leqslant p_1(\cdot)\leqslant c\leqslant p(\cdot)\leqslant\dfrac{6}{5-4\alpha}$, $s_1(\cdot)=4-2\alpha+\dfrac{2\alpha}{\rho}-\dfrac{3}{p_1(\cdot)}$, $\dfrac{1}{2}<\alpha\leqslant1$, $1\leqslant q,\rho\leqslant\infty$ 且 $f\in\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}$, 则有

$$\|T_{\Omega,\alpha}(t)f\|_{\tilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|f\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$

其中 $\Omega\in\mathbb{R}$.

证明：由定义 2、引理 1 和 Fourier 乘子 $T_{\Omega,\alpha}(t)$ 有界，可得

$$\|T_{\Omega,\alpha}(t)f\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}=\|\{\|2^{js_1(\cdot)}\varphi_j\mathscr{F}[T_{\Omega,\alpha}(t)f]\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})}\lesssim$$

$$\|\{\|2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-t|\cdot|^{2\alpha}}\hat{f}\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})}\lesssim$$

$$\left\|\left\{\sum_{l=0,\pm1}\|2^{j(4-2\alpha-3/c)}\varphi_j\hat{f}\|_{L^c}\|2^{j(2\alpha/\rho+3/c-3/p_1(\cdot))}\varphi_{j+l}\mathrm{e}^{-t2^{2\alpha(j+l)}}\|_{L^{\rho}(0,\infty;L^{cp_1(\cdot)/(c-p_1(\cdot))})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\lesssim$$

$$\left\|\left\{\sum_{l=0,\pm1}\|2^{j(4-2\alpha-3/p(\cdot))}\varphi_j\hat{f}\|_{L^{p(\cdot)}}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\lesssim\|f\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$

其中第 4 行不等式的估计用到了如下估计结果：

$$\|2^{j(2\alpha/\rho+3/c-3/p_1(\cdot))}\varphi_{j+l}\mathrm{e}^{-t2^{2\alpha(j+l)}}\|_{L^{\rho}(0,\infty;L^{cp_1(\cdot)/(c-p_1(\cdot))})}\lesssim$$

$$\|2^{j2\alpha/\rho}\mathrm{e}^{-t2^{2\alpha(j+l)}}\|_{L^{\rho}(0,\infty)}\|2^{j(3/c-3/p_1(\cdot))}\varphi_{j+l}\|_{L^{cp_1(\cdot)/(c-p_1(\cdot))}}=$$

$$\|2^{j2\alpha/\rho}\mathrm{e}^{-t2^{2\alpha(j+l)}}\|_{L^{\rho}(0,\infty)}\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{2^{j(3/c-3/p_1(\boldsymbol{x}))}\varphi_{j+l}}{\lambda}\right|^{cp_1(\boldsymbol{x})/(c-p_1(\boldsymbol{x}))}\mathrm{d}\boldsymbol{x}\leqslant1\right\}\lesssim$$

$$\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{2^{j(3/c-3/p_1(\boldsymbol{x}))}\varphi_{j+l}}{\lambda}\right|^{cp_1(\boldsymbol{x})/(c-p_1(\boldsymbol{x}))}\mathrm{d}\boldsymbol{x}\leqslant1\right\}\lesssim$$

$$\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{\varphi_{j+l}}{\lambda}\right|^{cp_1(\boldsymbol{x})/(c-p_1(\boldsymbol{x}))}2^{-3j}\mathrm{d}\boldsymbol{x}\leqslant1\right\}\lesssim$$

$$\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{\varphi_l}{\lambda}\right|^{cp_1(2^j\boldsymbol{x})/(c-p_1(2^j\boldsymbol{x}))}\mathrm{d}\boldsymbol{x}\leqslant1\right\}\lesssim C.$$

引理 7　设 $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$，$2\leqslant p_1(\cdot)\leqslant c\leqslant p(\cdot)\leqslant\frac{6}{5-4\alpha}$，$s_1(\cdot)=4-2\alpha+\frac{2\alpha}{\rho}-\frac{3}{p_1(\cdot)}$，$\frac{1}{2}<\alpha\leqslant1$，$1\leqslant q,\rho\leqslant\infty$ 且 $f\in\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})$，则有

$$\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}f\mathrm{d}\tau\right\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|f\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})},$$

其中 $\Omega\in\mathbb{R}$.

证明：由定义 2、引理 1、引理 3、引理 4 和 Hausdorff-Young 不等式，可得

$$\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}f\mathrm{d}\tau\right\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}=\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathscr{F}\left[\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}f\mathrm{d}\tau\right]\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\lesssim$$

$$\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\hat{f}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\lesssim\|f\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})},$$

其中最后一个不等式的估计用到了如下估计结果：

$$\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\hat{f}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\lesssim$$

$$\left\|\int_0^t\|2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{2p_1(\cdot)/(2-p_1(\cdot))}}\|\varphi_j\hat{f}\|_{L^2}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim$$

$$\left\|\int_0^t\|2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{2p_1(\cdot)/(2-p_1(\cdot))}}\|\dot{\Delta}_jf\|_{L^2}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim$$

$$\left\|\int_0^t 2^{j(2\alpha/\rho+5/2-2\alpha)}\mathrm{e}^{-(t-\tau)2^{2\alpha j}}\|2^{-3j(2-p_1(\cdot))/(2p_1(\cdot))}\varphi_j\|_{L^{2p_1(\cdot)/(2-p_1(\cdot))}}\|\dot{\Delta}_jf\|_{L^2}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim$$

$$\left\|\int_0^t 2^{j(2\alpha/\rho+5/2-2\alpha)}\mathrm{e}^{-(t-\tau)2^{2\alpha j}}\|\dot{\Delta}_jf\|_{L^2}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim$$

$$\|2^{j(2\alpha/\rho+5/2-2\alpha)}\|\dot{\Delta}_jf\|_{L^2}\|_{L^{\rho}(0,\infty)}\|\mathrm{e}^{-t2^{2\alpha j}}\|_{L^1(0,\infty)}\lesssim\|2^{j(2\alpha/\rho+5/2-2\alpha)}\|\dot{\Delta}_jf\|_{L^2}\|_{L^{\rho}(0,\infty)}.$$

3　分数阶 Boussinesq-Coriolis 方程解的整体适定性

定理 1　设 $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$，$\frac{1}{2}<\alpha\leqslant1$，$2\leqslant p(\cdot)\leqslant\frac{6}{5-4\alpha}$且 $1\leqslant q,\rho\leqslant\infty$，存在一个足够小的常数 $\varepsilon>0$，使得 $\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}<\varepsilon$，则此时方程(1)有唯一的整体解

$$(\boldsymbol{u},\theta)\in \widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap \widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap \widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q}).$$

进一步，设 $p_1(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$，$s_1(\cdot)\in C_{\log}(\mathbb{R}^3)$ 且 $s_1(\cdot)=4-2\alpha+\dfrac{2\alpha}{\rho}-\dfrac{3}{p_1(\cdot)}$，若存在一个常数 $c>0$，使得 $2\leqslant p_1(\cdot)\leqslant c\leqslant p(\cdot)\leqslant 6/(5-4\alpha)$，则上述解仍满足

$$(\boldsymbol{u},\theta)\in \mathscr{C}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap \widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q}).$$

证明：设 $M,\delta>0$，

$$X=\{(\boldsymbol{u},\theta):\ \|\boldsymbol{u}\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})}+\|\theta\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})}\leqslant M,$$
$$\|\boldsymbol{u}\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+\|\theta\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\leqslant\delta\},$$

在该空间上定义度量为

$$d\left(\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix},\begin{pmatrix}\boldsymbol{\omega}\\ v\end{pmatrix}\right)=\|\boldsymbol{u}-\boldsymbol{\omega}\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+$$
$$\|\theta-v\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})},$$

易见(X,d)是一个度量空间. 考虑映射：

$$\Phi:\ \begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix}\rightarrow \boldsymbol{A}_{\Omega,\alpha}(t)\begin{pmatrix}\boldsymbol{u}_0\\ \theta_0\end{pmatrix}-\int_0^t\boldsymbol{A}_{\Omega,\alpha}(t-\tau)\begin{pmatrix}\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-g\theta\boldsymbol{e}_3]\\ (\boldsymbol{u}\cdot\nabla)\theta-\boldsymbol{0}\end{pmatrix}\mathrm{d}\tau,$$

其中 $\boldsymbol{A}_{\Omega,\alpha}(t)=\begin{pmatrix}T_{\Omega,\alpha}(t) & 0\\ 0 & \mathrm{e}^{-t(-\Delta)^{\alpha}}\end{pmatrix}$，$\mathbb{P}:=I-\nabla(-\Delta)^{-1}$ 为零散度向量域上的 Helmholtz 投影. 下面证明 $\Phi:(X,d)\rightarrow(X,d)$是一个压缩映射.

首先，估计线性项 $T_{\Omega,\alpha}(t)\boldsymbol{u}_0$ 和 $\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0$. 根据引理 6，有

$$\|T_{\Omega,\alpha}(t)\boldsymbol{u}_0\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$

且 $\|\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}$，类似地，有

$$\|T_{\Omega,\alpha}(t)\boldsymbol{u}_0\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\lesssim\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},\qquad \|\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\lesssim\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}.$$

此外，对于 $\rho=\infty$ 和 $p_1(\cdot)=p(\cdot)$，$T_{\Omega,\alpha}(t)\boldsymbol{u}_0$ 和 $\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0$ 的估计如下：

$$\|T_{\Omega,\alpha}(t)\boldsymbol{u}_0\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})}\lesssim\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$
$$\|T_{\Omega,\alpha}(t)\boldsymbol{u}_0\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\lesssim\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$
$$\|\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})}\lesssim\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$
$$\|\mathrm{e}^{-t(-\Delta)^{\alpha}}\theta_0\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\lesssim\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}.$$

其次，估计剩余项. 根据定义 2、引理 2～引理 4 和 Hausdorff-Young 不等式，有

$$\begin{aligned}
&\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau\right\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}=\\
&\qquad\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathscr{F}\left[\int_0^t T_{\Omega,\alpha}(t-\tau)\,\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau\right]\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\lesssim\\
&\qquad\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\mathscr{F}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\leqslant\\
&\qquad\|\boldsymbol{u}\|_{\widetilde{L}^{\rho}(0,\infty;\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{L^{\infty}(0,\infty;L^{3/(2\alpha-1)})}\leqslant\|\boldsymbol{u}\|_{\widetilde{L}^{\rho}(0,\infty;\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{\widetilde{L}^{\infty}(0,\infty;\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}=\\
&\qquad\|\boldsymbol{u}\|_{\widetilde{L}^{\rho}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{\widetilde{L}^{\infty}(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})},
\end{aligned}\tag{6}$$

其中第 3 行不等式的估计用到了如下估计结果：

$$\begin{aligned}
&\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\mathscr{F}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty;L^{p_1(\cdot)})}\lesssim\\
&\quad\left\|\int_0^t\|2^{j(s_1(\cdot)+1)}\varphi_j\mathrm{e}^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{6p_1(\cdot)/(6-(5-4\alpha)p_1(\cdot))}}\|\dot{\Delta}_j(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^{6/(4\alpha+1)}}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim\\
&\quad\left\|\int_0^t 2^{j(2\alpha/\rho+5/2)}\mathrm{e}^{-(t-\tau)2^{2\alpha j}}\|2^{-3j(6-(5-4\alpha)p_1(\cdot))/(6p_1(\cdot))}\varphi_j\|_{L^{6p_1(\cdot)/(6-(5-4\alpha)p_1(\cdot))}}\|\dot{\Delta}_j(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^{6/(4\alpha+1)}}\mathrm{d}\tau\right\|_{L^{\rho}(0,\infty)}\lesssim
\end{aligned}$$

$$\left\|\int_0^t 2^{j(2\alpha/\rho+5/2)} \mathrm{e}^{-(t-\tau)2^{2\alpha j}} \|\dot{\Delta}_j(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^{6/(4\alpha+1)}} \mathrm{d}\tau\right\|_{L^\rho(0,\infty)} \lesssim$$

$$\|2^{j(2\alpha/\rho+5/2-2\alpha)} \|\dot{\Delta}_j(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^{6/(4\alpha+1)}}\|_{L^\rho(0,\infty)} \|2^{2\alpha j}\mathrm{e}^{-t2^{2\alpha j}}\|_{L^1(0,\infty)}.$$

同理，根据引理 7 和式(6)，可得

$$\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}g\theta\boldsymbol{e}_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|\theta\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})},$$

$$\left\|\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^\alpha}[(\boldsymbol{u}\cdot\nabla)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}.$$

此外，还有

$$\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})} \lesssim$$

$$\|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})},$$

$$\left\|\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}g\theta\boldsymbol{e}_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})} \lesssim \|\theta\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})},$$

$$\left\|\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^\alpha}[(\boldsymbol{u}\cdot\nabla)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})} \lesssim$$

$$\|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}.$$

最后，证明解的存在性和唯一性．设 $Y=\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})$，则有

$$\begin{aligned}\|\Phi(\boldsymbol{u})\|_Y+\|\Phi(\theta)\|_Y \lesssim\ & \|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\\ & \|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+\\ & \|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+\\ & \|\theta\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}.\end{aligned}$$

令 $\delta=M=2(\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}})<2C_\varepsilon$，若 ε 足够小，则有 $\|\Phi(\boldsymbol{u})\|_Y+\|\Phi(\theta)\|_Y\leqslant\dfrac{\delta}{2}+\dfrac{\delta}{2}=\delta$，且 $d\left(\Phi\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix},\Phi\begin{pmatrix}\boldsymbol{\omega}\\ v\end{pmatrix}\right)\leqslant\dfrac{1}{2}d\left(\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix},\begin{pmatrix}\boldsymbol{\omega}\\ v\end{pmatrix}\right)$．当 ε 足够小时，由 Banach 压缩映射原理可知三维分数阶 Boussinesq-Coriolis 方程具有唯一的整体解，且满足

$$(\boldsymbol{u},\theta)\in\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q}).$$

另一方面，设

$$Z=\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1}_{p_1(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}),$$

则有

$$\begin{aligned}\|\Phi(\boldsymbol{u},\theta)\|_Z=\|\Phi(\boldsymbol{u})\|_Z+\|\Phi(\theta)\|_Z\lesssim\ & \|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}+\\ & \|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\boldsymbol{u}\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+\\ & \|\boldsymbol{u}\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}+\\ & \|\theta\|_{\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{5/2-2\alpha}_{2,q})}.\end{aligned}$$

设

$$\delta=M=2(\|\boldsymbol{u}_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}\cap\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p_1(\cdot)}_{p_1(\cdot),q}}+\|\theta_0\|_{\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}\cap\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p_1(\cdot)}_{p_1(\cdot),q}})<2C_\varepsilon,$$

若 ε 足够小，则有 $\|\Phi(\boldsymbol{u},\theta)\|_Z\leqslant\dfrac{\delta}{2}+\dfrac{\delta}{2}=\delta$，且 $d\left(\Phi\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix},\Phi\begin{pmatrix}\boldsymbol{\omega}\\ v\end{pmatrix}\right)\leqslant\dfrac{1}{2}d\left(\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix},\begin{pmatrix}\boldsymbol{\omega}\\ v\end{pmatrix}\right)$.

当 ε 足够小时，由 Banach 压缩映射原理可知三维分数阶 Boussinesq-Coriolis 方程具有唯一的整体解，且满足

$$(\boldsymbol{u},\theta)\in\widetilde{L}^\rho(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})\cap\widetilde{L}^\infty(0,\infty;\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}).$$

注 1　变指数 Fourier-Besov 空间 $\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}$ 是方程(1)的临界空间．若 $\boldsymbol{u}(t,\boldsymbol{x})$ 是方程(1)的解，

则 $\boldsymbol{u}_\lambda(t,\boldsymbol{x})=\lambda^{2\alpha-1}\boldsymbol{u}(\lambda^{2\alpha}t,\lambda\boldsymbol{x})$也是方程(1)的一个解，且

$$\|\boldsymbol{u}(0,\boldsymbol{x})\|_{\mathscr{FB}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}} \sim \|\boldsymbol{u}_\lambda(0,\boldsymbol{x})\|_{\mathscr{FB}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}.$$

4 分数阶 Boussinesq-Coriolis 方程解的 Gevrey 类正则性

定理 2(**Gevrey 类正则性**) 设 $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathscr{P}_0(\mathbb{R}^3)$，$\frac{1}{2}<\alpha\leqslant 1$，$2\leqslant p_1(\cdot)\leqslant p(\cdot)\leqslant\frac{6}{5-4\alpha}$，$1\leqslant q<\frac{3}{2\alpha-1}$. 对任意初值$(\boldsymbol{u}_0,\theta_0)\in\mathscr{F}\dot{\mathscr{B}}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}(\mathbb{R}^3)$，若存在一个正常数 ϑ_0，使得 $\|(\boldsymbol{u}_0,\theta_0)\|_{\mathscr{FB}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}}<\vartheta_0$，则定理 1 的解是正则的，且满足

$$\left\|\begin{pmatrix}\mathrm{e}^{\sqrt{t}\,|D|^\alpha}\boldsymbol{u}\\ \mathrm{e}^{\sqrt{t}\,|D|^\alpha}\theta\end{pmatrix}\right\|_{\tilde{L}^\rho(0,\infty;\mathscr{FB}^{4-2\alpha-3/p_1(\cdot)+2\alpha/\rho}_{p_1(\cdot),q})\cap\tilde{L}^\rho(0,\infty;\mathscr{FB}^{2\alpha/\rho+5/2-2\alpha}_{2,q})\cap\tilde{L}^\infty(0,\infty;\mathscr{FB}^{5/2-2\alpha}_{2,q})}\lesssim\left\|\begin{pmatrix}\boldsymbol{u}_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{FB}^{4-2\alpha-3/p(\cdot)}_{p(\cdot),q}},$$

其中 $\mathrm{e}^{\sqrt{t}\,|D|^\alpha}$ 是一个 Fourier 乘子，其象征定义为 $\mathrm{e}^{\sqrt{t}\,|\xi|^\alpha}$.

证明：设$\begin{pmatrix}\boldsymbol{B}_u\\ \boldsymbol{B}_\theta\end{pmatrix}=\boldsymbol{H}_\alpha(t)\begin{pmatrix}\boldsymbol{u}\\ \theta\end{pmatrix}$，其中 $\boldsymbol{H}_\alpha(t)=\begin{pmatrix}\mathrm{e}^{\sqrt{t}\,|D|^\alpha} & 0\\ 0 & \mathrm{e}^{\sqrt{t}\,|D|^\alpha}\end{pmatrix}$，由式(5)得

$$\begin{pmatrix}\boldsymbol{B}_u\\ \boldsymbol{B}_\theta\end{pmatrix}=\boldsymbol{H}_\alpha(t)\boldsymbol{A}_{\Omega,\alpha}(t)\begin{pmatrix}\boldsymbol{u}_0\\ \theta_0\end{pmatrix}-\int_0^t\boldsymbol{H}_\alpha(t)\boldsymbol{A}_{\Omega,\alpha}(t-\tau)\begin{pmatrix}\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-g\theta\boldsymbol{e}_3]\\ (\boldsymbol{u}\cdot\nabla)\theta-\boldsymbol{0}\end{pmatrix}\mathrm{d}\tau.$$

类似定理 1 的证明，只需证以下结论. 由 $\mathrm{e}^{\sqrt{t}\,|\xi|^\alpha-\frac{1}{2}t|\xi|^{2\alpha}}=\mathrm{e}^{(\sqrt{t}\,|\xi|^\alpha-1)^2/2+1/2}\leqslant\mathrm{e}^{1/2}$和引理 5，有

$$\begin{aligned}
&\left\|\begin{pmatrix}\boldsymbol{B}_u\\ \boldsymbol{B}_\theta\end{pmatrix}\right\|_{\tilde{L}^\rho(0,\infty;\mathscr{FB}^{4-2\alpha-3/p_1(\cdot)+2\alpha/\rho}_{p_1(\cdot),q})}\leqslant\left\|\boldsymbol{H}_\alpha(t)\boldsymbol{A}_{\Omega,\alpha}(t)\begin{pmatrix}\boldsymbol{u}_0\\ \theta_0\end{pmatrix}\right\|_{\tilde{L}^\rho(0,\infty;\mathscr{FB}^{4-2\alpha-3/p_1(\cdot)+2\alpha/\rho}_{p_1(\cdot),q})}+\\
&\left\|\int_0^t\boldsymbol{H}_\alpha(t)\boldsymbol{A}_{\Omega,\alpha}(t-\tau)\begin{pmatrix}\mathbb{P}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-g\theta\boldsymbol{e}_3]\\ (\boldsymbol{u}\cdot\nabla)\theta-\boldsymbol{0}\end{pmatrix}\mathrm{d}\tau\right\|_{\tilde{L}^\rho(0,\infty;\mathscr{FB}^{4-2\alpha-3/p_1(\cdot)+2\alpha/\rho}_{p_1(\cdot),q})}\lesssim\\
&\left\|\left\|\begin{pmatrix}\mathrm{e}^{\sqrt{t}\,|\xi|^\alpha-t|\xi|^{2\alpha}}2^{i(4-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_i\hat{\boldsymbol{u}}_0\\ \mathrm{e}^{\sqrt{t}\,|\xi|^\alpha-t|\xi|^{2\alpha}}2^{i(4-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_i\hat{\theta}_0\end{pmatrix}\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\|_{l^q}+\\
&\|\,\|2^{i(5-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_iK\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\|_{l^q}\lesssim\\
&\left\|\left\|\begin{pmatrix}\mathrm{e}^{t|\xi|^{2\alpha}/2}2^{i(4-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_i\hat{\boldsymbol{u}}_0\\ \mathrm{e}^{t|\xi|^{2\alpha}/2}2^{i(4-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_i\hat{\boldsymbol{u}}_0\end{pmatrix}\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\|_{l^q}+\\
&\left\|\left\|2^{i(5-2\alpha-3/p_1(\cdot)+2\alpha/\rho)}\varphi_i\int_0^t\mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}/2}\begin{pmatrix}\mathscr{F}[\boldsymbol{u}\otimes\boldsymbol{u}]-\mathscr{F}[g\otimes\theta\boldsymbol{e}_3]\\ \mathscr{F}[\boldsymbol{u}\otimes\theta]-\hat{\boldsymbol{0}}\end{pmatrix}\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\|_{l^q},
\end{aligned}$$

其中，

$$K=\int_0^t\mathrm{e}^N\int_{\mathbb{R}^3}\mathrm{e}^L\begin{pmatrix}\hat{\boldsymbol{u}}(\boldsymbol{\xi}-\boldsymbol{\zeta},\tau)\otimes\hat{\boldsymbol{u}}(\boldsymbol{\zeta},\tau)-\hat{g}(\boldsymbol{\xi}-\boldsymbol{\zeta},\tau)\otimes\mathscr{F}[\theta\boldsymbol{e}_3](\boldsymbol{\zeta},\tau)\\ \hat{\boldsymbol{u}}(\boldsymbol{\xi}-\boldsymbol{\zeta},\tau)\otimes\hat{\theta}(\boldsymbol{\zeta},\tau)-\hat{\boldsymbol{0}}\end{pmatrix}\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\tau,$$

$$N=-\frac{1}{2}(t-\tau)|\boldsymbol{\xi}|^{2\alpha},\qquad L=\sqrt{t}\,|\boldsymbol{\xi}|^\alpha-\frac{1}{2}(t-\tau)|\boldsymbol{\xi}|^{2\alpha}-\sqrt{\tau}(|\boldsymbol{\xi}-\boldsymbol{\zeta}|^\alpha+|\boldsymbol{\zeta}|^\alpha).$$

参 考 文 献

[1] KANG K, LEE J, NGUYEN D D. Global Well-Posedness and Stability of the 2D Boussinesq Equations with Partial Dissipation Near a Hydrostatic Equilibrium [J]. Journal of Differential Equations, 2024, 393(1): 1-57.

[2] IBRAHIM S, SULAIMAN T A, YUSUF A, et al. Wave Propagation to the Doubly Dispersive Equation and the Improved Boussinesq Equation [J/OL]. Optical and Quantum Electronics, (2023-11-23)[2024-01-11]. https://doi.org/10.1007/s11082-023-05571-5.

[3] SUN J Y, CUI S B. Sharp Well-Posedness and Ill-Posedness of the Three-Dimensional Primitive Equations of Geophysics in Fourier-Besov Spaces [J]. Nonlinear Analysis: Real World Applications, 2019, 48: 445-465.

[4] WANG W H. Global Well-Posedness and Analyticity for the 3D Fractional Magnetohydrodynamics Equations in Variable Fourier-Besov Spaces [J]. Zeitschrift für Angewandte Mathematik und Physik，2019，70(6)：163-1-163-16.

[5] ABIDIN M Z，CHEN J C. Global Well-Posedness for Fractional Navier-Stokes Equations in Variable Exponent Fourier-Besov-Morrey Spaces [J]. Acta Mathematica Scientia：Series B (English Edition)，2021，41(1)：164-176.

[6] ABIDIN M Z，ULLAH N，OMER O A. Global Well-Posedness and Analyticity of the Primitive Equations of Geophysics in Variable Exponent Fourier-Besov Spaces [J]. Symmetry，2022，14(1)：165-1-165-12.

[7] SUN J Y，LIU C L，YANG M H. Global Solutions to 3D Rotating Boussinesq Equations in Besov Spaces [J]. Journal of Dynamics and Differential Equations，2020，32(2)：589-603.

[8] SUN X C，WU Y L，XU G T. Global Well-Posedness for the 3D Rotating Boussinesq Equations in Variable Exponent Fourier-Besov Spaces [J]. AIMS Mathematics，2023，8(11)：27065-27079.

[9] 汝少雷. 不可压 Navier-Stokes 方程在变指标函数空间上的整体适定性 [J]. 中国科学：数学，2018，48(10)：1427-1442.（RU S L. Global Well-Posedness of the Incompressible Navier-Stokes Equations in Function Spaces with Variable Exponents [J]. Scientia Sinica：Mathematics，2018，48(10)：1427-1442.）

[10] OUIDIRNE F，AZANZAL A，ALLALOU C，et al. Well-Posedness and Analyticity for the Viscous Primitive Equations of Geophysics in Critical Fourier-Besov-Morrey Spaces with Variable Exponents [J]. International Journal of Nonlinear Analysis and Applications，2023，14(1)：2915-2929.

[11] ABIDIN M Z，MARWAN M，ULLAH N，et al. Well-Posedness in Variable-Exponent Function Spaces for the Three-Dimensional Micropolar Fluid Equations [J/OL]. Journal of Mathematics，(2023-12-26)[2024-07-07]. doi：10.1155/2023/4083997.

[12] ABIDIN M Z，CHEN J C. Global Well-Posedness of the Generalized Rotating Magnetohydrodynamics Equations in Variable Exponent Fourier-Besov Spaces [J]. Journal of Applied Analysis and Computation，2021，11(3)：1177-1190.

[13] OUIDIRNE F，ALLALOU C，OUKESSOU M. Analyticity for the Fractional Navier-Stokes Equations in Critical Fourier-Besov-Morrey Spaces with Variable Exponents [J/OL]. Boletim da Sociedade Paranaense de Matemática (3rd Série)，(2024-05-02)[2024-07-07]. https://doi.org/10.5269/bspm.62956.

[14] OUIDIRNE F，SRHIRI H，ALLALOU C，et al. Global Existence for the 3-D Generalized Micropolar Fluid System in Critical Fourier-Besov Spaces with Variable Exponent [J]. Nonlinear Dynamics and Systems Theory，2023，23(3)：338-347.

[15] AZANZAL A，ALLALOU C，MELLIANI S. Well-Posedness，Analyticity and Time Decay of the 3D Fractional Magneto-Hydrodynamics Equations in Critical Fourier-Besov-Morrey Spaces with Variable Exponent [J]. Journal of Elliptic and Parabolic Equations，2022，8(2)：723-742.

[16] ABIDIN M Z，CHEN J C. Global Well-Posedness and Analyticity of Generalized Porous Medium Equation in Fourier-Besov-Morrey Spaces with Variable Exponent [J/OL]. Mathematics，(2021-02-28)[2024-07-07]. https://doi.org/10.3390/math9050498.

[17] JACOB N. Pseudo Differential Operators and Markov Processes：Vol. 3 [M]. London：Imperial College Press，2005：1-474.

[18] ALMEIDA A，HÄSTÖ P. Besov Spaces with Variable Smoothness and Integrability [J]. Journal of Functional Analysis，2010，258(5)：1628-1655.

[19] ABIDIN M Z，CHEN J C. Global Well-Posedness of Generalized Magnetohydrodynamics Equations in Variable Exponent Fourier-Besov-Morrey Spaces [J]. Acta Mathematica Sinica (English Series)，2022，38(12)：2187-2198.

[20] WANG W H，WU G. Global Mild Solution of the Generalized Navier-Stokes Equations with the Coriolis Force [J]. Applied Mathematics Letters，2018，76(1)：181-186.

（责任编辑：赵立芹）