Vol. 42 No. 5　　安徽工业大学学报(自然科学版)　　第42卷 第5期
September 2025　　Journal of Anhui University of Technology (Natural Science)　　2025年 9月

文章编号: 1671-7872(2025)05-0528-06

一类不连续 Sturm−Liouville 问题的特征值与特征函数零点估计

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摘要: 针对具有周期边界条件的不连续 Sturm−Liouville 问题 (SLPs), 本文首先基于常微分方程初值理论推导出不连续 SLPs 两个线性无关解的渐进估计; 继而运用 Gronwall 不等式、特征值性质及解的渐进估计式, 建立不连续 SLPs 特征值的渐进估计形式; 最后通过 Prufer 变换证明不连续 SLPs 问题第n个特征值对应的特征函数在$(0,c)\cup(c,\pi)$内有n个零点。本研究为不连续 SLPs 特征值下标的精确计算及解的振荡性分析提供了重要理论依据。

关键词: Sturm−Liouville 问题; 转移条件; 特征值; 周期边界条件; Prufer 变换; Gronwall 不等式

中图分类号: O 175.9　　**文献标志码:** A　　**doi:** 10.12415/j.issn.1671−7872.24006

Esimation of Eigenvalues and Zeros of Eigenfunction for a Class of Discontinuous Sturm−Liouville Problems

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Abstract: For discontinuous Sturm–Liouville problems (SLPs) with periodic boundary conditions, the asymptotic estimates of two linearly independent solutions were first derived based on the initial value theory of ordinary differential equations. Subsequently, the asymptotic form of eigenvalues for discontinuous SLPs was established by employing Gronwall's inequality, eigenvalue properties, and the asymptotic solution estimates. Finally, it was proved through the Prufer transformation that the eigenfunction corresponding to the n-th eigenvalue possessed exactly n zeros within the interval $(0,c)\cup(c,\pi)$.This study provides important theoretical foundations for the precise calculation of eigenvalue indices and the oscillatory analysis of solutions for discontinuous SLPs.

Keywords: Sturm−Liouville problems; transmission conditions; eigenvalue; periodic boundary condition; Prufer transform; gronwall inequality

文中研究由二阶微分方程 (1) 和周期边界条件 (2), (3) 及转移条件 (4), (5) 构成的不连续 Sturm−Liouville 问题 (Sturm−Liouville problems, SLPs)。

$$-y''+q(x)y=\lambda y, x\in I=(0,c)\cup(c,\pi) \tag{1}$$

$$y(0)=y(\pi) \tag{2}$$

$$y'(0)=y'(\pi) \tag{3}$$

$$y(c+0)=ay(c-0) \tag{4}$$

收稿日期: 2024-01-05

基金项目: 安徽省高校自然科学基金重点项目 (2022AH050290; 2022AH050329)

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引文格式: 张艳霞, 张志强, 刘延年, 等. 一类不连续 Sturm−Liouville 问题的特征值与特征函数零点估计 [J]. 安徽工业大学学报 (自然科学版), 2025, 42(5):528-533.

$$y'(c+0)=\frac{1}{a}y'(c-0) \tag{5}$$

其中$q(x)$为$I=(0,c)\cup(c,\pi)$上的连续函数；$\lambda\in C$，为谱参数。

许多重要的物理问题, 如含质量点的弦振动问题、波衍射现象 、水下声波遇到障碍物的反射传播问题, 以及多层复合材料薄板的热传导问题等, 均可建模为具有转移条件的 SLPs, 即不连续 SLPs。当前不连续 SLPs 研究的主要方向包括: 特征值的分布规律和计算方法[1−3]、特征函数的振荡特性与完备性分析[4−6]、逆谱问题研究[7−11] 及特征值的高效数值计算技术[12−13] 等关键领域。现有文献中关于不连续 SLPs 的研究主要集中于两类边界条件: 分离型边界条件[1,2,4,7−11] 或含谱参数的分离型边界条件[5,14]。已有研究[3] 表明, 对于分离型边界条件的不连续 SLPs, 其谱结构仅包含可数个实数单重特征值, 这些特征值有下界而无上界。尽管在周期边界下, 无论是定型还是不定型 SLPs 的特征值性质及上界估计等问题已有较多研究[14−16], 然而对于混合型边界条件下不连续 SLPs 的研究却较少, 且缺乏系统性结论。鉴于此, 本文将不连续 SLPs 的研究拓展至周期混合型边界条件, 探讨其特征值的渐近估计及相应特征函数零点分布规律。

1　预备知识

设在$(0,c)$上, $\varphi_1(x,\lambda)$是式 (1) 的满足初始条件$\varphi_1(0,\lambda)=1,\varphi_1'(0,\lambda)=0$的解。在$(c,\pi)$上, $\varphi_2(x,\lambda)$是式 (1) 的满足初始条件$\varphi_2(c+0,\lambda)=a\varphi_1(c-0,\lambda)$, $\varphi_2'(c+0,\lambda)=\frac{1}{a}\varphi_1'(c-0,\lambda)$的解。令

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda),x\in(0,c)\\ \varphi_2(x,\lambda),x\in(c,\pi)\end{cases} \tag{6}$$

则$\varphi(x,\lambda)$是式 (1) 的满足转移条件 (4) 和 (5) 的解。

设在$(0,c)$上, $\psi_1(x,\lambda)$是式 (1) 的满足初始条件$\psi_1(0,\lambda)=0,\psi_1'(0,\lambda)=1$的解；在$(c,\pi)$上, $\psi_2(x,\lambda)$是式 (1) 的满足初始条件$\psi_2(c+0,\lambda)=a\psi_1(c-0,\lambda)$, $\psi_2'(c+0,\lambda)=\frac{1}{a}\psi_1'(c-0,\lambda)$的解。令

$$\psi(x,\lambda)=\begin{cases}\psi_1(x,\lambda),x\in(0,c)\\ \psi_2(x,\lambda),x\in(c,\pi)\end{cases} \tag{7}$$

则$\psi(x,\lambda)$是式 (1) 的满足转移条件 (4) 和 (5) 的解。$\varphi(x,\lambda)$与$\psi(x,\lambda)$在I上是线性无关的, 且关于λ是整函数。式 (1) 的任一解可表示为:

$$y(x,\lambda)=c_1\varphi(x,\lambda)+c_2\psi(x,\lambda)$$

引理 1　定义$\varphi(x,\lambda),\psi(x,\lambda)$分别如式 (6), (7), 令

$$D(\lambda)=\varphi_2(\pi,\lambda)\psi_2'(\pi,\lambda)-\varphi_2(\pi,\lambda)-\psi_2'(\pi,\lambda)+1-\varphi_2'(\pi,\lambda)\psi_2(\pi,\lambda) \tag{8}$$

则λ是不连续 SLPs(1)~(5) 的特征值的充分必要条件是$D(\lambda)=0$。

引理 2　不连续 SLPs(1)~(5) 具有可数个离散的特征值, 这些特征值均为实数且在无穷远处无有限聚点。

引理 1, 2 的证明见文献 [17]。

2　不连续 Sturm−Liouville 问题特征值的估计

引理 3　设在$(0,c)$上, $\varphi_1(x,\lambda)$, $\psi_1(x,\lambda)$是方程 (1) 的满足初始条件$\varphi_1(0,\lambda)=1,\varphi_1'(0,\lambda)=0$, $\psi_1(0,\lambda)=0$, $\psi_1'(0,\lambda)=1$的解。令$\lambda=s^2$, $s=\sigma+i\tau$, 则当$|\lambda|\to\infty$时, $\varphi_1(x,\lambda)$, $\psi_1(x,\lambda)$有如下估计式:

$$\varphi_1(x,\lambda)=\cos(sx)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right) \tag{9}$$

$$\varphi_1'(x,\lambda)=-s\sin(sx)+O\left(\mathrm{e}^{|\tau|x}\right) \tag{10}$$

$$\psi_1(x,\lambda)=\frac{\sin(sx)}{s}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|^2}\right) \tag{11}$$

$$\psi_1'(x,\lambda)=\cos(sx)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right) \tag{12}$$

引理 3 的证明见文献 [18−20]。

定理 1　设在(c,π)上, $\varphi_2(x,\lambda)$, $\psi_2(x,\lambda)$分别是式 (1) 的满足初始条件 (13)~(16) 的解。

$$\varphi_2(c+0,\lambda)=a\varphi_1(c-0,\lambda) \tag{13}$$

$$\varphi_2'(c+0,\lambda)=\frac{1}{a}\varphi_1'(c-0,\lambda) \tag{14}$$

$$\psi_2(c+0,\lambda)=a\psi_1(c-0,\lambda) \tag{15}$$

$$\psi_2'(c+0,\lambda)=\frac{1}{a}\psi_1'(c-0,\lambda) \tag{16}$$

令$\lambda=s^2$, $s=\sigma+i\tau$, 则当$|\lambda|\to\infty$时, $\varphi_2(x,\lambda)$, $\psi_2(x,\lambda)$有如下估计式:

$$\varphi_2(x,\lambda)=a\cos(sc)\cos s(x-c)-\frac{1}{a}\sin(sc)\sin s(x-c)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right) \tag{17}$$

$$\varphi_2'(x,\lambda)=-\left[as\cos(sc)\sin s(x-c)+\frac{s}{a}\sin(sc)\cos s(x-c)\right]+O\left(\mathrm{e}^{|\tau|x}\right) \tag{18}$$

$$\psi_2(x,\lambda)=\frac{a}{s}\sin(sc)\cos s(x-c)+\frac{1}{as}\cos(sc)\sin s(x-c)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|^2}\right) \tag{19}$$

$$\psi_2'(x,\lambda)=-a\sin(sc)\sin s(x-c)+\frac{1}{a}\cos(sc)\cos s(x-c)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right) \tag{20}$$

证明 在(c,π)上, 由于$\varphi_2(x,\lambda)$是方程 (1) 的解, 则$\varphi_2(x,\lambda)$满足如下积分方程:

$$\varphi_2(x,\lambda)=A\cos(sx)+\frac{B}{s}\sin(sx)+\int_c^x\frac{\sin s(x-t)}{s}q(t)\varphi_2(t,\lambda)\,\mathrm{d}t \tag{21}$$

$$\varphi_2'(x,\lambda)=-As\sin(sx)+B\cos(sx)+\int_c^x\cos s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t \tag{22}$$

又$\varphi_2(x,\lambda)$满足初始条件 (13), 即

$$\varphi_2(c+0,\lambda)=A\cos(sc)+B\frac{\sin(sc)}{s}=a\varphi_1(c-0,\lambda)$$

$$\varphi_2'(c+0,\lambda)=-As\sin(sc)+B\cos(sc)=\frac{1}{a}\varphi_1'(c-0,\lambda)$$

所以

$$\varphi_2(x,\lambda)=a\varphi_1(c-0,\lambda)\cos s(x-c)+\frac{1}{as}\sin s(x-c)\varphi_1'(c-0,\lambda)+\frac{1}{s}\int_c^x\sin s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t \tag{23}$$

由式 (9), (10) 可得:

$$\varphi_1(c-0,\lambda)=\cos(sc)+O\left(\frac{\mathrm{e}^{|\tau|c}}{|s|}\right)$$

$$\varphi_1'(c-0,\lambda)=-s\sin(sc)+O\left(\mathrm{e}^{|\tau|c}\right)$$

则

$$\begin{aligned}\varphi_2(x,\lambda)=&a\left[\cos(sc)+O\left(\frac{\mathrm{e}^{|\tau|c}}{|s|}\right)\right]\cos s(x-c)+\\&\frac{1}{as}\sin s(x-c)\left[-s\sin(sc)+O\left(\mathrm{e}^{|\tau|c}\right)\right]+\\&\frac{1}{s}\int_c^x\sin s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t=\\&a\cos(sc)\cos s(x-c)-\frac{1}{a}\sin(sc)\sin s(x-c)+\\&\frac{1}{s}\int_c^x\sin s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right)\end{aligned} \tag{24}$$

令$F(x,\lambda)=\mathrm{e}^{-|\tau|(x-c)}\varphi_2(x,\lambda)$, 由于$|\mathrm{e}^{-|\tau|(x-c)}\sin s(x-c)|\le 1$, $|\mathrm{e}^{-|\tau|(x-c)}\cos s(x-c)|\le 1$, 可得:

$$|F(x,\lambda)|\leqslant a+\frac{1}{a}+\frac{1}{|s|}\int_c^\pi|q(t)||F(x,\lambda)|\,\mathrm{d}t+\left|O\left(\frac{1}{|s|}\right)\right| \tag{25}$$

由 Gronwall 不等式得:

$$|F(x,\lambda)|\leqslant\left(a+\frac{1}{a}+\left|O\left(\frac{1}{|s|}\right)\right|\right)\mathrm{e}^{\frac{1}{|s|}\int_c^\pi|q(t)|\mathrm{d}t} \tag{26}$$

当$|\lambda|\to\infty$时

$$|F(x,\lambda)|\leqslant M$$

其中M为正常数。因此

$$\varphi_2(x,\lambda)=\mathrm{e}^{|\tau|(x-c)}F(x,\lambda)=O\left(\mathrm{e}^{|\tau|(x-c)}\right)=O\left(\mathrm{e}^{|\tau|x}\right) \tag{27}$$

将式 (27) 代入式 (24), 又由于

$$\frac{\left|\frac{1}{s}\int_c^x\sin s(x-t)q(t)O(\mathrm{e}^{|\tau|t})\,\mathrm{d}t\right|}{\frac{\mathrm{e}^{|\tau|x}}{|s|}}\leqslant\frac{1}{\mathrm{e}^{|\tau|x}}\int_c^\pi|\sin s(x-t)||q(t)|\left|O\left(\mathrm{e}^{|\tau|t}\right)\right|\mathrm{d}t\leqslant\frac{A}{\mathrm{e}^{|\tau|x}}\int_c^\pi\mathrm{e}^{|\tau|(x-t)}\mathrm{e}^{|\tau|t}\mathrm{d}t=A(\pi-c)$$

因此可得式 (17) 成立。将式 (23) 两端对x求导可得:

$$\varphi_2'(x,\lambda)=-as\varphi_1(c-0,\lambda)\sin s(x-c)+\frac{1}{a}\cos s(x-c)\varphi_1'(c-0,\lambda)+\int_c^x\cos s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t \tag{28}$$

由$\varphi_1'(c-0,\lambda)=-s\sin(sc)+O(\mathrm{e}^{|\tau|c})$可得

$$\varphi_2'(x,\lambda)=-as\cos(sc)\sin s(x-c)-\frac{s}{a}\sin(sc)\cos s(x-c)+\int_c^x\cos s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t+O\left(\mathrm{e}^{|\tau|x}\right) \tag{29}$$

将式 (27) 代入式 (29), 又由于

$$\frac{\int_c^x\cos s(x-t)q(t)O(\mathrm{e}^{|\tau|t})\,\mathrm{d}t}{\mathrm{e}^{|\tau|x}}\leqslant\frac{1}{\mathrm{e}^{|\tau|x}}\int_c^\pi\mathrm{e}^{|\tau|(x-t)}A\mathrm{e}^{|\tau|t}\mathrm{d}t=A(\pi-c)$$

所以$\int_c^x\cos s(x-t)q(t)\varphi_2(t,\lambda)\,\mathrm{d}t=O(\mathrm{e}^{|\tau|x})$。因此式 (18) 成立。

在(c,π)上, $\psi_2(x,\lambda)$是方程 (1) 的解, 则

$$\psi_2(x,\lambda)=A\cos sx+\frac{B}{s}\sin(sx)+\int_c^x\frac{\sin s(x-t)}{s}q(t)\psi_2(t,\lambda)\,\mathrm{d}t$$

$$\psi_2'(x,\lambda)=-As\sin(sx)+B\cos(sx)+\int_c^x\cos s(x-t)q(t)\psi_2(t,\lambda)\,\mathrm{d}t$$

由初始条件 (15), (16) 式得:

$$A=a\cos(sc)\psi_1(c-0,\lambda)-\frac{1}{as}\sin(sc)\psi_1'(c-0,\lambda)$$

$$B=\frac{1}{a}\cos(sc)\psi_1'(c-0,\lambda)+as\sin(sc)\psi_1(c-0,\lambda)$$

所以

$$\psi_2(x,\lambda)=a\psi_1(c-0,\lambda)\cos s(x-c)+\frac{1}{as}\sin s(x-c)\psi_1'(c-0,\lambda)+\frac{1}{s}\int_c^x\sin s(x-t)q(t)\psi_2(t,\lambda)\,\mathrm{d}t \tag{30}$$

$$\psi_2'(x,\lambda)=-a\sin(sc)\sin s(x-c)+\frac{1}{a}\cos(sc)\cos s(x-c)+\int_c^x\cos s(x-t)q(t)\psi_2(t,\lambda)\,\mathrm{d}t \tag{31}$$

由式 (11), (12) 可得:

$$\psi_2(x,\lambda)=a\cos s(x-c)\frac{\sin(sc)}{s}+\frac{1}{as}\sin s(x-c)\cos(sc)+\frac{1}{s}\int_c^x\sin s(x-t)q(t)\psi_2(t,\lambda)\,\mathrm{d}t+O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|^2}\right) \tag{32}$$

令$H(x,\lambda)=s\mathrm{e}^{-|\tau|(x-c)}\psi_2(x,\lambda)$, 则

$$\begin{aligned}&|H(x,\lambda)|\leqslant a+\frac{1}{a}+\\&\frac{1}{|s|^2}\int_c^{\pi}|\sin s(x-t)||q(t)||H(t,\lambda)|\left|\mathrm{e}^{|\tau|(x-c)}\right|\mathrm{d}t\leqslant\\&a+\frac{1}{a}+\frac{1}{|s|^2}\int_c^{\pi}|q(t)||H(t,\lambda)|\mathrm{d}t+O\left(\frac{1}{|s|}\right)\end{aligned}\tag{33}$$

再由 Gronwall 不等式可得:

$$|H(x,\lambda)|\le\left(a+\frac{1}{a}+\left|O\left(\frac{1}{|s|}\right)\right|\right)\mathrm{e}^{\frac{1}{|s|^2}\int_c^{\pi}|q(t)|\mathrm{d}t}\tag{34}$$

当$|\lambda|\to\infty$时, $|H(x,\lambda)|\le M$, 所以

$$\psi_2(x,\lambda)=\frac{\mathrm{e}^{|\tau|(x-c)}}{s}H(x,\lambda)=O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right)\tag{35}$$

将式 (35) 代入式 (32), 又由

$$\begin{aligned}&\frac{\left|\frac{1}{s}\int_c^x\sin s(x-t)q(t)O\left(\frac{\mathrm{e}^{|\tau|x}}{|s|}\right)\mathrm{d}t\right|}{\frac{\mathrm{e}^{|\tau|x}}{|s|^2}}\leqslant\\&\frac{1}{\mathrm{e}^{|\tau|x}}\int_c^{\pi}\left|\mathrm{e}^{|\tau|(x-t)}\right||q(t)|\left|O\left(\mathrm{e}^{|\tau|t}\right)\right|\mathrm{d}t\leqslant\\&\frac{A}{\mathrm{e}^{|\tau|x}}\int_c^{\pi}\mathrm{e}^{|\tau|(x-t)}\mathrm{e}^{|\tau|t}\mathrm{d}t=A(\pi-c)\end{aligned}$$

因此式 (19) 成立。同样的方法对式 (31) 进行估计可得式 (20) 成立。

定理 2　定义$\varphi(x,\lambda)$, $\psi(x,\lambda)$分别如式 (6), (7), 设$\alpha_0=\arccos\frac{2}{a+\frac{1}{a}}$, $\lambda_n=s_n^2$是不连续 SLPs(1)~(5) 的第n个特征值, 则对充分大的n有如下估计式:

$$s_n=\frac{1}{\pi}(\pm\alpha_0+2n\pi)+O\left(\frac{1}{n}\right)=\pm\frac{1}{\pi}\alpha_0+2n+O\left(\frac{1}{n}\right)\tag{36}$$

特别当$a=1$时, $s_n=2n+O\left(\frac{1}{n}\right)$。

证明　定义$\varphi(x,\lambda)$, $\psi(x,\lambda)$分别如式 (9), (10), 由定理 1 知, $\varphi(x,\lambda)$, $\psi(x,\lambda)$满足估计式 (17)~(20), 即

$$\begin{aligned}\varphi_2(\pi,\lambda)=&a\cos(sc)\cos s(\pi-c)-\\&\frac{1}{a}\sin(sc)\sin s(\pi-c)+O\left(\frac{1}{|s|}\right)\end{aligned}\tag{37}$$

$$\begin{aligned}\varphi_2'(\pi,\lambda)=&-as\cos(sc)\sin s(\pi-c)-\\&\frac{s}{a}\sin(sc)\cos s(\pi-c)+O(1)\end{aligned}\tag{38}$$

$$\begin{aligned}\psi_2(\pi,\lambda)=&\frac{a}{s}\sin(sc)\cos s(\pi-c)+\\&\frac{1}{as}\cos(sc)\sin s(\pi-c)+O\left(\frac{1}{|s|^2}\right)\end{aligned}\tag{39}$$

$$\begin{aligned}\psi_2'(\pi,\lambda)=&-a\sin(sc)\sin s(\pi-c)+\\&\frac{1}{a}\cos(sc)\cos s(\pi-c)+O\left(\frac{1}{|s|}\right)\end{aligned}\tag{40}$$

设λ是不连续 SLPs(1)~(5) 的特征值, 则λ满足式 (8) 且λ为实数。将式 (37)~(40) 代入式 (8) 得:

$$\begin{aligned}D(\lambda)=&-a^2\sin(sc)\cos(sc)\sin s(\pi-c)\cos s(\pi-c)+\\&\sin^2(sc)\sin^2s(\pi-c)+\cos^2(sc)\cos^2s(\pi-c)-\\&\frac{1}{a^2}\sin(sc)\cos(sc)\sin s(\pi-c)\cos s(\pi-c)+O\left(\frac{1}{|s|}\right)+\\&O\left(\frac{1}{|s|^2}\right)+\Big[a^2\sin(sc)\cos(sc)\sin s(\pi-c)\cos s(\pi-c)+\\&\cos^2(sc)\sin^2s(\pi-c)+\sin^2(sc)\cos^2s(\pi-c)+\\&\frac{1}{a^2}\sin(sc)\cos(sc)\sin s(\pi-c)\cos s(\pi-c)\Big]+\\&O\left(\frac{1}{|s|}\right)+O\left(\frac{1}{|s|^2}\right)-a\cos(sc)\cos s(\pi-c)+\\&\frac{1}{a}\sin(sc)\sin s(\pi-c)+O\left(\frac{1}{|s|}\right)+\\&a\sin(sc)\sin s(\pi-c)-\frac{1}{a}\cos(sc)\cos s(\pi-c)+\\&O\left(\frac{1}{|s|}\right)+1=\cos^2s(\pi-c)+\sin^2s(\pi-c)+\\&1-a\cos(s\pi)-\frac{1}{a}\cos(s\pi)+O\left(\frac{1}{|s|}\right)+\\&O\left(\frac{1}{|s|^2}\right)=2-\left(a+\frac{1}{a}\right)\cos(s\pi)+\\&O\left(\frac{1}{|s|}\right)+O\left(\frac{1}{|s|^2}\right)=0\end{aligned}\tag{41}$$

对充分大的s, $2-\left(a+\frac{1}{a}\right)\cos(s\pi)+O\left(\frac{1}{|s|}\right)+O\left(\frac{1}{|s|^2}\right)=0$的解靠近$2k\pm\frac{1}{\pi}\alpha_0$。设$s_n\pi=\pm\alpha_0+\delta_n+2n\pi$, 这里$\cos\alpha_0=\frac{2}{a+\frac{1}{a}}$, 则

$$2-\left(a+\frac{1}{a}\right)\cos(\pm\alpha_0+\delta_n+2n\pi)+O\left(\frac{1}{|s|}\right)+O\left(\frac{1}{|s|^2}\right)=0\tag{42}$$

即

$$\begin{aligned}2-\left(a+\frac{1}{a}\right)&[\cos(\pm\alpha_0+2n\pi)\cos\delta_n-\\&\sin(\pm\alpha_0+2n\pi)\sin\delta_n]+O\left(\frac{1}{n}\right)=0\end{aligned}\tag{43}$$

$$2-2\cos\delta_n\mp\left(a-\frac{1}{a}\right)\sin\delta_n+O\left(\frac{1}{n}\right)=0\tag{44}$$

因此$\lim\limits_{n\to\infty}\frac{2-2\cos\delta_n\mp\left(a-\frac{1}{a}\right)\sin\delta_n}{\frac{1}{\mathrm{n}}}=C\neq0$, 所以$\delta_n=O\left(\frac{1}{n}\right)$, 因此式 (36) 成立。

特别当$a=1$时, cos $\alpha_0=1$, α_0=0, $s_n=2n+O\left(\frac{1}{n}\right)$。

3　不连续 Sturm–Liouville 问题特征函数的零点个数

设在$(0,c)$上, $\varphi_1(x,\lambda)$是微分方程 (1) 的满足初始条件$\varphi_1(0,\lambda)=c_1,\varphi_1'(0,\lambda)=c_2$, 其中$c_1$, c_2不全为零。在(c,π)上, $\varphi_2(x,\lambda)$是式 (1) 的满足初始条件$\varphi_2(c+0,\lambda)=$

$a\varphi_1(c-0,\lambda),\varphi_2'(c+0,\lambda)=\frac{1}{a}\varphi_1'(c-0,\lambda)$的解。令

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda),x\in(0,c)\\ \varphi_2(x,\lambda),x\in(c,\pi)\end{cases} \tag{45}$$

则$\varphi(x,\lambda)$是式 (1) 的满足转移条件 (4) 和 (5) 的解。令

$$\begin{cases}\varphi_1(x,\lambda)=\rho_1(x,\lambda)\sin\theta_1(x,\lambda)\\ \varphi_1'(x,\lambda)=\rho_1(x,\lambda)\cos\theta_1(x,\lambda)\end{cases} \tag{46}$$

$$\begin{cases}\varphi_2(x,\lambda)=\rho_2(x,\lambda)\sin\theta_2(x,\lambda)\\ \varphi_2'(x,\lambda)=\rho_2(x,\lambda)\cos\theta_2(x,\lambda)\end{cases} \tag{47}$$

定理 3 设$\varphi(x,\lambda)$的定义如式 (45), 若存在λ_n使得$\varphi_1(0,\lambda_n)$=$\varphi_2(\pi,\lambda_n)$,$\varphi_1'(0,\lambda_n)$=$\varphi_2'(\pi,\lambda_n)$, 则$\lambda_n$是不连续 SLPs(1)~(5) 的特征值, $\varphi(x,\lambda_n)$是对应的特征函数, 特征值λ_n满足式 (48)。

$$\theta_2(\pi,\lambda_n)=n\pi+\alpha\,(n=0,1,2,\cdots),\text{其中}\alpha=\arccos\frac{c_2}{\sqrt{c_1^2+c_2^2}} \tag{48}$$

且第n个特征值λ_n对应于的特征函数$\varphi(x,\lambda_n)$在$(0,c)\cup(c,\pi)$内有n个零点$(n=0,1,2,\cdots)$。

证明 若存在λ_n使得$\varphi_1(0,\lambda_n)=\varphi_2(\pi,\lambda_n)$,$\varphi_1'(0,\lambda_n)=\varphi_2'(\pi,\lambda_n)$, 可得$\varphi(0,\lambda_n)=\varphi(\pi,\lambda_n)$, $\varphi'(0,\lambda_n)$=$\varphi'(\pi,\lambda_n)$, 即$\varphi(x,\lambda_n)$满足边界条件 (2), (3), 所以λ_n是不连续 SLPs(1)~(5) 的特征值, $\varphi(x,\lambda_n)$是对应的特征函数。当$x\in(0,c)$时, 将式 (46) 代入式 (1) 得:

$$\rho_1'(x,\lambda)=\frac{1}{2}(1+q-\lambda)\rho_1(x,\lambda)\sin2\theta_1(x,\lambda) \tag{49}$$

$$\theta_1'(x,\lambda)=\cos^2\theta_1(x,\lambda)+(\lambda-q)\sin^2\theta_1(x,\lambda) \tag{50}$$

由$\varphi_1(0,\lambda)=c_1,\varphi_1'(0,\lambda)=c_2$及式 (46) 得$\theta_1(0,\lambda)=\alpha$, $\rho_1(0,\lambda)=\sqrt{c_1^2+c_2^2}$, 其中$\alpha=\arccos\frac{c_2}{\sqrt{c_1^2+c_2^2}}$。由式 (49) 得:

$$\ln\rho_1(x,\lambda)=\frac{1}{2}\int_0^x(1+q-\lambda)\sin\,2\theta_1(t,\lambda)\mathrm{d}t+\ln\rho_1(0,\lambda)$$

所以

$$\rho_1(x,\lambda)=\mathrm{e}^{\frac{1}{2}\int_0^x(1+q-\lambda)\sin\,2\theta_1(t,\lambda)\mathrm{d}t+\ln\,\rho_1(0,\lambda)}>0 \tag{51}$$

当$x\in(c,\pi)$时, 将式 (47) 代入式 (1) 得:

$$\rho_2'(x,\lambda)=\frac{1}{2}(1+q-\lambda)\rho_2(x,\lambda)\sin2\theta_2(x,\lambda) \tag{52}$$

$$\theta_2'(x,\lambda)=\cos^2\theta_2(x,\lambda)+(\lambda-q)\sin^2\theta_2(x,\lambda) \tag{53}$$

$$\rho_2(x,\lambda)=\mathrm{e}^{\ln\,\rho_2(\pi,\lambda)-\frac{1}{2}\int_x^\pi(1+q-\lambda)\sin2\theta_2(t,\lambda)\mathrm{d}t}>0$$

由$\varphi_1(0,\lambda_n)=\varphi_2(\pi,\lambda_n),\varphi_1'(0,\lambda_n)=\varphi_2'(\pi,\lambda_n)$及式 (47) 可得:

$$\rho_2(\pi,\lambda_n)\sin\theta_2(\pi,\lambda_n)=c_1,\rho_2(\pi,\lambda_n)\cos\theta_2(\pi,\lambda_n)=c_2 \tag{54}$$

所以特征值λ_n满足式 (48), 且

$$\rho_2(c+0,\lambda_n)\sin\theta_2(c+0,\lambda_n)=a\rho_1(c-0,\lambda_n)\sin\theta_1(c-0,\lambda) \tag{55}$$

$$\rho_2(c+0,\lambda_n)\cos\theta_2(c+0,\lambda_n)=\frac{1}{a}\rho_1(c-0,\lambda_n)\cos\theta_1(c-0,\lambda_n) \tag{56}$$

$\varphi(x,\lambda_n)$在$(0,c)\cup(c,\pi)$的零点个数等于$\varphi_1(x,\lambda_n)$在$(0,c)$内零点个数与$\varphi_2(x,\lambda_n)$在(c,π)内零点个数之和。由式 (46) 知$\varphi_1(x,\lambda_n)$在$(0,c)$内的零点是使$\sin\theta_1(x,\lambda_n)=0$的$x$, 即$\theta_1(x,\lambda_n)=k\pi$。由文献 [13] 中引理 2.2.2 知$\theta_1(x,\lambda)$关于$\lambda$是增函数, 非负, 且$\lim\limits_{\lambda\to-\infty}\theta_1(c-0,\lambda)=0$, $\lim\limits_{\lambda\to+\infty}\theta_1(c-0,\lambda)=+\infty$。设$\theta_1(c-0,\lambda_n)=m\,(0<m<+\infty)$, 则存在正整数$k_0$, 使得$k_0\pi<m<(k_0+1)\pi$。

设$\theta_1(x_0,\lambda_n)=k\pi$, 将其代入式 (50) 得$\theta_1'(x_0,\lambda_n)=1$, 故$\theta_1(x,\lambda_n)$在$x_0$附近关于$x$单增, 即$\theta_1(x,\lambda_n)$只能取 1 次$k\pi$。若存在$x_1$使$\theta_1(x_1,\lambda_\mathrm{n})=k\pi$, 则$\theta_1'(x_1,\lambda_n)<0$矛盾。所以$\theta_1(x,\lambda_n)$在$(0,c)$内可取$\pi,2\pi,3\pi,\cdots,k_0\pi$, 因此$\varphi_1(x,\lambda_n)$在$(0,c)$内有$k_0$个零点, $x_1,x_2,\cdots,x_{k_0}$。

进一步估算$\varphi_2(x,\lambda_n)$在(c,π)内的零点个数。由式 (47) 知$\varphi_2(x,\lambda_n)$的零点是使$\sin\,\theta_2(x,\lambda_n)=0$的$x$, 即为$\theta_2(x,\lambda_n)=k\pi$的点$x$, 且$\theta_2(x,\lambda_n)$只能取 1 次$k\pi$。由式 (55), (56) 可得:

$$\cot\theta_2(c+0,\lambda_n)=\frac{1}{a^2}\cot\theta_1(c-0,\lambda_n) \tag{57}$$

由$\theta_1(c-0,\lambda_n)=m$, 且$k_0\pi<m<(k_0+1)\pi$可得:

$$\theta_2(c+0,\lambda_n)=\operatorname{arcot}\left[\frac{1}{a^2}\cot\theta_1(c-0,\lambda)\right]+k_0\pi=\beta_0+k_0\pi\,(0<\beta_0<\pi)$$

由$\theta_2(\pi,\lambda_n)=n\pi+\alpha\,(n=0,1,2,\cdots)$知$\theta_2(x,\lambda_n)$在$(c,\pi)$内可取到$(k_0+1)\pi,(k_0+2)\pi,\cdots,n\pi$。因此$\varphi_2(x,\lambda)$在$(c,\pi)$内有$n-k_0$个零点。所以$\varphi(x,\lambda_n)$在$(0,c)\cup(c,\pi)$内有$n$个零点。

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责任编辑: 丁吉海