第 64 卷　第 2 期　　吉 林 大 学 学 报 (理 学 版)　　Vol. 64　No. 2
2026 年 3 月　　Journal of Jilin University (Science Edition)　　Mar　2026

doi：10.13413/j.cnki.jdxblxb.2025195

带有非局部 q-积分与广义反周期边界条件的分数阶 q-差分方程

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摘要：讨论一类带有非局部积分和广义反周期边界条件的非线性 Caputo 型分数阶 q-差分方程解的存在性和稳定性. 首先，通过 Banach 压缩映像原理给出该边值问题解的存在性和唯一性证明；其次，给出该问题的 Ulam 稳定性结果；最后，通过实例验证所得结果的有效性.

关键词：Caputo 型分数阶 q-差分方程；广义反周期边界条件；Hyers-Ulam 稳定性；Banach 压缩映像原理

中图分类号：O175.8　　**文献标志码**：A　　**文章编号**：1671-5489(2026)02-0236-07

Fractional q-Difference Equations with Nonlocal q-Integral and Generalized Anti-periodic Boundary Conditions

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Abstract: We discussed the existence and stability of solutions for a class of nonlinear Caputo-type fractional q-difference equations with nonlocal integrals and generalized anti-periodic boundary conditions. Firstly, by using the Banach contraction mapping principle, we gave proof of the existence and uniqueness of solutions to the boundary value problem. Secondly, we gave Ulam stability results for this problem. Finally, the validity of the obtained results was verified through an example.

Keywords: Caputo-type fractional q-difference equation; generalized anti-periodic boundary condition; Hyers-Ulam stability; Banach contraction mapping principle

q-微积分(量子微积分)在经济学、工程、物理学、计算机科学和空气动力学等领域应用广泛. 目前，关于非线性分数阶 q-差分方程各类边值问题的研究已取得了许多成果[1-7]. 例如：Ahmad 等[6]利用 Banach 不动点定理和 Leray-Schauder 非线性抉择证明了具有非局部边界条件 q-差分方程解的存在性；Liang 等[7]用偏序集不动点理论给出了在多点边界条件下 q-差分方程正解的存在性结果.

近年来，关于反周期边界条件下非线性分数阶 q-差分方程的研究也取得了一些进展[8-15]. 例如，文献[13]考虑如下反周期边值问题：

$$\begin{cases} {}^{\mathrm{C}}D_q^{\alpha}u(t)=f(t,u(t)), \quad t\in[0,1], \\ u(0)=-u(1), \quad {}^{\mathrm{C}}D_q^{\gamma}u(0)=-{}^{\mathrm{C}}D_q^{\gamma}u(1), \end{cases}$$

先通过线性边值问题的解确定了与该问题等价的积分方程，然后用基本的不动点定理得到了关于该问

收稿日期：2025-06-24.

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基金项目：国家自然科学基金(批准号：10971084).

题解的存在性和唯一性结果，其中 ${}^{C}D_q^\alpha$ 表示 α 阶 Caputo 分数阶 q-导数，$1<\alpha\leqslant 2$，$f\in C([0,1]\times[0,+\infty),\mathbb{R})$. 文献[14]考虑如下非局部积分边值问题：

$$\begin{cases} D_q^\alpha u(t)=f(t,u(t)), \quad t\in[0,1], \\ u(0)=0, \quad u(1)=\mu I_q^\sigma u(\eta), \end{cases}$$

给出了其对应线性边值问题解的表达式，研究了 Green 函数的性质，并借助广义的 Banach 压缩映像原理、Krasnoselskii's 不动点定理，再结合单调迭代方法，证明了该问题存在正解，最后给出了若干应用实例，其中 $q\in(0,1)$，D_q^α 表示 α 阶 Riemann-Liouville 分数阶 q-导数，$1<\alpha\leqslant 2$，$0<\beta\leqslant 2$，$0<\eta<1$，$\mu>0$，$f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$. 文献[15]考虑如下分数阶微分方程的混合边值问题：

$$\begin{cases} {}^{C}D_0^q\ u(t)=f(t,u(t)), \quad t\in[0,1], \quad 1<q<2, \\ u(1)=\mu\int_0^1 u(s)\mathrm{d}s, \quad u'(0)+u'(1)=0, \end{cases}$$

采用压缩映像原理、Krasnoselskii's 不动点定理和拓扑度方法，证明了该问题解的存在性和唯一性.

受上述研究工作的启发，本文考虑如下边界条件含有非局部积分和广义反周期条件的 Caputo 型分数阶非线性 q-差分方程：

$$\begin{cases} {}^{C}D_q^\alpha u(t)=f(t,u(t),I_q^\beta u(t)), \quad t\in[t_0,t_1], \\ u(t_1)=\mu I_q^\sigma u(\eta), \quad a\,{}^{C}D_q^\gamma u(t_0)+b\,{}^{C}D_q^\gamma u(t_1)=\lambda, \end{cases} \tag{1}$$

借助 Banach 压缩映像原理和 Lipschitz 条件，确定边值问题(1)解存在唯一的条件，并证明边值问题(1)具有 Hyers-Ulam 稳定性和 Hyers-Ulam-Rassias 稳定性，其中 ${}^{C}D_q^\alpha$ 表示 α 阶 Caputo 分数阶 q-导数，I_q^β 表示 β 阶 Riemann-Liouville 分数阶 q-积分，常数 $q\in(0,1)$，$1<\alpha\leqslant 2$，$0<\gamma\leqslant 1$，$\beta,\sigma>0$，$0\leqslant t_0<\eta<t_1$，$a,b>0$，$\lambda,\mu\in\mathbb{R}$，$f\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$. 本文假设 $\Delta=\Gamma_q(\sigma+1)-\mu\eta^\sigma\neq 0$.

1　预备知识

关于 Caputo 型分数阶 q-导数和 Riemann-Liouville 型分数阶 q-积分等基本概念与结果参见文献[9,12,14].

引理 1[12]　设 $\alpha>0$，则 $I_q^\alpha\,{}^{C}D_q^\alpha g(x)=g(x)-\sum_{k=0}^{m-1}c_k x^k$，其中 $m\geqslant\alpha$ 且 $m\in\mathbb{N}$.

应用引理 1 可得如下结果.

引理 2　设 $1<\alpha\leqslant 2$，$0<\gamma\leqslant 1$，函数 $h\in C([t_0,t_1],\mathbb{R})$，则线性边值问题

$$\begin{cases} {}^{C}D_q^\alpha u(t)=h(t), \quad t\in[t_0,t_1], \\ u(t_1)=\mu I_q^\sigma u(\eta), \quad a\,{}^{C}D_q^\gamma u(t_0)+b\,{}^{C}D_q^\gamma u(t_1)=\lambda \end{cases}$$

有唯一解

$$\begin{aligned} u(t)=&I_q^\alpha h(t)+\frac{\mu\Gamma_q(\sigma+1)}{\Delta}I_q^{\sigma+1}h(\eta)-\frac{\Gamma_q(\sigma+1)}{\Delta}I_q^\alpha h(t_1)+ \\ &\left[\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}I_q^{\alpha-\gamma}h(t_1)+ \\ &\left[t-\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)\right]\frac{\lambda\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}= \\ &\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^t(t-qs)^{(\alpha-1)}h(s)\mathrm{d}_qs+\frac{\mu\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha+\sigma)}\int_{t_0}^\eta(\eta-qs)^{(\alpha+\sigma-1)}h(s)\mathrm{d}_qs- \\ &\frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}h(s)\mathrm{d}_qs+ \\ &\left[\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\frac{1}{\Gamma_q(\alpha-\gamma)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}h(s)\mathrm{d}_qs+ \\ &\left[t-\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)\right]\frac{\lambda\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}. \end{aligned}$$

在数值分析、最优化理论、非线性分析等研究中，稳定性是非常重要的理论，包括指数稳定、Lyapunov稳定、渐近稳定等. 本文讨论Ulam稳定性.

定义1 若存在$C_f>0$，且对任意的$\varepsilon>0$及满足不等式

$$|{}^{C}D_q^{\alpha}v(t)-f(t,v(t),I_q^{\beta}v(t))|\leqslant\varepsilon,\qquad t\in[t_0,t_1] \tag{2}$$

的每个解$v\in C([t_0,t_1],\mathbb{R})$，都存在边值问题(1)的解$u\in C([t_0,t_1],\mathbb{R})$，使得

$$|u(t)-v(t)|\leqslant C_f\varepsilon,$$

则称边值问题(1)是Hyers-Ulam稳定的.

定义2 设$\psi\in C([t_0,t_1],\mathbb{R}^+)$，若存在$C_{f,\psi}>0$，且对任意的$\varepsilon>0$及满足不等式

$$|{}^{C}D_q^{\alpha}v(t)-f(t,v(t),I_q^{\beta}v(t))|\leqslant\varepsilon\psi(t),\qquad t\in[t_0,t_1] \tag{3}$$

的每个解$v\in C([t_0,t_1],\mathbb{R})$，都存在边值问题(1)的解$u\in C([t_0,t_1],\mathbb{R})$，使得

$$|u(t)-v(t)|\leqslant C_{f,\psi}\,\varepsilon\psi(t),$$

则称边值问题(1)关于ψ是Hyers-Ulam-Rassias稳定的.

由定义1和定义2可得如下两个引理.

引理3 函数$v\in C([t_0,t_1],\mathbb{R})$是不等式(2)的解当且仅当存在$\Psi\in C([t_0,t_1],\mathbb{R})$，满足：

1) $|\Psi(t)|\leqslant\varepsilon$, $t\in[t_0,t_1]$；

2) ${}^{C}D_q^{\alpha}v(t)=f(t,v(t),I_q^{\beta}v(t))+\Psi(t)$, $t\in[t_0,t_1]$.

引理4 函数$v\in C([t_0,t_1],\mathbb{R})$是不等式(3)的解当且仅当存在$\Psi\in C([t_0,t_1],\mathbb{R})$，满足：

1) $|\Psi(t)|\leqslant\varepsilon\psi(t)$, $t\in[t_0,t_1]$；

2) ${}^{C}D_q^{\alpha}v(t)=f(t,v(t),I_q^{\beta}v(t))+\Psi(t)$, $t\in[t_0,t_1]$.

本文假设条件如下：

(H_1) 存在$p,r\in C([t_0,t_1],\mathbb{R}^+)$，使得对任何$t\in[t_0,t_1]$及$x_1,y_1,x_2,y_2\in\mathbb{R}$，有

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqslant p(t)|x_1-x_2|+r(t)|y_1-y_2|; \tag{4}$$

(H_2) 存在$\psi\in C([t_0,t_1],\mathbb{R}^+)$和$L_\psi>0$，使得对任何$t\in[t_0,t_1]$，有$I_q^{\alpha}\psi(t)\leqslant L_\psi\psi(t)$.

2 存在性与唯一性

对于$U=\{u: u\in C([t_0,t_1],\mathbb{R})\}$，规定范数$\|u\|=\max\limits_{t\in[t_0,t_1]}|u(t)|$，则$(U,\|\cdot\|)$是一个Banach空间. 对任意的$u\in U$，$t\in[t_0,t_1]$，定义算子$\Phi$: $U\to U$为

$$\begin{aligned}\Phi u(t)=&\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t}(t-qs)^{(\alpha-1)}f(s,u(s),I_q^{\beta}u(s))\mathrm{d}_qs+\\&\frac{\mu\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha-\sigma-1)}f(s,u(s),I_q^{\beta}u(s))\mathrm{d}_qs-\\&\frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}f(s,u(s),I_q^{\beta}u(s))\mathrm{d}_qs+\\&\left[\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\times\\&\frac{1}{\Gamma_q(\alpha-\gamma)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}f(s,u(s),I_q^{\beta}u(s))\mathrm{d}_qs+\\&\left[t-\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)\right]\frac{\lambda\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}.\end{aligned}$$

因为引理2给出了线性边值问题解的形式，所以要证明非线性边值问题(1)解的存在性等价于证明积分方程$u(t)=\Phi u(t)$解的存在性. 因此，只需证明Φ存在不动点. 为讨论方便，记

$$\begin{aligned}\rho=&\frac{(|\Delta|+\Gamma_q(\sigma+1))(t_1^{\alpha}-t_0^{\alpha})}{|\Delta|\Gamma_q(\alpha+1)}+\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|}\frac{\eta^{\alpha+\sigma}-t_0^{\alpha+\sigma}}{\Gamma_q(\alpha+\sigma+1)}+\\&\left(t_1+\frac{\Gamma_q(\sigma+1)}{|\Delta|}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|\right)\frac{b\Gamma_q(2-\gamma)(t_1^{\alpha+\gamma}-t_0^{\alpha+\gamma})}{(at_0^{1-\gamma}+bt_1^{1-\gamma})\Gamma_q(\alpha+\gamma+1)},\end{aligned}$$

$$\delta=\left[t_1+\frac{\Gamma_q(\sigma+1)}{\Delta}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|\right]\frac{|\lambda|\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}.$$

定理 1　假设条件(H_1)成立，若 $\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)<1$，则边值问题(1)存在唯一解.

证明：设 $U_R=\{u: u\in C([t_0,t_1],\mathbb{R}),\ \|u\|\leqslant R\}$，其中 $R\geqslant\frac{M\rho+\delta}{1-\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)}$，$M=\max\limits_{t\in[t_0,t_1]}|f(t,0,0)|$.

首先，证明 Φ 是 U_R 到自身的映射. 对任意的 $t\in[t_0,t_1]$，$u\in U_R$，有

$$\begin{aligned}|\Phi u(t)|\leqslant&\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t}(t-qs)^{(\alpha-1)}|f(s,u(s),I_q^\beta u(s))|\mathrm{d}_qs+\\&\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}|f(s,u(s),I_q^\beta u(s))|\mathrm{d}_qs+\\&\frac{\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}|f(s,u(s),I_q^\beta u(s))|\mathrm{d}_qs+\\&\left[\frac{\Gamma_q(\sigma+1)}{|\Delta|}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|+t\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\times\\&\frac{1}{\Gamma_q(\alpha-\gamma)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),I_q^\beta u(s))|\mathrm{d}_qs+\\&\left[t+\frac{\Gamma_q(\sigma+1)}{\Delta}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|\right]\frac{|\lambda|\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\leqslant\\&\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t-qs)^{(\alpha-1)}[p(s)|u(s)|+r(s)|I_q^\beta u(s)|+M]\mathrm{d}_qs+\\&\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}[p(s)|u(s)|+r(s)|I_q^\beta u(s)|+M]\mathrm{d}_qs+\\&\frac{\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}[p(s)|u(s)|+r(s)|I_q^\beta u(s)|+M]\mathrm{d}_qs+\\&\left[\frac{\Gamma_q(\sigma+1)}{|\Delta|}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|+t_1\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\times\\&\frac{1}{\Gamma_q(\alpha-\gamma)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}[p(s)|u(s)|+r(s)|I_q^\beta u(s)|+M]\mathrm{d}_qs+\\&\left[t_1+\frac{\Gamma_q(\sigma+1)}{|\Delta|}\left|t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right|\right]\frac{|\lambda|\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\leqslant\\&\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)R+M\rho+\delta\leqslant R,\end{aligned}$$

故 $\|\Phi u\|\leqslant R$，即 $\Phi: U_R\to U_R$.

其次，证明 Φ 是压缩映射. 对任意的 $t\in[t_0,t_1]$，$u,v\in U_R$，有

$$\begin{aligned}|\Phi u(t)-\Phi v(t)|\leqslant&\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t}(t-qs)^{(\alpha-1)}|f(s,u(s),I_q^\beta u(s))-f(s,v(s),I_q^\beta v(s))|\mathrm{d}_qs+\\&\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}|f(s,u(s),I_q^\beta u(s))-f(s,v(s),I_q^\beta v(s))|\mathrm{d}_qs+\\&\frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}|f(s,u(s),I_q^\beta u(s))-f(s,v(s),I_q^\beta v(s))|\mathrm{d}_qs+\\&\left|\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right|\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\frac{1}{\Gamma_q(\alpha-\gamma)}\times\\&\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),I_q^\beta u(s))-f(s,v(s),I_q^\beta v(s))|\mathrm{d}_qs\leqslant\end{aligned}$$

$$\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}[p(s)|u(s)-v(s)|+r(s)|I_q^\sigma u(s)-I_q^\sigma v(s)|]\mathrm{d}_qs+$$
$$\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}[p(s)|u(s)-v(s)|+r(s)|I_q^\sigma u(s)-I_q^\sigma v(s)|]\mathrm{d}_qs+$$
$$\frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}[p(s)|u(s)-v(s)|+r(s)|I_q^\sigma u(s)-I_q^\sigma v(s)|]\mathrm{d}_qs+$$
$$\left|\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right|\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\frac{1}{\Gamma_q(\alpha-\gamma)}\times$$
$$\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}[p(s)|u(s)-v(s)|+r(s)|I_q^\sigma u(s)-I_q^\sigma v(s)|]\mathrm{d}_qs\leqslant$$
$$\left[\left(1+\frac{\Gamma_q(\sigma+1)}{|\Delta|}\right)\frac{t_1^\alpha-t_0^\alpha}{\Gamma_q(\alpha+1)}+\frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|}\frac{\eta^{\alpha+\sigma}-t_0^{\alpha-\sigma}}{\Gamma_q(\alpha+\sigma+1)}+\right.$$
$$\left.\left(t_1+\frac{\Gamma_q(\sigma+1)}{|\Delta|}\left|t_1-\frac{\mu\eta^{\sigma-1}}{\Gamma_q(\sigma+2)}\right|\right)\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\frac{t_1^{\alpha-\gamma}-t_0^{\alpha-\gamma}}{\Gamma_q(\alpha-\gamma+1)}\right]\times$$
$$\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)\|u-v\|=$$
$$\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)\|u-v\|,$$

因为 $\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)<1$，所以 Φ 是压缩映射. 根据 Banach 压缩映像原理，系统(1)存在唯一解.

3 Ulam 稳定性

先给出并证明边值问题(1)的 Hyers-Ulam 稳定性结果.

定理2 假设条件(H_1)成立，若 $\rho\left(\|p\|+\frac{\|r\|(t_1^\beta-t_0^\beta)}{\Gamma_q(\beta+1)}\right)<1$，则边值问题(1)是 Hyers-Ulam 稳定的.

证明：设 $v\in C([t_0,t_1],\mathbb{R})$ 是不等式(2)的任意解. 根据引理3，存在 $\Psi\in C([t_0,t_1],\mathbb{R})$，满足下列条件：

1) $|\Psi(t)|\leqslant\varepsilon$, $t\in[t_0,t_1]$;

2) ${}^{\mathrm{C}}D_q^\alpha v(t)=f(t,v(t),I_q^\beta v(t))+\Psi(t)$, $t\in[t_0,t_1]$.

考虑边值问题

$$\begin{cases}{}^{\mathrm{C}}D_q^\alpha v(t)=f(t,v(t),I_q^\beta v(t))+\Psi(t), & t\in[t_0,t_1],\\ v(t_1)=\mu I_q^\sigma v(\eta), & a^{\mathrm{C}}D_q^\gamma v(t_0)+b^{\mathrm{C}}D_q^\gamma v(t_1)=\lambda.\end{cases}\tag{5}$$

根据引理2，边值问题(5)的解满足

$$v(t)=\frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t}(t-qs)^{(\alpha-1)}[f(s,v(s),I_q^\beta v(s))+\Psi(s)]\mathrm{d}_qs+$$
$$\frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}[f(s,v(s),I_q^\beta v(s))+\Psi(s)]\mathrm{d}_qs+$$
$$\frac{\mu\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}[f(s,v(s),I_q^\beta v(s))+\Psi(s)]\mathrm{d}_qs-$$
$$\left[\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right]\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\times$$
$$\frac{1}{\Gamma_q(\alpha-\gamma)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}[f(s,v(s),I_q^\beta v(s))+\Psi(s)]\mathrm{d}_qs+$$
$$\left[t-\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)\right]\frac{\lambda\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}.$$

根据定理 1，设 $u\in C([t_0,t_1],\mathbb{R})$是边值问题(1)的解，则对任何 $t\in[t_0,t_1]$，有

$$
\begin{aligned}
|u(t)-v(t)| \leqslant & \frac{1}{\Gamma_q(\alpha)}\int_{t_0}^{t}(t-qs)^{(\alpha-1)}[|f(s,u(s),I_q^{\beta}u(s))-f(s,v(s),I_q^{\beta}v(s))|+|\Psi(s)|]\mathrm{d}_qs+ \\
& \frac{|\mu|\Gamma_q(\sigma+1)}{|\Delta|\Gamma_q(\alpha+\sigma)}\int_{t_0}^{\eta}(\eta-qs)^{(\alpha+\sigma-1)}[|f(s,u(s),I_q^{\beta}u(s))-f(s,v(s),I_q^{\beta}v(s))|+|\Psi(s)|]\mathrm{d}_qs+ \\
& \frac{\Gamma_q(\sigma+1)}{\Delta\Gamma_q(\alpha)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha-1)}[|f(s,u(s),I_q^{\beta}u(s))-f(s,v(s),I_q^{\beta}v(s))|+|\Psi(s)|]\mathrm{d}_qs+ \\
& \left|\frac{\Gamma_q(\sigma+1)}{\Delta}\left(t_1-\frac{\mu\eta^{\sigma+1}}{\Gamma_q(\sigma+2)}\right)-t\right|\frac{b\Gamma_q(2-\gamma)}{at_0^{1-\gamma}+bt_1^{1-\gamma}}\frac{1}{\Gamma_q(\alpha-\gamma)}\times \\
& \int_{t_0}^{t_1}(t_1-qs)^{(\alpha-\gamma-1)}[|f(s,u(s),I_q^{\beta}u(s))-f(s,v(s),I_q^{\beta}v(s))|+|\Psi(s)|]\mathrm{d}_qs\leqslant \\
& \rho\left(\|p\|+\frac{\|r\|(t_1^{\beta}-t_0^{\beta})}{\Gamma_q(\beta+1)}\right)\|u-v\|+\rho\varepsilon,
\end{aligned}
$$

故

$$
\|u-v\|\leqslant\frac{\rho\varepsilon}{1-\rho\left(\|p\|+\dfrac{\|r\|(t_1^{\beta}-t_0^{\beta})}{\Gamma_q(\beta+1)}\right)},
$$

即 $\|u-v\|\leqslant C_f\varepsilon$，其中

$$
C_f=\frac{\rho}{1-\rho\left(\|p\|+\dfrac{\|r\|(t_1^{\beta}-t_0^{\beta})}{\Gamma_q(\beta+1)}\right)}.
$$

因此边值问题(1)是 Hyers-Ulam 稳定的.

同理，应用定理 1、引理 2 和引理 4，可证明如下边值问题(1)的 Hyers-Ulam-Rassias 稳定性结果.

定理 3　假设条件(H_1)和(H_2)成立，若 $\rho\left(\|p\|+\dfrac{\|r\|(t_1^{\beta}-t_0^{\beta})}{\Gamma_q(\beta+1)}\right)<1$，则边值问题(1)关于 ψ 是 Hyers-Ulam-Rassias 稳定的.

4　应用实例

考虑下列边值问题解的存在性及 Ulam 稳定性：

$$
\begin{cases}
{}^{C}D_{0.5}^{1.5}u(t)=\dfrac{e^{t}u(t)}{100(1+(u(t))^2)}+\dfrac{t^2+e^{t}}{120}\sin I_{0.5}^{1.5}u(t), \quad 0\leqslant t\leqslant 1, \\
u(1)+I_{0.5}u(0.5)=0, \quad ({}^{C}D_{0.5}^{0.25}u)(0)+\sqrt{2}({}^{C}D_{0.5}^{0.25}u)(1)=5,
\end{cases}
\tag{6}
$$

其中 $q=0.5$，$\alpha=1.5$，$\beta=1.5$，$\eta=0.5$，$\mu=-1$，$\sigma=1$，$\gamma=0.25$，$a=1$，$b=\sqrt{2}$，$\lambda=5$，$t_0=0$，$t_1=1$.

设

$$
f(t,x,y)=\frac{e^{t}x}{100(1+x^2)}+\frac{t^2+e^{t}}{120}\sin y, \quad p(t)=\frac{e^{t}}{100}, \quad r(t)=\frac{t^2+e^{t}}{120},
$$

则条件(H_1)成立，且 $\|p\|=\dfrac{e}{100}$，$\|r\|=\dfrac{1+e}{120}$. 因为 $\Delta=1.5$，故

$$
\rho=\left(1+\frac{\Gamma_{0.5}(2)}{1.5}\right)\frac{1}{\Gamma_{0.5}(1.5)}+\frac{0.5^{2.5}\Gamma_{0.5}(2)}{1.5\Gamma_{0.5}(3.5)}+\left(1+\frac{\Gamma_{0.5}(2)}{1.75}\left(1-\frac{0.5^3}{\Gamma_{0.5}(3)}\right)\right)\frac{\Gamma_{0.5}(1.75)}{\Gamma_{0.5}(2.25)},
$$

从而 $\rho\left(\|p\|+\dfrac{\|r\|(t_1^{\beta}-t_0^{\beta})}{\Gamma_q(\beta+1)}\right)<1$. 根据定理 1 和定理 2 知，边值问题(6)存在唯一解，并且是 Hyers-Ulam 稳定的.

设 $\psi(t)=t^2+1$，则对任何 $t\in[0,1]$，$I_q^{\alpha}\psi(t)\leqslant L_{\psi}\psi(t)$，其中 $L_{\psi}=\dfrac{2}{\Gamma_{0.5}(1.5)}$. 故条件$(H_2)$成立，由定理 3 知，边值问题(6)关于 $\psi(t)=t^2+1$ 是 Hyers-Ulam-Rassias 稳定的.

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