doi: 10.13413/j.cnki.jdxblxb.2024327

套代数上的一类非线性局部广义 Jordan 三重中心化子

刘昕卓，张建华
(陕西师范大学 数学与统计学院，西安 710119)

摘要：设 H 是实数域或复数域 $\mathbb{F}$ 上的 Hilbert 空间，$\mathscr{N}$ 为 H 上的非平凡套，$\tau(\mathscr{N})$ 为相应的套代数，并且 ϕ: $\tau(\mathscr{N})\to\tau(\mathscr{N})$ 是一个映射(无可加和线性假设). 利用代数分解的方法证明如果对任意的 $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\tau(\mathscr{N})$ 且 $\boldsymbol{XYZ}=0$，有 $3\phi(\boldsymbol{X}\circ\boldsymbol{Y}\circ\boldsymbol{Z})=\phi(\boldsymbol{X})\circ\boldsymbol{Y}\circ\boldsymbol{Z}+\boldsymbol{X}\circ\phi(\boldsymbol{Y})\circ\boldsymbol{Z}+\boldsymbol{X}\circ\boldsymbol{Y}\circ\phi(\boldsymbol{Z})$，则 ϕ 是可加的中心化子.

关键词：套代数；非线性映射；中心化子
中图分类号：O177.1　　**文献标志码**：A　　**文章编号**：1671-5489(2025)03-0752-05

A Class of Nonlinear Local Generalized Jordan Triple Centralizers on Nest Algebras

LIU Xinzhuo, ZHANG Jianhua
(*School of Mathematics and Statistics, Shaanxi Normal University, Xi'an* 710119, *China*)

Abstract: Let H be a Hilbert space over the real or complex field $\mathbb{F}$, $\mathscr{N}$ be a non-trivial nest on H and $\tau(\mathscr{N})$ be the associated nest algebra. Suppose ϕ: $\tau(\mathscr{N})\to\tau(\mathscr{N})$ is a map without additivity and linearity. With the help of method of algebratic decomposition, we prove that if $3\phi(\boldsymbol{X}\circ\boldsymbol{Y}\circ\boldsymbol{Z})=\phi(\boldsymbol{X})\circ\boldsymbol{Y}\circ\boldsymbol{Z}+\boldsymbol{X}\circ\phi(\boldsymbol{Y})\circ\boldsymbol{Z}+\boldsymbol{X}\circ\boldsymbol{Y}\circ\phi(\boldsymbol{Z})$ for any $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\tau(\mathscr{N})$ with $\boldsymbol{XYZ}=0$, then ϕ is an additive centralizer.

Keywords: nest algebra; nonlinear map; centralizer

1　引言与预备知识

设 $\mathscr{A}$ 是一个含单位元的环或代数，ϕ: $\mathscr{A}\to\mathscr{A}$ 是一个可加或线性映射，若对任意的 $\boldsymbol{X},\boldsymbol{Y}\in\mathscr{A}$，有 $\phi(\boldsymbol{XY})=\phi(\boldsymbol{X})\boldsymbol{Y}=\boldsymbol{X}\phi(\boldsymbol{Y})$ 或 $\phi(\boldsymbol{X}^2)=\phi(\boldsymbol{X})\boldsymbol{X}=\boldsymbol{X}\phi(\boldsymbol{X})$，则 ϕ 分别称为中心化子或 Jordan 中心化子. 注意到，中心化子一定是 Jordan 中心化子，但通常 Jordan 中心化子不一定是中心化子[1]. 若对任意的 $\boldsymbol{X}\in\mathscr{A}$，有 $(m+n)\phi(\boldsymbol{X}^2)=m\phi(\boldsymbol{X})\boldsymbol{X}+n\boldsymbol{X}\phi(\boldsymbol{X})$，其中 $m,n\in N$ 且 $m+n\neq 0$，则称 ϕ 为 (m,n)-Jordan 中心化子.

关于环或代数中满足什么条件的映射是中心化子的研究一直备受关注. 如 Vukman[2] 研究2-无挠自由半素环 $\mathscr{A}$ 上的可加映射 ϕ，证明了对任意的 $\boldsymbol{X}\in\mathscr{A}$，有 $2\phi(\boldsymbol{X}^2)=\boldsymbol{X}\phi(\boldsymbol{X})+\phi(\boldsymbol{X})\boldsymbol{X}$，则 ϕ 是中心化子. 杨翠等[3]在套代数上证明了 (m,n)-Jordan 中心化子是中心化子. 进一步，如果映射 ϕ: $\mathscr{A}\to\mathscr{A}$ 没有

收稿日期：2024-08-19.
第一作者简介：刘昕卓(2001—)，女，汉族，硕士研究生，从事算子代数的研究，E-mail: liuxinzhuo2022@163.com. **通信作者简介**：张建华(1965—)，男，汉族，博士，教授，博士生导师，从事算子代数的研究，E-mail: jhzhang@snnu.edu.cn.
基金项目：国家自然科学基金(批准号：11771261).

可加性或线性性，则 ϕ 称为非线性映射. 杨翠等[4]研究了因子 von Neumann 代数上的一类非线性映射 ϕ，满足

$$2m\phi(\boldsymbol{XY})+2n\phi(\boldsymbol{YX})=m\phi(\boldsymbol{X})\boldsymbol{Y}+m\boldsymbol{X}\phi(\boldsymbol{Y})+n\phi(\boldsymbol{Y})\boldsymbol{X}+n\boldsymbol{Y}\phi(\boldsymbol{X}),$$

其中 $(m+n)(m-n)\neq 0$，则 ϕ 是可加的中心化子. 设 F：$\mathscr{A}\to\mathscr{A}$ 是一个映射，Q 是 $\mathscr{A}$ 的一个子集，如果对任意的 $\boldsymbol{X},\boldsymbol{Y}\in\mathscr{A}$ 且 $F(\boldsymbol{X},\boldsymbol{Y})\in Q$，有

$$\phi(\boldsymbol{X}\circ\boldsymbol{Y})=\phi(\boldsymbol{X})\boldsymbol{Y}+\phi(\boldsymbol{Y})\boldsymbol{X}=\boldsymbol{X}\phi(\boldsymbol{Y})+\boldsymbol{Y}\phi(\boldsymbol{X}),$$

则 ϕ 称为 $\mathscr{A}$ 上局部 Jordan 中心化子. Liu[5]研究表明，若特殊三角代数上的局部点 $\boldsymbol{XY}=G$，满足 $\phi(\boldsymbol{X}\circ\boldsymbol{Y})=\phi(\boldsymbol{X})\boldsymbol{Y}+\phi(\boldsymbol{Y})\boldsymbol{X}=\boldsymbol{X}\phi(\boldsymbol{Y})+\boldsymbol{Y}\phi(\boldsymbol{X})$，则 ϕ 是中心化子，其中 $\boldsymbol{X},\boldsymbol{Y}\in\mathscr{A}$，$G$ 为三角代数上给定的点. 受文献[6-10]的启发，如果对任意的 $\boldsymbol{X},\boldsymbol{Y}\,\boldsymbol{Z}\in\mathscr{A}$，满足

$$3\phi(\boldsymbol{X}\circ\boldsymbol{Y}\circ\boldsymbol{Z})=\phi(\boldsymbol{X})\circ\boldsymbol{Y}\circ\boldsymbol{Z}+\boldsymbol{X}\circ\phi(\boldsymbol{Y})\circ\boldsymbol{Z}+\boldsymbol{X}\circ\boldsymbol{Y}\circ\phi(\boldsymbol{Z}),\tag{1}$$

则 ϕ 称为广义 Jordan 三重中心化子，其中 $\boldsymbol{X}\circ\boldsymbol{Y}\circ\boldsymbol{Z}=(\boldsymbol{X}\circ\boldsymbol{Y})\circ\boldsymbol{Z}$. 显然，中心化子一定是广义 Jordan 三重中心化子，但广义 Jordan 三重中心化子不一定是中心化子. 本文研究套代数上的非线性局部广义 Jordan 三重中心化子.

设 H 是实数域或复数域 $\mathbb{F}$ 上的 Hilbert 空间，$B(H)$ 表示 H 上全体有界线性算子构成的代数. 如果 $\mathscr{N}$ 是 $B(H)$ 中的一个包含 0 和 $\boldsymbol{I}$ 的全序投影族，且在强算子拓扑下是闭的，则称 $\mathscr{N}$ 是一个套. 与套 $\mathscr{N}$ 相对应的套代数记为 $\tau(\mathscr{N})$，并定义为 $\tau(\mathscr{N})=\{\boldsymbol{A}\in B(H)：\boldsymbol{PAP}=\boldsymbol{AP}，\boldsymbol{P}\in\mathscr{N}\}$. 如果套 $\mathscr{N}$ 至少含有一个非平凡投影，则称套 $\mathscr{N}$ 是非平凡的. 否则，称套 $\mathscr{N}$ 是一个平凡套. 设 $\boldsymbol{P}_1\in\mathscr{N}$ 为一个非平凡投影，记 $\boldsymbol{P}_2=\boldsymbol{I}-\boldsymbol{P}_1$ 且 $\boldsymbol{A}_{ij}=\boldsymbol{P}_i\tau(\mathscr{N})\boldsymbol{P}_j\,(1\leqslant i\leqslant j\leqslant 2)$，则 $\tau(\mathscr{N})=\boldsymbol{A}_{11}\oplus\boldsymbol{A}_{12}\oplus\boldsymbol{A}_{22}$.

2　主要结果

定理 1　设 H 是实数域或复数域 $\mathbb{F}$ 上的 Hilbert 空间，$\mathscr{N}$ 为 H 上的非平凡套，$\tau(\mathscr{N})$ 为相应的套代数，并且 ϕ：$\tau(\mathscr{N})\to\tau(\mathscr{N})$ 是一个映射(无可加和线性假设). 如果对任意的 $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\tau(\mathscr{N})$ 且 $\boldsymbol{XYZ}=0$，式(1)成立，则 ϕ 是可加的中心化子.

为证明定理 1，需要以下几个引理.

引理 1[11]　设 $\boldsymbol{X}_{ij}\in\mathscr{A}_{ij}\,(1\leqslant i\leqslant j\leqslant 2)$，则下列结论成立：

1) 若 $\boldsymbol{X}_{11}\mathscr{A}_{12}=0$，则 $\boldsymbol{X}_{11}=0$；

2) 若 $\mathscr{A}_{12}\boldsymbol{X}_{22}=0$，则 $\boldsymbol{X}_{22}=0$；

引理 2　$\phi(0)=0$.

证明：在式(1)中取 $\boldsymbol{X}=\boldsymbol{Y}=\boldsymbol{Z}=0$，可得 $\phi(0)=0$.

引理 3　对任意的 $\boldsymbol{X}_{ij}\in\mathscr{A}_{ij}\,(1\leqslant i\leqslant j\leqslant 2)$，有：

1) $\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_1=\boldsymbol{P}_2\phi(\boldsymbol{P}_1)\boldsymbol{P}_2=0$，$\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_2=0$；

2) $\boldsymbol{P}_1\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2=\boldsymbol{X}_{11}\phi(\boldsymbol{P}_1)\boldsymbol{P}_2$，$\boldsymbol{P}_1\phi(\boldsymbol{X}_{22})\boldsymbol{P}_2=-\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{22}$.

证明：1) 在式(1)中，取 $\boldsymbol{X}=\boldsymbol{P}_1$，$\boldsymbol{Y}=\boldsymbol{P}_2$，$\boldsymbol{Z}=\boldsymbol{P}_1$，则有

$$\begin{aligned}0&=3\phi(0)=3\phi(\boldsymbol{P}_1\circ\boldsymbol{P}_2\circ\boldsymbol{P}_1)=\phi(\boldsymbol{P}_1)\circ\boldsymbol{P}_2\circ\boldsymbol{P}_1+\boldsymbol{P}_1\circ\phi(\boldsymbol{P}_2)\circ\boldsymbol{P}_1+\boldsymbol{P}_1\circ\boldsymbol{P}_2\circ\phi(\boldsymbol{P}_1)=\\&4\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_1+\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_2,\end{aligned}$$

可得

$$\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_1=0,\qquad \boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{P}_2)\boldsymbol{P}_2=0.\tag{2}$$

类似地，可得 $\boldsymbol{P}_2\phi(\boldsymbol{P}_1)\boldsymbol{P}_2=0$，从而结论 1)成立.

2) 对任意的 $\boldsymbol{X}_{11}\in\mathscr{A}_{11}$，由于 $\boldsymbol{X}_{11}\boldsymbol{P}_1\boldsymbol{P}_2=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{X}_{11}$，$\boldsymbol{Y}=\boldsymbol{P}_1$，$\boldsymbol{Z}=\boldsymbol{P}_2$，再结合引理 3 中 1)，则有

$$\begin{aligned}0&=3\phi(\boldsymbol{X}_{11}\circ\boldsymbol{P}_1\circ\boldsymbol{P}_2)=\phi(\boldsymbol{X}_{11})\circ\boldsymbol{P}_1\circ\boldsymbol{P}_2+\boldsymbol{X}_{11}\circ\phi(\boldsymbol{P}_1)\circ\boldsymbol{P}_2+\boldsymbol{X}_{11}\circ\boldsymbol{P}_1\circ\phi(\boldsymbol{P}_2)=\\&\boldsymbol{P}_1\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2-\boldsymbol{X}_{11}\phi(\boldsymbol{P}_1)\boldsymbol{P}_2,\end{aligned}$$

从而得

$$\boldsymbol{P}_1\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2=\boldsymbol{X}_{11}\phi(\boldsymbol{P}_1)\boldsymbol{P}_2.\tag{3}$$

对任意的 $\boldsymbol{X}_{22}\in\mathscr{A}_{22}$，由于 $\boldsymbol{X}_{22}\boldsymbol{P}_1\boldsymbol{P}_1=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{X}_{22}$，$\boldsymbol{Y}=\boldsymbol{P}_1$，$\boldsymbol{Z}=\boldsymbol{P}_1$，则有

$$\begin{aligned}0=3\phi(\boldsymbol{X}_{22}\circ\boldsymbol{P}_1\circ\boldsymbol{P}_1)=\phi(\boldsymbol{X}_{22})\circ\boldsymbol{P}_1\circ\boldsymbol{P}_1+\boldsymbol{X}_{22}\circ\phi(\boldsymbol{P}_1)\circ\boldsymbol{P}_1+\boldsymbol{X}_{22}\circ\boldsymbol{P}_1\circ\phi(\boldsymbol{P}_1)=\\4\boldsymbol{P}_1\phi(\boldsymbol{X}_{22})\boldsymbol{P}_1+\boldsymbol{P}_1\phi(\boldsymbol{X}_{22})\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{22},\end{aligned}\tag{4}$$

对式(4)左乘 $\boldsymbol{P}_1$、右乘 $\boldsymbol{P}_2$，可得

$$\boldsymbol{P}_1\phi(\boldsymbol{X}_{22})\boldsymbol{P}_2=-\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{22}.\tag{5}$$

从而结论2)成立. 证毕.

引理4 $\phi(\mathscr{A}_{12})\subseteq\mathscr{A}_{12}$，$\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_1+\boldsymbol{P}_2\phi(\boldsymbol{P}_2)\boldsymbol{P}_2=\lambda\boldsymbol{I}$，其中 $\lambda\in\mathbb{F}$.

证明：对任意的 $\boldsymbol{X}_{12}\in\mathscr{A}_{12}$，由于 $\boldsymbol{P}_1\boldsymbol{X}_{12}\boldsymbol{P}_1=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_1$，$\boldsymbol{Y}=\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{P}_1$，则有

$$\begin{aligned}3\phi(\boldsymbol{X}_{12})=3\phi(\boldsymbol{P}_1\circ\boldsymbol{X}_{12}\circ\boldsymbol{P}_1)=\phi(\boldsymbol{P}_1)\circ\boldsymbol{X}_{12}\circ\boldsymbol{P}_1+\boldsymbol{P}_1\circ\phi(\boldsymbol{X}_{12})\circ\boldsymbol{P}_1+\boldsymbol{P}_1\circ\boldsymbol{X}_{12}\circ\phi(\boldsymbol{P}_1)=\\4\boldsymbol{P}_1\phi(\boldsymbol{X}_{12})\boldsymbol{P}_1+\boldsymbol{P}_1\phi(\boldsymbol{X}_{12})\boldsymbol{P}_2+2\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{12},\end{aligned}$$

这蕴含了 $\boldsymbol{P}_1\phi(\boldsymbol{X}_{12})\boldsymbol{P}_1=0$，$\boldsymbol{P}_1\phi(\boldsymbol{X}_{12})\boldsymbol{P}_2=\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{12}$，从而有

$$\phi(\boldsymbol{X}_{12})=\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{12}=\boldsymbol{P}_1\phi(\boldsymbol{X}_{12})\boldsymbol{P}_2\in\mathscr{A}_{12}.\tag{6}$$

在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_2$，$\boldsymbol{Y}=\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{P}_2$，并结合式(5)及引理3中1)，则有

$$\begin{aligned}3\phi(\boldsymbol{X}_{12})=3\phi(\boldsymbol{P}_2\circ\boldsymbol{X}_{12}\circ\boldsymbol{P}_2)=\phi(\boldsymbol{P}_2)\circ\boldsymbol{X}_{12}\circ\boldsymbol{P}_2+\boldsymbol{P}_2\circ\phi(\boldsymbol{X}_{12})\circ\boldsymbol{P}_2+\boldsymbol{P}_2\circ\boldsymbol{X}_{12}\circ\phi(\boldsymbol{P}_2)=\\\phi(\boldsymbol{X}_{12})+2\boldsymbol{X}_{12}\phi(\boldsymbol{P}_2)\boldsymbol{P}_2,\end{aligned}$$

从而可得

$$\phi(\boldsymbol{X}_{12})=\boldsymbol{X}_{12}\phi(\boldsymbol{P}_2)\boldsymbol{P}_2.\tag{7}$$

结合式(6),(7)，可得 $\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{X}_{12}=\boldsymbol{X}_{12}\phi(\boldsymbol{P}_2)\boldsymbol{P}_2$. 根据文献[3]中定理2.1知，存在 $\lambda\in\mathbb{F}$，使得 $\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_1+\boldsymbol{P}_2\phi(\boldsymbol{P}_2)\boldsymbol{P}_2=\lambda\boldsymbol{I}$. 证毕.

引理5 $\phi(\mathscr{A}_{ii})\subseteq\mathscr{A}_{ii}+\mathscr{A}_{12}$，$i=1,2$.

证明：对任意的 $\boldsymbol{X}_{11}\in\mathscr{A}_{11}$，由于 $\boldsymbol{X}_{11}\boldsymbol{P}_2\boldsymbol{P}_2=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{X}_{11}$，$\boldsymbol{Y}=\boldsymbol{P}_2$，$\boldsymbol{Z}=\boldsymbol{P}_2$，则有

$$\begin{aligned}0=3\phi(\boldsymbol{X}_{11}\circ\boldsymbol{P}_2\circ\boldsymbol{P}_2)=\phi(\boldsymbol{X}_{11})\circ\boldsymbol{P}_2\circ\boldsymbol{P}_2+\boldsymbol{X}_{11}\circ\phi(\boldsymbol{P}_2)\circ\boldsymbol{P}_2+\boldsymbol{X}_{11}\circ\boldsymbol{P}_2\circ\phi(\boldsymbol{P}_2)=\\4\boldsymbol{P}_2\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2+\boldsymbol{X}_{11}\phi(\boldsymbol{P}_2)\boldsymbol{P}_2.\end{aligned}$$

根据引理3可得 $\boldsymbol{P}_2\phi(\boldsymbol{X}_{11})\boldsymbol{P}_2=0$. 表明 $\phi(\boldsymbol{X}_{11})\in\mathscr{A}_{11}+\mathscr{A}_{12}$. 类似地，可证 $\phi(\boldsymbol{X}_{22})\in\mathscr{A}_{12}+\mathscr{A}_{22}$. 证毕.

引理6 存在 $\lambda\in\mathbb{F}$，使得对任意的 $\boldsymbol{X}_{ij}\in\mathscr{A}_{ij}$，有 $\phi(\boldsymbol{X}_{ij})=\lambda\boldsymbol{X}_{ij}$ $(1\leqslant i\leqslant j\leqslant 2)$.

证明：由式(6),(7)及引理4，对任意的 $\boldsymbol{X}_{12}\in\mathscr{A}_{12}$，可得 $\phi(\boldsymbol{X}_{12})=\phi(\boldsymbol{P}_1)\boldsymbol{X}_{12}=\boldsymbol{X}_{12}\phi(\boldsymbol{P}_2)$，取 $\lambda=\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_1+\boldsymbol{P}_2\phi(\boldsymbol{P}_2)\boldsymbol{P}_2$，则有

$$\phi(\boldsymbol{X}_{12})=(\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_1+\boldsymbol{P}_2\phi(\boldsymbol{P}_2)\boldsymbol{P}_2)\boldsymbol{X}_{12}=\boldsymbol{X}_{12}(\boldsymbol{P}_1\phi(\boldsymbol{P}_1)\boldsymbol{P}_1+\boldsymbol{P}_2\phi(\boldsymbol{P}_2)\boldsymbol{P}_2).$$

即 $\phi(\boldsymbol{X}_{12})=\lambda\boldsymbol{X}_{12}$. 对任意的 $\boldsymbol{X}_{ij}\in\mathscr{A}_{ij}$，由于 $\boldsymbol{P}_2\boldsymbol{X}_{12}\boldsymbol{X}_{11}=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_2$，$\boldsymbol{Y}=\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{X}_{11}$，则有

$$\begin{aligned}3\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})=3\phi(\boldsymbol{P}_2\circ\boldsymbol{X}_{12}\circ\boldsymbol{X}_{11})=\\\phi(\boldsymbol{P}_2)\circ\boldsymbol{X}_{12}\circ\boldsymbol{X}_{11}+\boldsymbol{P}_2\circ\phi(\boldsymbol{X}_{12})\circ\boldsymbol{X}_{11}+\boldsymbol{P}_2\circ\boldsymbol{X}_{12}\circ\phi(\boldsymbol{X}_{11})=\\\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})+\boldsymbol{X}_{11}\phi(\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{11})\boldsymbol{X}_{12}.\end{aligned}$$

表明 $2\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})=\boldsymbol{X}_{11}\phi(\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{11})\boldsymbol{X}_{12}$. 由式(7)有 $\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})=\boldsymbol{X}_{11}\phi(\boldsymbol{X}_{12})$，则

$$\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})=\phi(\boldsymbol{X}_{11})\boldsymbol{X}_{12}.\tag{8}$$

即 $\phi(\boldsymbol{X}_{11}\boldsymbol{X}_{12})=\phi(\boldsymbol{P}_1)\boldsymbol{X}_{11}\boldsymbol{X}_{12}=\phi(\boldsymbol{X}_{11})\boldsymbol{X}_{12}$. 由引理1中1)和引理5，有 $\phi(\boldsymbol{X}_{11})=\phi(\boldsymbol{P}_1)\boldsymbol{X}_{11}$. 即 $\phi(\boldsymbol{X}_{11})=\lambda\boldsymbol{X}_{11}$. 类似地，可证 $\phi(\boldsymbol{X}_{22})=\lambda\boldsymbol{X}_{22}$. 证毕.

引理7 对任意的 $\boldsymbol{X}_{ij}\in\mathscr{A}_{ij}$ $(1\leqslant i\leqslant j\leqslant 2)$，有：

1) $\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})=\phi(\boldsymbol{X}_{11})+\phi(\boldsymbol{X}_{12})$；

2) $\phi(\boldsymbol{X}_{12}+\boldsymbol{X}_{22})=\phi(\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{22})$.

证明：1) 对任意的 $\boldsymbol{X}_{ij},\boldsymbol{Y}_{ij}\in\mathscr{A}_{ij}$ $(1\leqslant i\leqslant j\leqslant 2)$，由于 $\boldsymbol{Y}_{12}(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_1=0$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{Y}_{12}$，$\boldsymbol{Y}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{P}_1$，则有

$$\begin{aligned}3\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{12})=3\phi(\boldsymbol{Y}_{12}\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{P}_1)=\phi(\boldsymbol{Y}_{12})\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{P}_1+\\\boldsymbol{Y}_{12}\circ\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{P}_1+\boldsymbol{Y}_{12}\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\phi(\boldsymbol{P}_1)=\\2\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{12})+\boldsymbol{Y}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{Y}_{12}.\end{aligned}$$

从而

$$\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{12})=\boldsymbol{Y}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2+\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{Y}_{12}. \tag{9}$$

同理在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_2$，$\boldsymbol{Y}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{Y}_{12}$，则有

$$\begin{aligned}0=&3\phi(\boldsymbol{P}_2\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{12})=\\&\phi(\boldsymbol{P}_2)\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{12}+\boldsymbol{P}_2\circ\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{12}+\boldsymbol{P}_2\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\phi(\boldsymbol{Y}_{12})=\\&\boldsymbol{Y}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2,\end{aligned}$$

从而

$$\boldsymbol{Y}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2=0. \tag{10}$$

将式(10)代入式(9)可得

$$\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{12})=\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{Y}_{12}. \tag{11}$$

由式(11)和引理 1 中 1)有

$$\phi(\boldsymbol{X}_{11})=\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_1. \tag{12}$$

在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_2$，$\boldsymbol{Y}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}$，$\boldsymbol{Z}=\boldsymbol{Y}_{11}$，则有

$$\begin{aligned}3\phi(\boldsymbol{Y}_{11}\boldsymbol{X}_{12})=&3\phi(\boldsymbol{P}_2\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{11})=\\&\phi(\boldsymbol{P}_2)\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{11}+\boldsymbol{P}_2\circ\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\boldsymbol{Y}_{11}+\boldsymbol{P}_2\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\circ\phi(\boldsymbol{Y}_{11})=\\&2\phi(\boldsymbol{Y}_{11}\boldsymbol{X}_{12})+\boldsymbol{Y}_{11}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2,\end{aligned}$$

从而 $\phi(\boldsymbol{Y}_{11}\boldsymbol{X}_{12})=\boldsymbol{Y}_{11}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2$. 同理可得

$$\phi(\boldsymbol{X}_{12})=\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2. \tag{13}$$

由引理 1 中 2)和式(10),(12),(13)，可得

$$\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})-\phi(\boldsymbol{X}_{11})-\phi(\boldsymbol{X}_{12})=\boldsymbol{P}_2\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})\boldsymbol{P}_2=0.$$

于是可得 $\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12})=\phi(\boldsymbol{X}_{11})+\phi(\boldsymbol{X}_{12})$. 类似可证结论 2)成立. 证毕.

引理 8　对任意的 $\boldsymbol{X}_{ij},\boldsymbol{Z}_{ij}\in\mathscr{A}_{ij}(1\leqslant i\leqslant j\leqslant 2)$，有 $\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})=\phi(\boldsymbol{X}_{11})+\phi(\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{22})$.

证明：对任意的 $\boldsymbol{X}_{ij},\boldsymbol{Z}_{ij}\in\mathscr{A}_{ij}(1\leqslant i\leqslant j\leqslant 2)$，在式(1)中取 $\boldsymbol{X}=\boldsymbol{P}_2$，$\boldsymbol{Y}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22}$，$\boldsymbol{Z}=\boldsymbol{Z}_{12}$，则有

$$\begin{aligned}3\phi(2\boldsymbol{Z}_{12}\boldsymbol{X}_{22})=&3\phi(\boldsymbol{P}_2\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{Z}_{12})=\phi(\boldsymbol{P}_2)\circ(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{Z}_{12}+\\&\boldsymbol{P}_2\circ\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{Z}_{12}+(2\boldsymbol{X}_{22}+\boldsymbol{X}_{12})\circ\phi(\boldsymbol{Z}_{12})=\\&4\phi(\boldsymbol{Z}_{12}\boldsymbol{X}_{22})+\boldsymbol{Z}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})+\boldsymbol{Z}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\boldsymbol{P}_2,\end{aligned}$$

整理得 $\phi(\boldsymbol{Z}_{12}\boldsymbol{X}_{22})=\boldsymbol{Z}_{12}\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\boldsymbol{P}_2$. 由引理 5 和引理 3 中 2)，可得

$$\phi(\boldsymbol{X}_{22})=\boldsymbol{P}_2\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\boldsymbol{P}_2. \tag{14}$$

类似可证

$$\phi(\boldsymbol{X}_{11})=\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\boldsymbol{P}_1. \tag{15}$$

在式(1)中取 $\boldsymbol{X}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22}$，$\boldsymbol{Y}=\boldsymbol{P}_1$，$\boldsymbol{Z}=\boldsymbol{P}_2$，则有

$$\begin{aligned}3\phi(\boldsymbol{X}_{12})=&3\phi((\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{P}_1\circ\boldsymbol{P}_2)=\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{P}_1\circ\boldsymbol{P}_2+\\&(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\phi(\boldsymbol{P}_1)\circ\boldsymbol{P}_2+(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\circ\boldsymbol{P}_1\circ\phi(\boldsymbol{P}_2)=\\&\boldsymbol{P}_1\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})\boldsymbol{P}_2+2\phi(\boldsymbol{X}_{12}),\end{aligned}$$

整理得

$$\phi(\boldsymbol{X}_{12})=\boldsymbol{P}_1(\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22}))\boldsymbol{P}_2. \tag{16}$$

由式(14)～(16)可得

$$\phi(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})=\phi(\boldsymbol{X}_{11})+\phi(\boldsymbol{X}_{12})+\phi(\boldsymbol{X}_{22}).$$

证毕.

下面证明定理 1. 首先，由引理 6 和引理 8 知，ϕ 在套代数上是可加的. 其次，证明 ϕ 是中心化子. 对任意的 $\boldsymbol{X},\boldsymbol{Y}\in\tau(\mathscr{N})$，设 $\boldsymbol{X}=\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22}$，$\boldsymbol{Y}=\boldsymbol{Y}_{11}+\boldsymbol{Y}_{12}+\boldsymbol{Y}_{22}$，其中 $\boldsymbol{X}_{11},\boldsymbol{Y}_{11}\in\mathscr{A}_{11}$，$\boldsymbol{X}_{12},\boldsymbol{Y}_{12}\in\mathscr{A}_{12}$，$\boldsymbol{X}_{22},\boldsymbol{Y}_{22}\in\mathscr{A}_{22}$. 则由引理 6 可得

$$\begin{aligned}\phi(\boldsymbol{X}\boldsymbol{Y})=&\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{11}+\boldsymbol{X}_{11}\boldsymbol{Y}_{12}+\boldsymbol{X}_{12}\boldsymbol{Y}_{22}+\boldsymbol{X}_{22}\boldsymbol{Y}_{22})=\\&\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{11})+\phi(\boldsymbol{X}_{11}\boldsymbol{Y}_{12})+\phi(\boldsymbol{X}_{12}\boldsymbol{Y}_{22})+\phi(\boldsymbol{X}_{22}\boldsymbol{Y}_{22})=\end{aligned}$$

$$\begin{aligned}&\lambda \boldsymbol{X}_{11}\boldsymbol{Y}_{11}+\lambda \boldsymbol{X}_{11}\boldsymbol{Y}_{12}+\lambda \boldsymbol{X}_{12}\boldsymbol{Y}_{22}+\lambda \boldsymbol{X}_{22}\boldsymbol{Y}_{22}=\\&\lambda(\boldsymbol{X}_{11}+\boldsymbol{X}_{12}+\boldsymbol{X}_{22})(\boldsymbol{Y}_{11}+\boldsymbol{Y}_{12}+\boldsymbol{Y}_{22})=\\&\lambda \boldsymbol{X}\boldsymbol{Y}=\boldsymbol{X}\lambda \boldsymbol{Y}=\phi(\boldsymbol{X})\boldsymbol{Y}=\boldsymbol{X}\phi(\boldsymbol{Y}).\end{aligned}$$

从而 ϕ 在套代数上是可加的中心化子. 定理 1 证毕.

参 考 文 献

[1] GHAHRAMANI H. On Centralizers of Banach Algebras [J]. Bulletin of the Malaysian Mathematical Sciences Society, 2015, 38(1): 155-164.

[2] VUKMAN J. An Identity Related to Centralizers in Semiprime Rings [J]. Commentations Mathmaticas Universitatis Carolinae, 1999, 40(3): 447-456.

[3] 杨翠, 张建华. 套代数上的广义 Jordan 中心化子 [J]. 数学学报(中文版), 2010, 53(5): 975-980. (YANG C, ZHANG J H. Generalized Jordan Centralizers on Nest Algebras [J]. Acta Mathematica Sinica (Chinese Series), 2010, 53(5): 975-980.)

[4] 杨翠, 吴冰, 刘珍. 因子 von Neumann 代数上的非线性中心化子 [J]. 华中师范大学学报(自然科学版), 2020, 54(3): 352-355. (YANG C, WU B, LIU Z. Nonlinear Centralizers on Factor von Neumann Algebras [J]. Journal of Central China Normal University (Natural Sciences), 2020, 54(3): 352-355.)

[5] LIU L. On Jordan Centralizers of Triangular Algebras [J]. Banach Journal of Mathematical Analysis, 2016, 10(2): 223-234.

[6] VUKMAN J. On (m,n)-Jordan Centralizers in Rings and Algebras [J]. Glasnik Matematički, 2010, 45(1): 43-53.

[7] ZHAO X P. Nonlinear Lie Triple Derivations by Local Actions on Triangular Algebras [J]. Journal of Algebra and Its Applications, 2023, 22(3): 2350059-1-2350059-15.

[8] QI X F, HOU J C. Characterizing Centralizers and Generalized Derivations on Triangular Algebras by Acting on Zero Product [J]. Acta Mathematica Sinica (English Series), 2013, 29(7): 1245-1256.

[9] ZHANG J H, WU B W, CAO H X. Lie Triple Derivations of Nest Algebras [J]. Linear Algebra and Its Applications, 2006, 416(2/3): 559-567.

[10] 孟利花, 张建华. 三角代数上的一类局部可导非线性映射 [J]. 吉林大学学报(理学版), 2017, 55(1): 43-47. (MENG L H, ZHANG J H. A Class of Local Derivable Nonlinear Maps on Triangular Algebras [J]. Journal of Jilin University (Science Edition), 2017, 55(1): 43-47.)

[11] LING Z C, LU F Y. Jordan Maps of Nest Algebras [J]. Linear Algebra and Its Applications, 2004, 387: 361-368.

(责任编辑: 赵立芹)