

热量方程对基底扰动的连续依赖性

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摘要: 利用先验估计和微分不等式技术, 讨论基底扰动对半无限圆柱体内热传导稳定性的影响, 证明随扰动幅度的减小, 解呈指数式收敛到与完全平坦基底对应的解. 该结果对热传导模型的稳健性以及微小几何不规则性对传热预测的潜在影响有一定的理论意义.

关键词: 先验界; 扰动参数; 结构稳定性; 连续依赖性

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Continuous Dependence of Heat Equation on Substrate Disturbance

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Abstract: By using prior estimates and differential inequality technique, the author discussed the influence of base perturbations on the stability of heat conduction inside a semi infinite cylinder. The author proves that as the disturbance amplitude decreases, the solution converges exponentially to the solution corresponding to a completely flat base. This result has certain theoretical significance for the robustness of heat conduction model and the potential impact of small geometric irregularities on heat transfer prediction.

Keywords: a priori bound; disturbance parameter; structural stability; continuous dependence

Horgan 等^[1]考虑了定义在三维半无限圆柱体上热量方程的渐近问题. 假设解在时间 $t=T$ 处的值与解在 $t=0$ 处的值成正比, 文献[1]给出了下列方程解 $u(t, \mathbf{x})$ 的空间衰减估计:

$$u_t = \Delta u + \gamma u, \quad \text{在 } [0, T] \times R \text{ 上,} \quad (1)$$

$$u = 0, \quad \text{在 } [0, T] \times \partial D \times \{x_3 > 0\} \text{ 上,} \quad (2)$$

$$u = 0, \quad \text{在 } \{t=0\} \times R \text{ 上,} \quad (3)$$

其中: γ 是一个大于零的常数; $R = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq 0\}$; D 是坐标平面 x_1Ox_2 上充分光滑的有界区域, ∂D 为 D 的光滑边界足以使用散度定理. 记 $D(z)$ 为 R 在距离原点 z 处的横截面, 即

$$D(z) = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = z \geq 0\}.$$

显然, $D=D(0)$. 当特征函数是完备的时, Horgan 等^[2]获得了方程(1)-(3)解的衰减率.

本文考虑方程(1)-(3)参数的扰动对解扰动的影响. 该研究的起因是在模型构建过程中, 方程的系数、边界参数和初始条件会不可避免地出现误差. 因此, 必须考虑这些误差是否会对方程的解产生重大干扰. 事实上, 该研究首先由 Ames 等^[3]提出, 并以结构稳定性命名. 起初的研究主要集中在有界区域上各种类型偏微分方程的结构稳定性^[4-12], 后续将该项研究推广到定义在半无穷柱体上偏微分方

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程的结构稳定性. 这种研究依赖于柱体入口处的初始数据和柱体侧面的条件, 目前已取得很多研究成果^[13-19].

但实际上, 柱体的基底也很容易受到扰动. 因此, 考虑基底的扰动是否会对方程的解产生重大扰动非常必要. Knops 等^[20]考虑了棱柱形半无限圆柱体, 并建立了均匀线弹性材料位移对基底几何数据的连续依赖关系. 文献^[21-22]分别研究了二元热量方程和伪抛物方程在棱柱形半无限圆柱体上的空间性质, 得到了解的指数式衰减界. 本文考虑棱柱形半无限圆柱体上的连续依赖性.

假设柱体 R 的基底受到微小扰动, 受扰动的基底记为 D_f , 即

$$D_f = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = f(x_1, x_2) \geq 0\},$$

其中 $f(x_1, x_2)$ 称为扰动函数, 满足

$$|f(x_1, x_2)| \leq \epsilon, \tag{4}$$

ϵ 称为扰动参数, 表示代表偏离 $D(0)$ 的最大扰动. 以 D_f 为基底的柱体记为

$$R(f) = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq f(x_1, x_2)\}.$$

假设 u^* 是方程(1)-(3)在 $R(f)$ 上的扰动解. 本文讨论基底扰动对半无限圆柱体内热传导稳定性和行为的影响. 本文用微分不等式技术建立热方程解对基底扰动的连续依赖性, 证明随扰动幅度的减小, 解呈指数收敛到与完全平坦基底对应的解. 该结果对热传导模型的稳健性以及微小几何不规则性对传热预测的潜在影响有一定理论意义.

1 先验估计

方程(1)-(3)未受扰动的解 u 在柱体基底上是给定的, 即

$$u(t, \mathbf{x}) = g_1(t, x_1, x_2), \quad 0 \leq t \leq T, \quad (x_1, x_2) \in D(0). \tag{5}$$

方程(1)-(3)受扰动的解在柱体基底上也是给定的, 即

$$u^*(t, \mathbf{x}) = g_2(t, x_1, x_2), \quad 0 \leq t \leq T, \quad x_3 = f(x_1, x_2), \quad (x_1, x_2) \in D(0). \tag{6}$$

由于基底受的扰动是微小的, 所以给定函数 g_1 和 g_2 的差随扰动参数趋近于零. 为得到解的先验界, 本文引入以下辅助函数:

$$\mathcal{G}_1(t, \mathbf{x}) = g_1(t, x_1, x_2) \exp\{-\alpha x_3\}, \quad \mathcal{G}_2(t, \mathbf{x}) = g_2(t, x_1, x_2) \exp\{-\sigma(x_3 - f)\}, \tag{7}$$

其中 $\sigma > 0$.

引理 1 若 $g_2 \in H^1([0, T] \times D(f))$, 则

$$\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla u^*(\tau)\|_{L^2(R(f))}^2 d\tau d\eta \leq d_1(t),$$

其中

$$d_1(t) = \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla \mathcal{G}_2(\tau)\|_{L^2(R(f))}^2 d\tau d\eta + \exp\{2\gamma t\} \int_0^t \exp\{-4\gamma\tau\} \|\mathcal{G}_2(\tau)\|_{L^2(R(f))}^2 d\tau + \frac{1}{2\gamma^2} [1 - \exp\{-2\gamma t\}]^2 \int_0^t \|\mathcal{G}_{2,\tau}(\tau)\|_{L^2(R(f))}^2 d\tau + \frac{1}{2} [1 - \exp\{-2\gamma t\}]^2 \int_0^t \|\mathcal{G}_2(\tau)\|_{L^2(R(f))}^2 d\tau.$$

证明: 由方程组(1)-(3), 有 $\int_0^t \int_{R(f)} (u_\tau^* - \Delta u^* - \gamma u^*) u^* d\mathbf{x} d\tau = 0$. 利用散度定理, 可得

$$\frac{1}{2} \|u^*(t)\|_{L^2(R(f))}^2 + \int_0^t \|\nabla u^*(\tau)\|_{L^2(R(f))}^2 d\tau - \gamma \int_0^t \|u^*(\tau)\|_{L^2(R(f))}^2 d\tau = - \int_0^t \int_{D(f)} \frac{\partial u^*}{\partial x_3} u^* dA d\tau. \tag{8}$$

若令 $h(t) = \int_0^t \|u^*(\tau)\|_{L^2(R(f))}^2 d\tau$, 则式(8)可写为

$$h'(t) - 2\gamma h(t) = - \int_0^t \|\nabla u^*(\tau)\|_{L^2(R(f))}^2 d\tau - \int_0^t \int_{D(f)} \frac{\partial u^*}{\partial x_3} u^* dA d\tau. \tag{9}$$

利用 Grownall 不等式, 由式(9)可得

$$\begin{aligned} & \exp\{-2\gamma t\} \int_0^t \|u^*(\tau)\|_{L^2(R(f))}^2 d\tau + \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla u^*(\tau)\|_{L^2(R(f))}^2 d\tau d\eta = \\ & - \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{D(f)} \frac{\partial u^*}{\partial x_3} u^* dA d\tau d\eta. \end{aligned} \tag{10}$$

由式(9)可知, \mathcal{G}_2 和 g_2 具有相同的边界条件. 由方程(1)-(3)和式(6)有

$$\begin{aligned}
& - \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{D(\zeta)} \frac{\partial u^*}{\partial x_3} u^* \, dA d\tau d\eta = - \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{D(\zeta)} \frac{\partial u^*}{\partial x_3} \mathcal{G}_2 \, dA d\tau d\eta = \\
& \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} \nabla (\nabla u^* \mathcal{G}_2) \, dx d\tau d\eta = \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} \nabla u^* \cdot \nabla \mathcal{G}_2 \, dx d\tau d\eta + \\
& \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} (u_\tau - \gamma u) \mathcal{G}_2 \, dx d\tau d\eta = \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} \nabla u^* \cdot \nabla \mathcal{G}_2 \, dx d\tau d\eta + \\
& \int_0^t \exp\{-2\gamma\eta\} \int_{R(\zeta)} u^* \mathcal{G}_2 \, dx d\eta + \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} u^* \mathcal{G}_{2\tau} \, dx d\tau d\eta - \\
& \gamma \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \int_{R(\zeta)} u^* \mathcal{G}_2 \, dx d\tau d\eta \doteq \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4. \tag{11}
\end{aligned}$$

应用 Schwarz 不等式, 可得

$$\mathcal{F}_1 \leq \frac{1}{2} \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla u^*(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau d\eta + \frac{1}{2} \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla \mathcal{G}_2(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau d\eta, \tag{12}$$

$$\mathcal{F}_2 \leq \frac{1}{2} \exp\{-2\gamma t\} \int_0^t \|\nabla u^*(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau + \frac{1}{2} \exp\{2\gamma t\} \int_0^t \exp\{-4\gamma\tau\} \|\mathcal{G}_2(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau, \tag{13}$$

$$\begin{aligned}
\mathcal{F}_3 & \leq \int_0^t \exp\{-2\gamma\eta\} d\eta \int_0^\eta \int_{R(\zeta)} |u^* \mathcal{G}_{2,\tau}| \, dx d\tau \leq \\
& \frac{1}{2\gamma} (1 - \exp\{-2\gamma t\}) \left(\int_0^t \|\nabla u^*(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau \int_0^t \|\mathcal{G}_{2,\tau}(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau \right)^{1/2} \leq \\
& \frac{1}{4} \exp\{-2\gamma t\} \int_0^t \|\nabla u^*(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau + \frac{1}{4\gamma^2} (1 - \exp\{-2\gamma t\})^2 \int_0^t \|\mathcal{G}_{2,\tau}(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau, \tag{14}
\end{aligned}$$

$$\mathcal{F}_4 \leq \frac{1}{4} \exp\{-2\gamma t\} \int_0^t \|\nabla u^*(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau + \frac{1}{4} (1 - \exp\{-2\gamma t\})^2 \int_0^t \|\mathcal{G}_2(\tau)\|_{L^2(R(\zeta))}^2 \, d\tau. \tag{15}$$

将式(12)~(15)代入式(11)可得结论成立. 证毕.

注 1 本文不仅需要扰动解 u^* 的先验界, 而且需要 u 的先验界. 由于 u 是方程(1)-(3)和式(5)未受扰动的解, 因此只需在引理 1 中取 $f=0$, 令 g_1 代替 g_2 , 即可得 u 的先验界. 因此, 有如下引理:

引理 2 若 $g_1 \in H^1([0, T] \times D)$, 则

$$\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla u(\tau)\|_{L^2(R)}^2 \, d\tau d\eta \leq d_2(t),$$

其中

$$\begin{aligned}
d_2(t) & = \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\nabla \mathcal{G}_1(\tau)\|_{L^2(R)}^2 \, d\tau d\eta + \exp\{2\gamma t\} \int_0^t \exp\{-4\gamma\tau\} \|\mathcal{G}_1(\tau)\|_{L^2(R)}^2 \, d\tau + \\
& \frac{1}{2\gamma^2} (1 - \exp\{-2\gamma t\})^2 \int_0^t \|\mathcal{G}_{1\tau}(\tau)\|_{L^2(R)}^2 \, d\tau + \frac{1}{2} (1 - \exp\{-2\gamma t\})^2 \int_0^t \|\mathcal{G}_1(\tau)\|_{L^2(R)}^2 \, d\tau.
\end{aligned}$$

令 w 表示方程(1)-(3)受扰动的解 u^* 和未受扰动的解 u 之差, 即

$$w = u - u^*, \quad x_3 > \epsilon, \tag{16}$$

则 w 满足

$$w_t = \Delta w + \gamma w, \quad \text{在 } [0, T] \times R(\epsilon) \text{ 上}, \tag{17}$$

$$w = 0, \quad \text{在 } [0, T] \times \partial D \times \{x_3 > \epsilon\} \text{ 上}, \tag{18}$$

$$w = 0, \quad \text{在 } \{t=0\} \times R(\epsilon) \text{ 上}. \tag{19}$$

对于基底上的给定数据, 可做以下延拓:

$$u(t, \mathbf{x}) = g_1(t, x_1, x_2), \quad t \in [0, T], \quad -\epsilon \leq x_3 \leq 0, \quad (x_1, x_2) \in D, \tag{20}$$

$$u^*(t, \mathbf{x}) = g_2(t, x_1, x_2), \quad -\epsilon \leq x_3 \leq f(x_1, x_2), \quad t \in [0, T], \quad (x_1, x_2) \in D. \tag{21}$$

当 $-\epsilon \leq x_3 \leq \epsilon$ 时, 令

$$w(t, \mathbf{x}) = u(t, \mathbf{x}) - u^*(t, \mathbf{x}), \quad t \in [0, T], \quad (x_1, x_2) \in D. \tag{22}$$

特别地, 有

$$\omega(t, x_1, x_2, -\epsilon) = g_1(t, x_1, x_2) - g_2(t, x_1, x_2), \quad t \in [0, T], \quad (x_1, x_2) \in D. \quad (23)$$

由式(16)和式(22)以及三角不等式, 可得

$$\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau d\eta \leq \int_0^t \exp\{-2\gamma\eta\} \times \int_0^\eta \left(\left\| \frac{\partial u}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 + \left\| \frac{\partial u^*}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 \right) d\tau d\eta. \quad (24)$$

利用引理 1 和引理 2、式(20)和式(21), 由式(24)可得

$$\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau d\eta \leq \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left(\left\| \frac{\partial u}{\partial x_3}(\tau) \right\|_{L^2(R)}^2 + \left\| \frac{\partial u^*}{\partial x_3}(\tau) \right\|_{L^2(R(f))}^2 \right) d\tau d\eta \leq d_1(t) + d_2(t). \quad (25)$$

于是可得以下引理.

引理 3 若 $g_1 \in H^1([0, T] \times D)$, $g_2 \in H^1([0, T] \times D(f))$, 则

$$\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau d\eta \leq d_3(t),$$

其中 $d_3(t) = d_1(t) + d_2(t)$.

2 辅助函数及其性质

下面引入辅助函数:

$$\varphi_\tau + \Delta\varphi = -\omega, \quad 0 < \tau < t, \quad \text{在 } R(x_3) \text{ 上}, \quad (26)$$

$$\varphi(\tau, \mathbf{x}) = 0, \quad 0 < \tau < t, \quad \text{在 } \partial D \times \{x_3\} \text{ 上}, \quad (27)$$

$$\varphi(\tau, \mathbf{x}) = 0, \quad 0 < \tau < t, \quad \text{在 } D(0) \text{ 上}, \quad (28)$$

$$\varphi(t, \mathbf{x}) = 0, \quad \text{在 } R(x_3) \text{ 上}, \quad (29)$$

$$\varphi, \nabla\varphi \rightarrow 0 \text{ (一致地对 } x_1, x_2, \tau), \quad x_3 \rightarrow \infty, \quad (30)$$

其中 $x_3 > \epsilon$.

建立能量函数

$$V(t, x_3) = \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau, \quad x_3 \geq \epsilon. \quad (31)$$

于是, 可得 $-\frac{\partial}{\partial x_3} V(t, x_3) = \int_0^t \|\omega(\tau)\|_{L^2(D(x_3))}^2 d\tau$. 下面给出辅助函数 φ 的一些性质.

引理 4 若 $\varphi \in H^1([0, t] \times R(x_3))$, 则

$$\int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau \leq V(t, x_3), \quad x_3 \geq \epsilon.$$

证明: 由方程(26)可得恒等式 $\int_0^t \int_{R(x_3)} \varphi_\tau(\varphi_\tau + \Delta\varphi + \omega) dx d\tau = 0$. 利用散度定理、Schwarz 不等式和方程(27)~(30), 可得

$$\begin{aligned} \int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau &= -\frac{1}{2} \|\nabla\varphi(0)\|_{L^2(R(x_3))}^2 + \int_0^t \int_{R(x_3)} \omega\varphi_\tau dx d\tau \leq \\ &\left(\int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2} \leq \\ &\frac{1}{2} \int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau + \frac{1}{2} \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau, \end{aligned} \quad (32)$$

结合式(31)和式(32)可知结论成立.

引理 5 若 $\varphi \in H^1([0, t] \times R(x_3))$, 则

$$\int_0^t \|\nabla\varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau \leq \frac{1}{\lambda_1} V(t, x_3),$$

其中 λ_1 是固定模问题

$$\begin{aligned} \Delta_2 \phi - \lambda \phi &= 0, \quad \text{在 } D \text{ 上,} \\ \phi &= 0, \quad \text{在 } \partial D \text{ 上} \end{aligned}$$

的第一特征值, 且 Δ_2 是二维的 Laplace 算子.

证明: 由方程(26)可得恒等式 $\int_0^t \int_{R(x_3)} \varphi(\varphi_\tau + \Delta\varphi + w) dx d\tau = 0$. 利用散度定理、Schwarz 不等式和方程(27)~(30), 可得

$$\begin{aligned} \int_0^t \|\nabla \varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau &= -\frac{1}{2} \|\varphi(0)\|_{L^2(R(x_3))}^2 + \int_0^t \int_{R(x_3)} w\varphi dx d\tau \leq \\ &= \left(\int_0^t \|\varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2} \leq \\ &= \frac{1}{\sqrt{\lambda_1}} \left(\int_0^t \|\nabla_2 \varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2}. \end{aligned} \tag{33}$$

利用 Young 不等式, 由式(33)可得

$$\int_0^t \|\nabla \varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau \leq \frac{1}{2} \int_0^t \|\nabla_2 \varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau + \frac{1}{2\lambda_1} \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau. \tag{34}$$

从而结论成立.

引理 6 若 $\varphi \in H^1([0, t] \times R(x_3))$, 则

$$\int_0^t \left\| \frac{\partial \varphi}{\partial x_3}(\tau) \right\|_{L^2(D(x_3))}^2 d\tau \leq \frac{1}{\sqrt{\lambda_1}} (2 + \delta) V(t, x_3).$$

证明: 设 δ 是一个大于零的常数. 由方程(26)可得恒等式

$$\int_0^t \int_{R(x_3)} \left(\frac{\partial \varphi}{\partial x_3} - \delta \varphi_\tau \right) (\varphi_\tau + \Delta\varphi + w) dx d\tau = 0. \tag{35}$$

在式(35)中利用散度定理和方程(27)~(30), 可得

$$\begin{aligned} \frac{1}{2} \delta \|\nabla \varphi(0)\|_{L^2(R(x_3))}^2 d\tau + \delta \int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau + \frac{1}{2} \int_0^t \left\| \frac{\partial \varphi}{\partial x_3}(\tau) \right\|_{L^2(D(x_3))}^2 d\tau = \\ \int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} \varphi_\tau dx d\tau + \int_0^t \int_{R(x_3)} \left(\frac{\partial \varphi}{\partial x_3} - \delta \varphi_\tau \right) w dx d\tau. \end{aligned} \tag{36}$$

利用 Schwarz 不等式、引理 4 和引理 5, 可得

$$\int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} \varphi_\tau dx d\tau \leq \left(\int_0^t \left\| \frac{\partial \varphi}{\partial x_3}(\tau) \right\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}} V(t, x_3), \tag{37}$$

$$\int_0^t \int_{R(x_3)} \frac{\partial \varphi}{\partial x_3} w dx d\tau \leq \left(\int_0^t \left\| \frac{\partial \varphi}{\partial x_3}(\tau) \right\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}} V(t, x_3), \tag{38}$$

$$-\delta \int_0^t \int_{R(x_3)} \varphi_\tau w dx d\tau \leq \delta \left(\int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}} \delta V(t, x_3). \tag{39}$$

把式(37)~(39)代入式(36)并舍弃式(36)中左边的第一项和第二项即可得结论.

3 连续依赖性定理

引理 7^[23] 若 $\phi \in H^1$, $\phi(0) = 0$, 则

$$\int_0^1 \phi^2(l) dl \leq \frac{4}{\pi^2} \int_0^1 \phi'(l)^2 dl. \tag{40}$$

定理 1(连续依赖性定理) 若 $g_1 \in H^1([0, T] \times R)$, $g_2 \in H^1([0, T] \times R(f))$, 且 $t < \frac{\pi}{4a_1\gamma}$, 则

$$\begin{aligned} \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|\omega(\tau)\|_{L^2(R(x_3))}^2 d\tau d\eta \leq & \left\{ \frac{32}{a_1\pi} d_3(t)\epsilon + \left(\frac{a_1\pi}{2} + \frac{2}{a_1} \right) \times \right. \\ & \left. \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \|(g_1 - g_2)(\tau)\|_{L^2(D)}^2 d\tau d\eta \right\} \exp\{-a_1(x_3 - \epsilon)\}, \quad x_3 \geq \epsilon. \end{aligned}$$

证明: 令 $x_3 \geq \epsilon$ 是 x_3 坐标轴上的一个稳定点. 由方程(26)~(31)、方程(17)和散度定理, 可得

$$\begin{aligned}
V(x_3, t) &= - \int_0^t \int_{R(x_3)} w(\varphi_\tau + \Delta\varphi) dx d\tau = \\
&= - \int_0^t \int_{R(x_3)} \varphi_\tau w dx d\tau + \int_0^t \int_{R(x_3)} \nabla w \cdot \nabla \varphi dx d\tau + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dA d\tau = \\
&= - \int_0^t \int_{R(x_3)} \varphi_\tau w dx d\tau - \int_0^t \int_{R(x_3)} \Delta w \varphi dx d\tau + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dA d\tau = \\
&= - \int_0^t \int_{R(x_3)} \varphi_\tau w dx d\tau - \int_0^t \int_{R(x_3)} \varphi w_\tau dx d\tau + \int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dA d\tau + \\
&= \gamma \int_0^t \int_{R(x_3)} \varphi w dx d\tau.
\end{aligned} \tag{41}$$

由式(19)和式(29), 易得

$$\int_0^t \int_{R(x_3)} (\varphi_\tau w + \varphi w_\tau) dx d\tau = 0. \tag{42}$$

利用 Hölder 不等式和 Young 不等式、式(31)和引理 6, 可得

$$\begin{aligned}
\int_0^t \int_{D(x_3)} w \frac{\partial \varphi}{\partial x_3} dA d\tau &\leq \left(\int_0^t \left\| \frac{\partial \varphi}{\partial x_3}(\tau) \right\|_{L^2(D(x_3))}^2 d\tau \int_0^t \|w(\tau)\|_{L^2(D(x_3))}^2 d\tau \right)^{1/2} \leq \\
&\leq \sqrt{\frac{1}{\sqrt{\lambda_1}}(2+\delta)} \sqrt{V(t, x_3)} \left(- \frac{\partial}{\partial x_3} V(t, x_3) \right)^{1/2} \leq \\
&= \frac{1}{4} V(t, x_3) + \frac{1}{\sqrt{\lambda_1}}(2+\delta) \left(- \frac{\partial}{\partial x_3} V(t, x_3) \right),
\end{aligned} \tag{43}$$

$$\gamma \int_0^t \int_{R(x_3)} \varphi w dx d\tau \leq \gamma \left(\int_0^t \|\varphi(\tau)\|_{L^2(R(x_3))}^2 d\tau \int_0^t \|w(\tau)\|_{L^2(R(x_3))}^2 d\tau \right)^{1/2}. \tag{44}$$

利用引理 7 和引理 4, 由式(44)可得

$$\gamma \int_0^t \int_{R(x_3)} \varphi w dx d\tau \leq \gamma \frac{2t}{\pi} \left(\int_0^t \|\varphi_\tau(\tau)\|_{L^2(R(x_3))}^2 d\tau V(t, x_3) \right)^{1/2} \leq \gamma \frac{2t}{\pi} V(t, x_3). \tag{45}$$

因为 $T < \frac{\pi}{8\gamma}$, 所以结合式(41)~(43), (45), 可得

$$V(t, x_3) \leq - \frac{1}{a_1} \frac{\partial}{\partial x_3} V(t, x_3), \quad x_3 \geq \epsilon, \tag{46}$$

其中 $\frac{1}{a_1} = \frac{1}{\sqrt{\lambda_1}}(2+\delta)$.

对式(46)从 ϵ 到 x_3 积分, 可得

$$V(t, x_3) \leq V(t, \epsilon) \exp\{-a_1(x_3 - \epsilon)\}, \quad x_3 \geq \epsilon. \tag{47}$$

由式(47)可知, 当 $x_3 \rightarrow \infty$ 时, 方程组(17)~(19)的解呈指数式衰减. 因为 $V(t, \epsilon)$ 是否依赖于扰动参数 ϵ 还不明确, 所以这种衰减并不严格. 因此, 下面推导 $V(t, \epsilon)$ 的显式上界.

在式(46)中取 $x_3 = \epsilon$ 并结合式(31), 有

$$\begin{aligned}
V(\epsilon, t) &\leq \frac{2}{a_1} \int_0^t \|w(\tau)\|_{L^2(D(\epsilon))}^2 dA d\tau = \\
&= \frac{2}{a_1} \int_0^t \int_{-\cdot}^{\cdot} \int_{D(x_3)} w \frac{\partial w}{\partial x_3} dx d\tau + \frac{2}{a_1} \int_0^t \|(g_1 - g_2)(\tau)\|_{L^2(D)}^2 d\tau \leq \\
&= \frac{2}{a_1} \left(\int_0^t \|w(\tau)\|_{L^2(D(x_3) \times [-\cdot, \cdot])}^2 d\tau \int_0^t \left\| \frac{\partial w}{\partial x_3}(\tau) \right\|_{L^2(D(x_3) \times [-\cdot, \cdot])}^2 d\tau \right)^{1/2} + \\
&= \frac{2}{a_1} \int_0^t \|(g_1 - g_2)(\tau)\|_{L^2(D)}^2 d\tau.
\end{aligned} \tag{48}$$

利用变量替换、式(23)和引理 7, 有

$$\int_0^t \|w(\tau)\|_{L^2(D(x_3) \times [-\cdot, \cdot])}^2 d\tau = \int_0^t \int_{-\cdot}^{\cdot} \|w(\tau, x_3)\|_{L^2(D(x_3))}^2 dx_3 d\tau \stackrel{\text{令 } y = (x_3 + \epsilon)/(2\epsilon)}{=}$$

$$\begin{aligned}
& 2\epsilon \int_0^t \int_0^1 [\| \omega(\tau, 2\epsilon y - \epsilon) \|_{L^2(D(x_3))}^2 - \| \omega(\tau, -\epsilon) \|_{L^2(D(x_3))}^2] dy d\tau + \\
& 2\epsilon \int_0^t \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau \leq \\
& \frac{32\epsilon^3}{\pi^2} \int_0^t \int_0^1 \left\| \frac{\partial \omega}{\partial y}(\tau, 2\epsilon y - \epsilon) \right\|_{L^2(D(x_3))}^2 dy d\tau + 2\epsilon \int_0^t \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau = \\
& \frac{16\epsilon^2}{\pi^2} \int_0^t \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(D(x_3) \times [-\epsilon, \epsilon])}^2 d\tau + 2\epsilon \int_0^t \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau. \tag{49}
\end{aligned}$$

把式(49)代入式(48)并结合 Schwarz 不等式, 可得

$$\begin{aligned}
V(\epsilon, t) & \leq \frac{32}{a_1 \pi} \epsilon \int_0^t \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(D(x_3) \times [-\epsilon, \epsilon])}^2 d\tau + \left(\frac{a_1 \pi}{2} + \frac{2}{a_1} \right) \int_0^t \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau \leq \\
& \frac{32}{a_1 \pi} \epsilon \int_0^t \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau + \left(\frac{a_1 \pi}{2} + \frac{2}{a_1} \right) \int_0^t \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau. \tag{50}
\end{aligned}$$

下面构造一个新函数:

$$\tilde{V}(t, x_3) = \int_0^t \exp\{-2\gamma\eta\} V(\eta, x_3) d\eta. \tag{51}$$

于是式(50)可写为

$$\begin{aligned}
\tilde{V}(t, \epsilon) & \leq \frac{32}{a_1 \pi} \epsilon \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau d\eta + \\
& \left(\frac{a_1 \pi}{2} + \frac{2}{a_1} \right) \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau d\eta. \tag{52}
\end{aligned}$$

利用引理 3, 由式(52)可得

$$\tilde{V}(t, \epsilon) \leq \frac{32}{a_1 \pi} d_3(t) \epsilon + \left(\frac{a_1 \pi}{2} + \frac{2}{a_1} \right) \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \| (g_1 - g_2)(\tau) \|_{L^2(D)}^2 d\tau d\eta. \tag{53}$$

另一方面, 结合式(51), 由式(31)和式(47), 可得

$$\tilde{V}(t, x_3) = \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \| \omega(\tau) \|_{L^2(R(x_3))}^2 d\tau d\eta, \tag{54}$$

$$\tilde{V}(t, x_3) \leq \tilde{V}(t, \epsilon) \exp\{-a_1(x_3 - \epsilon)\}, \quad x_3 \geq \epsilon, \tag{55}$$

结合式(53)~(55)即可得结论成立.

注 2 定理 1 表明, $\tilde{V}(t, x_3)$ 连续依赖于扰动参数 ϵ 基底数据, 即当 $\epsilon \rightarrow 0$ 时, 有 $u(t, x_3) \rightarrow u^*(t, x_3)$. 若 $\epsilon=0$, 定理 1 即为 Saint-Venant 型衰减结果.

注 3 在 R 的任意界面上, 方程(1)-(3)的解仍然具有连续依赖性结果. 计算可得

$$\begin{aligned}
& \int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \| \omega(\tau) \|_{L^2(D(x_3))}^2 d\tau d\eta = -2 \int_0^t \int_{R(x_3)} \omega \frac{\partial \omega}{\partial x_3} dx d\tau \leq \\
& 2\sqrt{\tilde{V}(x_3)} \left(\int_0^t \exp\{-2\gamma\eta\} \int_0^\eta \left\| \frac{\partial \omega}{\partial x_3}(\tau) \right\|_{L^2(R(-))}^2 d\tau d\eta \right)^{1/2}. \tag{56}
\end{aligned}$$

利用引理 3、式(56)和定理 1 可得连续依赖性结果.

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