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具有非局部竞争和时滞的广食性捕食者-食饵模型的 Hopf 分支

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摘要:利用特征方程根的分布分析方法,研究了正平衡点的稳定性和 Hopf 分支的存在性,得到了系统在发生 Hopf 分支时的时滞临界值。利用中心流形定理和规范型理论,确定了分支方向和分支周期解的稳定性。系统可能存在无食饵边界平衡点与正平衡点双稳、无食饵边界平衡点与周期解双稳 2 种双稳态,通过数值模拟验证了理论结果。

关键词:广食性捕食者;非局部竞争;时滞;Hopf 分支;双稳
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Hopf bifurcation in a diffusive generalist predator-prey system with nonlocal competition and time delay

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Abstract: The stability of positive equilibrium and the existence of Hopf bifurcation are studied by analyzing the distribution of eigenvalues. The critical time delay of Hopf bifurcation is obtained. Applying center manifold method and normal form theory, the direction of Hopf bifurcation and stability of the bifurcating periodic solution are discussed. It is shown that there are two types of bistability. The prey-free equilibrium and the positive equilibrium are both stable. The prey-free equilibrium and the periodic solution are both stable. Numerical simulations are presented to support the theoretical results.

Key words: generalist predator; nonlocal competition; time delay; Hopf bifurcation; bistability

0　引言

生物多样性是人类赖以生存与发展的重要基石,对生态系统的长期稳定有着举足轻重的意义。丰富而不同的生物种类在整个自然生态体系中扮演着各式各样的角色,它们相互作用形成了复杂的生态链。生物多样性的丧失可能导致物种的灭绝、环境的恶化、生态平衡的破坏。为了防止入侵物种对当地造成严重危害,引进入侵物种在原始产地的天敌来控制入侵物种的迅速蔓延[1]。入侵物种与其天敌之间是一种捕食-食饵的关系,对捕食-食饵系统进行动力学分析,能更好地认识入侵物种和天敌相互作用的规律,从而进行有效的生物控制。在捕食者-食饵系统中,根据捕食者所捕食物种的数量,可以将捕食者分为专食性捕食者和广食性捕食者。专食性捕食者以单一物种为食,它们控制入侵物种的能力受到食饵 Allee 效应的影响[2-3],而广食性捕食者有额外的食物来源,在没有特定食饵的情况下,依旧可以生存下来。研究表明,引入广食性捕食者是控制生物入侵的一种重要手段[4-5]。

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基于文献[2-3]的研究,文献[4]提出了寄主-寄生蜂模型:

$$\begin{cases}\dfrac{\partial u}{\partial t}=D\dfrac{\partial^2 u}{\partial x^2}+r_1u\left(1-\dfrac{u}{K_1}\right)-v\dfrac{Eu}{1+Ehu},\\ \dfrac{\partial v}{\partial t}=D\dfrac{\partial^2 v}{\partial x^2}+\gamma v\dfrac{Eu}{1+Ehu}+r_2v\left(1-\dfrac{v}{K_2}\right),\end{cases}\tag{1}$$

式中,$u=u(x,t)$为在位置 x 和时间 t(寄主)食饵的密度;$v=v(x,t)$为在位置 x 和时间 t(寄生蜂)捕食者的密度;D 为扩散速率;r_1 为食饵的生长速率;r_2 为捕食者的生长速率;K_1 为食饵的最大环境容纳量;K_2 为捕食者的最大环境容纳量;E 为食饵和捕食者的相遇率;h 为收获时间;γ 为广食性捕食者的转化率。

在 Magal 等[4]的研究基础上,Xiang 等[6]继续研究系统(1)对应的常微分系统的动力学行为,在对参数进行完整的定性和分支分析后,发现该模型表现出复杂的动力学和分支。此外,文献[4-8]也对系统(1)对应的常微分系统的动力学进行了全面的研究。Madec 等[9]考虑了以下无量纲系统:

$$\begin{cases}\dfrac{\partial u}{\partial t}=d_1\dfrac{\partial^2 u}{\partial x^2}+u(1-u)-\dfrac{Euv}{1+Ehu},\\ \dfrac{\partial v}{\partial t}=d_2\dfrac{\partial^2 v}{\partial x^2}+rv(1-v)+\alpha\dfrac{Eruv}{1+Ehu},\end{cases}\tag{2}$$

式中,d_1 为食饵的扩散系数,d_2 为捕食者的扩散系数,结果表明由广食性天敌诱导的双稳态可以逆转害虫的入侵。

种间和种内的相互作用可以看作是局部的[4,9],由于资源有限,一个种群不仅与同一地区的种群相互作用,同时也与相邻的种群相互作用,因此种间和种内的相互作用是非局部的[10]。Furter 等在文献[10]中首次将非局部作用纳入单种群的模型中,利用

$$\hat{u}=\int_{\Omega}G(x,y)u(y,t)\,\mathrm{d}y$$

表示非局部资源消耗,其中,$G(x,y)$是核函数,Ω 是空间域。

近年来,非局部效应对种群动力学影响受到广泛关注[11-12]。由于食饵和捕食者的生长都遵循 Logistic 增长,Yang 等[13]认为食饵和捕食者都具有非局部种内竞争,他们将食饵和捕食者的非局部种内竞争引入到系统(2)中,对具有非局部种内竞争的广食性捕食者-食饵系统的时空动力学进行了研究,分析了非局部竞争对共存平衡点稳定性的影响以及可能发生的 Turing 分支和 Hopf 分支。此外,还有许多学者对具有非局部竞争的系统进行了分支分析[14-17]。

种群中的许多行为不是瞬时完成的,而是有一定的时间滞后,如消化时滞、妊娠时滞等。Kuang[18]、Liu 等[19]和 Beretta 等[20]考虑了具有时滞的模型,其中 τ 可以看作捕食者的妊娠期或反应时间[21]。本文在 Yang 等[13]模型的基础上,考虑捕食者妊娠时滞的影响,得到以下模型:

$$\begin{cases}\dfrac{\partial u}{\partial t}=d_1\dfrac{\partial^2 u}{\partial x^2}+u(1-\hat{u})-\dfrac{Euv}{1+Ehu},\\ \dfrac{\partial v}{\partial t}=d_2\dfrac{\partial^2 v}{\partial x^2}+rv(1-\hat{v})+\alpha\dfrac{Ervu(x,t-\tau)}{1+Ehu(x,t-\tau)}, & x\in(0,l\pi),\ t>0,\\ u_x(0,t)=v_x(0,t)=0,\ u_x(l\pi,t)=v_x(l\pi,t)=0, & t>0,\\ u(x,t)=u_0(x,t)\geqslant 0,\ v(x,t)=v_0(x,t)\geqslant 0, & x\in[0,l\pi],\ t\in[-\tau,0],\end{cases}\tag{3}$$

式中,$\hat{u}=\dfrac{1}{l\pi}\displaystyle\int_0^{l\pi}u(y,t)\,\mathrm{d}y$, $\hat{v}=\dfrac{1}{l\pi}\displaystyle\int_0^{l\pi}v(y,t)\,\mathrm{d}y$, τ 为捕食者的妊娠时滞。

1　稳定性与分支分析

考虑系统(3)对应的常微分系统

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=u(1-u)-\dfrac{Euv}{1+Ehu},\\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=vr(1-v)+\alpha\dfrac{Ervu}{1+Ehu},\end{cases}\tag{4}$$

根据文献[9]的结果，系统(4)必存在平衡点$E_0(0,0)$、无捕食者边界平衡点$E_1(1,0)$和无食饵边界平衡点$E_2(0,1)$。

通过计算可知，系统(4)在$E_0(0,0)$处对应的特征方程为

$$(\lambda-1)(\lambda-r)=0,$$

则$\lambda_1=1>0,\ \lambda_2=r>0$，因此$E_0$不稳定。在$E_1(1,0)$处对应的特征方程为

$$(\lambda+1)\left[\lambda-\left(r+\frac{\alpha Er}{1+Eh}\right)\right]=0,$$

则$\lambda_1=-1<0,\ \lambda_2=r+\dfrac{\alpha Er}{1+Eh}>0$，因此$E_1$不稳定。

系统(4)在$E_2(0,1)$处对应的特征方程为

$$(\lambda+E-1)(\lambda+r)=0,$$

则$\lambda_1=1-E,\ \lambda_2=-r<0$。所以，当$E>1$时，$E_2$局部渐近稳定；当$E<1$时，$E_2$不稳定。

假设(H_0)：$E>1,\ Eh>1$，以及$\alpha<\dfrac{(Eh+1)^3-4E^2h(Eh+1)}{4E^2(Eh-1)}$成立时，系统(4)有2个正平衡点$E_3(u_3,v_3)$和$E_*(u_*,v_*)$，满足$u_3<u_*$且$v_3<v_*$。$E_3$是鞍点，$E_*$局部渐近稳定。

对于系统(3)，只需讨论E_2和E_*的稳定性。系统(3)在任一平衡点$(\bar{u},\bar{v})$处的线性化方程为

$$\frac{\partial}{\partial t}\begin{pmatrix}u(x,t)\\v(x,t)\end{pmatrix}=\boldsymbol{D}\begin{pmatrix}\Delta u(x,t)\\\Delta v(x,t)\end{pmatrix}+\boldsymbol{L}_1\begin{pmatrix}u(x,t)\\v(x,t)\end{pmatrix}+\boldsymbol{L}_2\begin{pmatrix}u(x,t-\tau)\\v(x,t-\tau)\end{pmatrix}+\boldsymbol{L}_3\begin{pmatrix}\hat{u}(x,t)\\\hat{v}(x,t)\end{pmatrix},\tag{5}$$

其中

$$\boldsymbol{D}=\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix},\quad \boldsymbol{L}_1=\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix},\quad \boldsymbol{L}_2=\begin{pmatrix}0&0\\b_{21}&0\end{pmatrix},\quad \boldsymbol{L}_3=\begin{pmatrix}c_{11}&0\\0&c_{22}\end{pmatrix},$$

$$a_{11}=1-\bar{u}-\frac{E\bar{v}}{(1+Eh\bar{u})^2},\quad a_{12}=-\frac{E\bar{u}}{1+Eh\bar{u}},\quad a_{22}=r(1-\bar{v})+\frac{\alpha Er\bar{u}}{1+Eh\bar{u}},\quad b_{21}=\frac{\alpha Er\bar{v}}{(1+Eh\bar{u})^2},\quad c_{11}=-\bar{u},\quad c_{22}=-r\bar{v}。$$

Laplace算子Δ的特征值为$-\dfrac{n^2}{l^2}(n=0,1,\cdots)$，其特征函数为

$$\gamma_0(x)=\sqrt{\frac{1}{l\pi}},\quad \gamma_n(x)=\sqrt{\frac{2}{l\pi}}\cos\frac{n}{l}x(n\geqslant1)。$$

令$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}a_n\\b_n\end{pmatrix}\cos\dfrac{n}{l}x\mathrm{e}^{\lambda t}$为方程(5)的解，注意到

$$\frac{1}{l\pi}\int_0^{l\pi}\cos\frac{n}{l}x\mathrm{d}x=\begin{cases}1, & n=0,\\0, & n\neq0,\end{cases}$$

因此，特征方程为

$$\begin{vmatrix}\lambda+\dfrac{n^2}{l^2}d_1-a_{11}-c_{11}\delta(n) & -a_{12}\\ -b_{21}\mathrm{e}^{-\lambda\tau} & \lambda+\dfrac{n^2}{l^2}d_2-a_{22}-c_{22}\delta(n)\end{vmatrix}=0,\tag{6}$$

其中$\delta(n)=\begin{cases}1, & n=0,\\0, & n\in\mathbf{N}。\end{cases}$

由式(6)，系统(3)在$E_2(0,1)$处对应的特征方程为

$$\left[\lambda+\frac{n^2}{l^2}d_1+E-1\right]\left[\lambda+\frac{n^2}{l^2}d_2+r\delta(n)\right]=0,\quad n=0,1,2,\cdots,$$

即

$$(\lambda+E-1)(\lambda+r)=0,\quad n=0,\tag{7}$$

$$\left(\lambda+\frac{n^2}{l^2}d_1+E-1\right)\left(\lambda+\frac{n^2}{l^2}d_2\right)=0,\quad n\in\mathbf{N}。\tag{8}$$

容易得到,当假设(H_0)成立时,

$$\lambda_{1,0}=-(E-1)<0,\quad \lambda_{2,0}=-r<0,\quad \lambda_{1,n}=-\left(\frac{n^2}{l^2}d_1+E-1\right)<0,\quad \lambda_{2,n}=-\frac{n^2}{l^2}d_2<0。$$

即特征方程(7)、(8)的特征根均具有负实部。

定理 1 若假设(H_0)成立,则系统(3)的平衡点 $E_2(0,1)$是局部渐近稳定的。

对于 $E_*(u_*,v_*)$,得到

$$a_{11}=\frac{E^2hu_*v_*}{(1+Ehu_*)^2}>0,\quad a_{12}=-\frac{Eu_*}{1+Ehu_*}<0,\quad a_{22}=0,\quad b_{21}=\frac{\alpha Erv_*}{(1+Ehu_*)^2}>0,\quad c_{11}=-u_*<0,\quad c_{22}=-rv_*<0,$$

对应的特征方程为

$$\lambda^2+A\lambda+B+C\mathrm{e}^{-\lambda\tau}=0,\quad n=0,\tag{9}$$

$$\lambda^2+A_n\lambda+B_n+C\mathrm{e}^{-\lambda\tau}=0,\quad n=1,2,\cdots,\tag{10}$$

其中

$$A=-(a_{11}+c_{11}+c_{22}),\quad B=c_{22}(a_{11}+c_{11}),\quad C=-a_{12}b_{21}>0,$$

$$A_n=\frac{n^2}{l^2}(d_1+d_2)-a_{11},\quad B_n=\frac{n^2}{l^2}d_2\left[\frac{n^2}{l^2}d_1-a_{11}\right]。$$

当 $\tau=0$ 时,特征方程(9)、(10)可化为

$$\lambda^2+A\lambda+B+C=0,\quad n=0,\tag{11}$$

$$\lambda^2+A_n\lambda+B_n+C=0,\quad n=1,2,\cdots。\tag{12}$$

假设(H_1):$A>0,\ B+C>0,\ A_n>0,\ B_n+C>0(n=1,2,\cdots)$。

当假设(H_1)成立时,特征方程(11)和(12)的根均具有负实部。

定理 2 当 $\tau=0$ 时,若假设(H_0)和(H_1)成立,则系统(3)的正平衡点 E_*是局部渐近稳定的。

下面讨论时滞 $\tau>0$ 时的情形。先考虑 $n=0$ 时对应的特征方程(9),将 $\lambda=\mathrm{i}\omega_0(\omega_0>0)$代入方程(9)中,分离实虚部后得到

$$\begin{cases}\omega_0^2-B=C\cos\omega_0\tau,\\ A\omega_0=C\sin\omega_0\tau,\end{cases}$$

方程两边平方后相加可得

$$\omega_0^4+(A^2-2B)\omega_0^2+B^2-C^2=0,\tag{13}$$

易得 $T=A^2-2B=(a_{11}+c_{11})^2+c_{22}^2>0$。由于 $D=B^2-C^2=(B+C)(B-C)$, $B+C>0$,因此,当 $B-C\geqslant 0$ 时,方程(13)没有正实根,从而方程(9)没有纯虚根;而当 $B-C<0$ 时,方程(13)有一个正根

$$\omega_0=\left[\frac{1}{2}(-T+\sqrt{T^2-4D})\right]^{\frac{1}{2}}。$$

假设(H_2):$B-C\geqslant 0$。

假设(H_3):$B-C<0$。

引理 1 如果假设(H_0)、(H_1)和(H_2)成立,则方程(9)没有纯虚根;如果假设(H_0)、(H_1)和(H_3)成立,则方程(9)当 $\tau=\tau_0^k$时,有一对简单纯虚根$\pm\mathrm{i}\omega_0$,其中

$$\omega_0=\left[\frac{1}{2}(-T+\sqrt{T^2-4D})\right]^{\frac{1}{2}},$$

$$\tau_0^k=\frac{1}{\omega_0}\left[\arccos\frac{\omega_0^2-B}{C}+2k\pi\right],\quad k=0,1,\cdots。$$

类似地,将 $\lambda=\mathrm{i}\omega_n(\omega_n>0)$代入方程(10),得到

$$\begin{cases}\omega_n^2-B_n=C\cos\omega_n\tau,\\ A_n\omega_n=C\sin\omega_n\tau,\end{cases}$$

以及

$$\omega_n^4+(A_n^2-2B_n)\omega_n^2+B_n^2-C^2=0。\tag{14}$$

显然 $T_n=A_n^2-2B_n=\left[\frac{n^2}{l^2}d_1-a_{11}\right]^2+\frac{n^4}{l^4}d_2^2>0$。由于 $D_n=B_n^2-C^2=(B_n+C)(B_n-C)$，$B_n+C>0$，因此，当 $B_n-C\geqslant 0$ 时，方程（14）没有正实根，故方程（10）没有纯虚根；而当 $B_n-C<0$ 时，方程（14）只有一个正根

$$\omega_n=\left[\frac{1}{2}\left(-T_n+\sqrt{T_n^2-4D_n}\right)\right]^{\frac{1}{2}}。$$

假设（H_4）：$B_n-C\geqslant 0$，$n=1,2,\cdots$。

假设（H_5）：$\exists n\in\mathbf{N}$ 使得 $B_n-C<0$。

记 $S=\{n\mid B_n-C<0,\ n\in\mathbf{N}\}$，因为 $\lim\limits_{n\to\infty}(B_n-C)\to+\infty$，所以 S 是一个有限集。

引理 2　若假设（H_0）、（H_1）和（H_4）成立，则方程（10）没有纯虚根；如果假设（H_0）、（H_1）和（H_5）成立，则方程（10）当 $\tau=\tau_n^k$ 时有一对纯虚根 $\pm\mathrm{i}\omega_n$，其中

$$\omega_n=\left[\frac{1}{2}\left(-T_n+\sqrt{T_n^2-4D_n}\right)\right]^{\frac{1}{2}},$$

$$\tau_n^k=\frac{1}{\omega_n}\left[\arccos\frac{\omega_n^2-B_n}{C}+2k\pi\right],\quad n\in S,\quad k=0,1,\cdots。$$

引理 3　如果假设（H_0）、（H_1）和（H_3）成立，那么 $\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]\Big|_{\tau=\tau_0^k}>0$（$k=0,1,\cdots$）；如果假设（$H_0$）、（$H_1$）和（$H_5$）成立，那么 $\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]\Big|_{\tau=\tau_n^k}>0$，其中 $n\in S$ 且 $k\in\mathbf{N}_0$。

证明　对方程（9）的两边关于 τ 求导，可得

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}=\frac{(2\lambda+A)\mathrm{e}^{\lambda\tau}}{C\lambda}-\frac{\tau}{\lambda},$$

从而

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}\Big|_{\tau=\tau_0^k}&=\frac{2\omega_0\cos\omega_0\tau+A\sin\omega_0\tau}{C\omega_0}\\&=\frac{2(\omega_0^2-B)+A^2}{C^2}\\&=\frac{\sqrt{T^2-4D}}{C^2}>0。\end{aligned}$$

对方程（10）的两边关于 τ 求导，可得

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}=\frac{(2\lambda+A_n)\mathrm{e}^{\lambda\tau}}{C\lambda}-\frac{\tau}{\lambda},$$

从而

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}\Big|_{\tau=\tau_n^k}&=\frac{2\omega_n\cos\omega_n\tau+A_n\sin\omega_n\tau}{C\omega_n}\\&=\frac{2(\omega_n^2-B_n)+A_n^2}{C^2}\\&=\frac{\sqrt{T_n^2-4D_n}}{C^2}>0。\end{aligned}$$

当假设（H_0）、（H_1）、（H_3）和（H_5）成立时，定义 $\tilde{\tau}\triangleq\min\{\tau_0^0,\tau_n^0,n\in S\}$。

定理 3　如果假设（H_0）和（H_1）成立，那么对于系统（3）有以下结论成立：

（1）若假设（H_2）和（H_4）成立，则当 $\tau\geqslant 0$ 时，正平衡点 $E_*(u_*,v_*)$ 局部渐近稳定；

(2) 若假设(H_3)或(H_5)成立,则当$\tau\in[0,\tilde{\tau})$时,$E_*(u_*,v_*)$局部渐近稳定,而当$\tau>\tilde{\tau}$时,$E_*(u_*,v_*)$不稳定;

(3)在每个τ_0^k和$\tau_n^k(n\in S,\ k\in\mathbf{N}_0)$处,系统(3)发生 Hopf 分支。

2　Hopf 分支方向与分支周期解的稳定性

令$\tilde{\tilde{u}}(x,t)=u(x,\tau t)-u_*,\ \tilde{\tilde{v}}(x,t)=v(x,\tau t)-v_*$,为了简化记号,去掉“≈”,则系统(3)可写为

$$\begin{cases}\dfrac{\partial u}{\partial t}=\tau[d_1\Delta u(x,t)+a_{11}u(x,t)+a_{12}v(x,t)+c_{11}\hat{u}(x,t)+\alpha_1u^2(x,t)+\alpha_2u(x,t)v(x,t)+\alpha_3\hat{u}(x,t)u(x,t)\\ \qquad+\alpha_4u^3(x,t)+\alpha_5u^2(x,t)v(x,t)]+h.o.t.,\\ \dfrac{\partial v}{\partial t}=\tau[d_2\Delta v(x,t)+b_{21}u(x,t-1)+c_{22}\hat{v}(x,t)+\alpha_6\hat{v}(x,t)v(x,t)+\alpha_7v(x,t)u(x,t-1)+\alpha_8u^2(x,t-1)\\ \qquad+\alpha_9u^3(x,t-1)+\alpha_{10}u^2(x,t-1)v(x,t)]+h.o.t.,\end{cases}\tag{15}$$

其中

$$\alpha_1=\frac{E^2hv_*}{(1+Ehu_*)^3},\quad \alpha_2=-\frac{E}{(1+Ehu_*)^2},\quad \alpha_3=-1,\quad \alpha_4=\frac{-E^3h^2v_*}{(1+Ehu_*)^4},\quad \alpha_5=\frac{E^2h}{(1+Ehu_*)^3},$$

$$\alpha_6=-r,\quad \alpha_7=\frac{\alpha rE}{(1+Ehu_*)^2},\quad \alpha_8=-\frac{\alpha rE^2hv_*}{(1+Ehu_*)^3},\quad \alpha_9=\frac{\alpha rE^3h^2v_*}{(1+Ehu_*)^4},\quad \alpha_{10}=-\frac{\alpha rE^2h}{(1+Ehu_*)^3}。$$

定义实值 Sobolev 空间$X:=\left\{(u,v)^{\mathrm{T}}:u,v\in H^2(0,l\pi),\ \dfrac{\partial u}{\partial x}(0,t)=\dfrac{\partial v}{\partial x}(l\pi,t)=0\right\}$,以及相应的复化空间

$$X_C:=X+\mathrm{i}X=\{x_1+\mathrm{i}x_2:x_1,x_2\in X\}。$$

定义内积为$\langle\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}\rangle:=\int_0^{l\pi}(\bar{u}_1v_1+\bar{u}_2v_2)\mathrm{d}x$,其中$\tilde{\boldsymbol{u}}=(u_1,u_2)^{\mathrm{T}},\ \tilde{\boldsymbol{v}}=(v_1,v_2)^{\mathrm{T}}\in X_C$,令$\mathscr{C}:=C([-1,0],X_{\mathrm{C}})$是赋上确界范数的相空间,令$\boldsymbol{\phi}_t\in\mathscr{C}$,其中$\boldsymbol{\phi}_t(\theta)=\boldsymbol{\phi}(t+\theta),\ -1\leqslant\theta\leqslant0$。记$\boldsymbol{\beta}_n^{(1)}=(\gamma_n(x),0)^{\mathrm{T}},\ \boldsymbol{\beta}_n^{(2)}=(0,\gamma_n(x))^{\mathrm{T}}$,$\boldsymbol{\beta}_n=\{\boldsymbol{\beta}_n^{(1)}(x),\boldsymbol{\beta}_n^{(2)}(x)\}$,则$\{\boldsymbol{\beta}_n^{(i)}(x)\}_{n\geqslant0}$为$\boldsymbol{D}\Delta$对应于特征值$-d_i\dfrac{n^2}{l^2}(i=1,2)$的特征函数,并且$\{\boldsymbol{\beta}_n^{(i)}(x)\}_{n\geqslant0}$是$X$的一组规范正交基。

令$\tau=\tilde{\tau}+\mu$,则当$\mu=0$时,系统发生 Hopf 分支,因此,系统(15)可写为

$$\frac{d\boldsymbol{U}(t)}{\mathrm{d}t}=(\tilde{\tau}+\mu)\boldsymbol{D}\Delta\boldsymbol{U}(t)+(\tilde{\tau}+\mu)\boldsymbol{L}(\boldsymbol{U}_t,\hat{\boldsymbol{U}}(t))+\boldsymbol{F}(\boldsymbol{U}_t,\hat{\boldsymbol{U}}(t),\mu),\tag{16}$$

其中

$$\boldsymbol{L}(\boldsymbol{\phi}(\theta),\boldsymbol{\phi}(0))=\boldsymbol{L}_1\boldsymbol{\phi}(0)+\boldsymbol{L}_2\boldsymbol{\phi}(-1)+\boldsymbol{L}_3\hat{\boldsymbol{\phi}}(0),$$

$$\boldsymbol{F}(\boldsymbol{\phi},\mu)=(\tilde{\tau}+\mu)\begin{pmatrix}\alpha_1\phi_1^2(0)+\alpha_2\phi_1(0)\phi_2(0)+\alpha_3\hat{\phi}_1(0)\phi_1(0)+\alpha_4\phi_1^3(0)+\alpha_5\phi_1^2(0)\phi_2(0)\\ \alpha_6\hat{\phi}_2(0)\phi_2(0)+\alpha_7\phi_2(0)\phi_1(-1)+\alpha_8\phi_1^2(-1)+\alpha_9\phi_1^3(-1)+\alpha_{10}\phi_1^2(-1)\phi_2(0)\end{pmatrix},$$

$\boldsymbol{\phi}=(\phi_1,\phi_2)^{\mathrm{T}}\in\mathscr{C}$,且$\hat{\boldsymbol{\phi}}=\dfrac{1}{l\boldsymbol{\pi}}\int_0^{l\pi}\boldsymbol{\phi}\mathrm{d}x$,$\boldsymbol{L}_1$、$\boldsymbol{L}_2$、$\boldsymbol{L}_3$由式(5)定义。

$\pm\mathrm{i}\omega_{n_0}\tilde{\tau}$是以下线性系统(17)和线性泛函微分方程(18)的一对简单纯虚根:

$$\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}=\tilde{\tau}\boldsymbol{D}\Delta\boldsymbol{U}(t)+\tilde{\tau}\boldsymbol{L}(\boldsymbol{U}_t,\hat{\boldsymbol{U}}(0)),\tag{17}$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t}=-\tilde{\tau}\boldsymbol{D}\frac{n^2}{l^2}z(t)+\tilde{\tau}\boldsymbol{L}(\boldsymbol{U}_t,\hat{\boldsymbol{U}}(0))。\tag{18}$$

由 Riesz 表示定理,存在 2×2 的矩阵函数$\boldsymbol{\eta}(\theta,\tilde{\tau})(\theta\in[-1,0])$,使得

$$-\tilde{\tau}\boldsymbol{D}\frac{n^2}{l^2}\boldsymbol{\phi}(0)+\tilde{\tau}\boldsymbol{L}(\boldsymbol{\phi})=\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta,\tilde{\tau})\boldsymbol{\phi}(\theta),\quad \boldsymbol{\phi}\in\mathscr{C}。$$

选择

$$\boldsymbol{\eta}(\theta,\tilde{\tau})=\begin{cases}\tilde{\tau}\left[-\dfrac{n^2}{l^2}\boldsymbol{D}+\boldsymbol{L}_1+\boldsymbol{L}_3\delta(n_0)\right], & \theta=0,\\ 0, & \theta\in(-1,0),\\ -\tilde{\tau}\boldsymbol{L}_2, & \theta=-1,\end{cases}$$

其中 $\boldsymbol{D}$、$\boldsymbol{L}_1$、$\boldsymbol{L}_2$、$\boldsymbol{L}_3$ 由式(5)中定义。

对于 $\boldsymbol{\phi}(\theta)\in C^1([-1,0],\mathbf{R}^2)$,定义 $A(\tilde{\tau})$ 为

$$A(\tilde{\tau})(\boldsymbol{\phi}(\theta))=\begin{cases}\dfrac{\mathrm{d}\boldsymbol{\phi}(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta,0)\boldsymbol{\phi}(\theta), & \theta=0,\end{cases}$$

对于 $\boldsymbol{\psi}=(\psi_1,\psi_2)\in C^1([0,1],(\mathbf{R}^2)^*)$,定义

$$A^*(\boldsymbol{\psi}(s))=\begin{cases}-\dfrac{\mathrm{d}\boldsymbol{\psi}(s)}{\mathrm{d}s}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}\boldsymbol{\psi}(-s)\mathrm{d}\boldsymbol{\eta}(s,0), & s=0。\end{cases}$$

此外,定义双线性形式

$$(\boldsymbol{\psi},\boldsymbol{\phi})=\boldsymbol{\psi}(0)\boldsymbol{\phi}(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\boldsymbol{\psi}(\xi-\theta)\mathrm{d}\boldsymbol{\eta}(\theta,\tilde{\tau})\boldsymbol{\phi}(\xi)\mathrm{d}\xi,\tag{19}$$

其中 $\boldsymbol{\phi}\in\mathscr{C}$, $\boldsymbol{\psi}\in\mathscr{C}^*$。

记 A 为解半群的无穷小生成元,A^* 是 A 在双线性形式(19)下的形式伴随算子,$A(\tilde{\tau})$ 和 A^* 对应于 $\mathrm{i}\omega_{n_0}\tilde{\tau}$ 的特征向量分别是

$$\boldsymbol{p}(\theta)=\boldsymbol{p}(0)\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}=(1,p_1)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta},\quad\theta\in[-1,0]$$

和

$$\boldsymbol{q}(s)=\boldsymbol{q}(0)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}s}=M(1,q_2)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}s},\quad s\in[0,1]。$$

通过计算,可得

$$p_1=\frac{1}{a_{12}}\left[\mathrm{i}\omega_{n_0}+\frac{n^2}{l^2}d_1-a_{11}-c_{11}\delta(n_0)\right],$$

$$q_2=\frac{a_{12}}{\mathrm{i}\omega_{n_0}+(n^2/l^2)d_2-c_{22}\delta(n_0)},$$

$$M=(1+p_1q_2+\tilde{\tau}q_2b_{21}\mathrm{e}^{-i\omega_{n_0}\tilde{\tau}})^{-1}。$$

相空间 $\mathscr{C}$ 可被分解为 $\mathscr{C}=P\oplus Q$,其中

$$P=\{\boldsymbol{p}(\theta)z\gamma_{n_0}+\bar{\boldsymbol{p}}(\theta)\bar{z}\gamma_{n_0}\mid z\in\mathbf{C}\},\quad Q=\{\boldsymbol{\phi}\in\mathscr{C}\mid(\boldsymbol{q}\gamma_{n_0},\boldsymbol{\phi})=0\text{ 且}(\bar{\boldsymbol{q}}\gamma_{n_0},\boldsymbol{\phi})=0\},$$

则系统(16)的解可分解为

$$\boldsymbol{U}_t=[\boldsymbol{p}(\theta)z+\bar{\boldsymbol{p}}(\theta)\bar{z}]\gamma_{n_0}+\boldsymbol{W}(t,\theta),\tag{20}$$

且 $\hat{\boldsymbol{U}}_t=\dfrac{1}{l\pi}\displaystyle\int_0^{l\pi}\boldsymbol{U}_t\mathrm{d}x$,其中 $z(t)=(\boldsymbol{q}\gamma_{n_0},\boldsymbol{U}_t)$, $\boldsymbol{W}(t,\theta)=\boldsymbol{U}_t(\theta)-2\mathrm{Re}\{z(t)\boldsymbol{p}(\theta)\gamma_{n_0}\}$。

存在中心流形,使得

$$\boldsymbol{W}(z,\bar{z})=\boldsymbol{W}_{20}(\theta)\frac{z^2}{2}+\boldsymbol{W}_{11}(\theta)z\bar{z}+\boldsymbol{W}_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots,\tag{21}$$

由此可得

$$\dot{z}(t)=\mathrm{i}\omega_{n_0}\tilde{\tau}z(t)+\boldsymbol{q}(0)\langle\boldsymbol{F}(0,u_t),\boldsymbol{\beta}_{n_0}\rangle。\tag{22}$$

将式(22)重写为

$$\dot{z}(t)=\mathrm{i}\omega_{n_0}\tilde{\tau}z(t)+g(z,\bar{z}),$$

其中

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots。\tag{23}$$

由式(20)、(21),可得

$$u_t(0)=(z+\bar{z})\gamma_{n_0}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+\cdots,$$

$$v_t(0)=(zp_1+\bar{z}\bar{p}_1)\gamma_{n_0}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+\cdots,$$

$$\hat{u}_t(0)=\delta(n_0)[(z+\bar{z})\gamma_{n_0}]+\frac{1}{l\pi}\int_0^{l\pi}W_{20}^{(1)}(0)\mathrm{d}x\frac{z^2}{2}+\frac{1}{l\pi}\int_0^{l\pi}W_{11}^{(1)}(0)\mathrm{d}xz\bar{z}+\frac{1}{l\pi}\int_0^{l\pi}W_{02}^{(1)}(0)\mathrm{d}x\frac{\bar{z}^2}{2}+\cdots,$$

$$\hat{v}_t(0)=\delta(n_0)[(zp_1+\bar{z}\bar{p}_1)\gamma_{n_0}]+\frac{1}{l\pi}\int_0^{l\pi}W_{20}^{(2)}(0)\mathrm{d}x\frac{z^2}{2}+\frac{1}{l\pi}\int_0^{l\pi}W_{11}^{(2)}(0)\mathrm{d}xz\bar{z}+\frac{1}{l\pi}\int_0^{l\pi}W_{02}^{(2)}(0)\mathrm{d}x\frac{\bar{z}^2}{2}+\cdots,$$

$$u_t(-1)=(z\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+\bar{z}\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}})\gamma_{n_0}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}+\cdots。$$

通过计算,得到

$$g_{20}=2\tilde{\tau}M\{\alpha_1+\alpha_2p_1+\alpha_3\delta(n_0)+q_2[\alpha_6p_1^2\delta(n_0)+\alpha_7p_1\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+\alpha_8\mathrm{e}^{-2\mathrm{i}\omega_{n_0}\tilde{\tau}}]\}\int_0^{l\pi}\gamma_{n_0}^3\mathrm{d}x,$$

$$g_{11}=\tilde{\tau}M\{2\alpha_1+\alpha_2(p_1+\bar{p}_1)+2\alpha_3\delta(n_0)+q_2[2\alpha_6p_1\bar{p}_1\delta(n_0)+\alpha_7(p_1\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}+\bar{p}_1\mathrm{e}^{-i\omega_{n_0}\tilde{\tau}})+2\alpha_8]\}\int_0^{l\pi}\gamma_{n_0}^3\mathrm{d}x,$$

$$g_{02}=2\tilde{\tau}M\{\alpha_1+\alpha_2\bar{p}_1+\alpha_3\delta(n_0)+q_2[\alpha_6\bar{p}_1^2\delta(n_0)+\alpha_7\bar{p}_1\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}+\alpha_8\mathrm{e}^{2i\omega_{n_0}\tilde{\tau}}]\}\int_0^{l\pi}\gamma_{n_0}^3\mathrm{d}x,$$

$$\begin{aligned}\frac{g_{21}}{2\tilde{\tau}M}=&\alpha_1\int_0^{l\pi}[2W_{11}^{(1)}(0)+W_{20}^{(1)}(0)]\gamma_{n_0}^2\mathrm{d}x+\alpha_2\int_0^{l\pi}\left[W_{11}^{(2)}(0)+\frac{W_{20}^{(2)}(0)}{2}+\frac{W_{20}^{(1)}(0)}{2}\bar{p}_1+W_{11}^{(1)}(0)p_1\right]\gamma_{n_0}^2\mathrm{d}x\\&+\alpha_3\int_0^{l\pi}\left[\frac{1}{l\pi}\int_0^{l\pi}W_{11}^{(1)}(0)\mathrm{d}x+\frac{1}{l\pi}\int_0^{l\pi}\frac{W_{20}^{(1)}(0)}{2}\mathrm{d}x+\frac{W_{20}^{(1)}(0)}{2}\delta(n_0)+W_{11}^{(1)}(0)\delta(n_0)\right]\gamma_{n_0}^2\mathrm{d}x\\&+q_2\alpha_6\int_0^{l\pi}\left[\frac{1}{l\pi}\bar{p}_1\int_0^{l\pi}\frac{W_{20}^{(2)}(0)}{2}\mathrm{d}x+\frac{1}{l\pi}p_1\int_0^{l\pi}W_{11}^{(2)}(0)\mathrm{d}x+\frac{W_{20}^{(2)}(0)}{2}\bar{p}_1\delta(n_0)+W_{11}^{(2)}(0)p_1\delta(n_0)\right]\gamma_{n_0}^2\mathrm{d}x\\&+q_2\alpha_7\int_0^{l\pi}\left[W_{11}^{(1)}(-1)p_1+\frac{W_{20}^{(1)}(-1)}{2}\bar{p}_1+\frac{W_{20}^{(2)}(0)}{2}\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}+W_{11}^{(2)}(0)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}\right]\gamma_{n_0}^2\mathrm{d}x\\&+q_2\alpha_8\int_0^{l\pi}[2W_{11}^{(1)}(-1)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+W_{20}^{(1)}(-1)\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}]\gamma_{n_0}^2\mathrm{d}x\\&+[3\alpha_4+\alpha_5(\bar{p}_1+2p_1)+3q_2\alpha_9\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+q_2\alpha_{10}(\bar{p}_1\mathrm{e}^{-2\mathrm{i}\omega_{n_0}\tilde{\tau}}+2p_1)]\int_0^{l\pi}\gamma_{n_0}^4\mathrm{d}x,\end{aligned}$$

并且有

$$\int_0^{l\pi}\gamma_{n_0}^3\mathrm{d}x=\begin{cases}\sqrt{\dfrac{1}{l\pi}} & n_0=0,\\ 0 & n_0\neq 0。\end{cases}$$

所以当 $n\neq 0$ 时,有 $g_{20}=g_{11}=g_{02}=0$。为了计算 g_{21},还需要计算 $\boldsymbol{W}_{20}(\theta)$ 和 $\boldsymbol{W}_{11}(\theta)$。

$\boldsymbol{W}(z(t),\bar{z}(t))$ 满足

$$\begin{aligned}\dot{\boldsymbol{W}}&=\dot{\boldsymbol{u}}_t-\dot{z}\boldsymbol{p}-\dot{\bar{z}}\bar{\boldsymbol{p}}\\&=\begin{cases}A\boldsymbol{W}-2\,\mathrm{Re}\{g(z,\bar{z})\boldsymbol{p}(\theta)\}\gamma_{n_0}, & \theta\in[-1,0),\\ A\boldsymbol{W}-2\,\mathrm{Re}\{g(z,\bar{z})\boldsymbol{p}(\theta)\}\gamma_{n_0}+\boldsymbol{F}, & \theta=0,\end{cases}\\&\triangleq A\boldsymbol{W}+\boldsymbol{H}(z,\bar{z},\theta),\end{aligned}\tag{24}$$

其中

$$\boldsymbol{H}(z,\bar{z})=\boldsymbol{H}_{20}(\theta)\frac{z^2}{2}+\boldsymbol{H}_{11}(\theta)z\bar{z}+\boldsymbol{H}_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots。$$

利用 $\dot{\boldsymbol{W}}=\dfrac{\partial \boldsymbol{W}(z,\bar{z})}{\partial z}\dot{z}+\dfrac{\partial \boldsymbol{W}(z,\bar{z})}{\partial \bar{z}}\dot{\bar{z}}$,得到

$$\begin{cases}(2\mathrm{i}\omega_{n_0}\tilde{\tau}-A)\boldsymbol{W}_{20}=\boldsymbol{H}_{20},\\ -A_U\boldsymbol{W}_{11}=\boldsymbol{H}_{11},\\ (-2\mathrm{i}\omega_{n_0}\tilde{\tau}-A)\boldsymbol{W}_{02}=\boldsymbol{H}_{02}。\end{cases}\tag{25}$$

由式(23)、(24)得到,当$-1\leqslant\theta<0$时,

$$\boldsymbol{H}_{20}(\theta)=-[g_{20}\boldsymbol{p}(\theta)+\bar{g}_{02}\bar{\boldsymbol{p}}(\theta)]\gamma_{n_0},$$
$$\boldsymbol{H}_{11}(\theta)=-[g_{11}\boldsymbol{p}(\theta)+\bar{g}_{11}\bar{\boldsymbol{p}}(\theta)]\gamma_{n_0},$$

由式(25)可得

$$\boldsymbol{W}_{20}(\theta)=\boldsymbol{E}_1\mathrm{e}^{2\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}+\frac{-g_{20}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}\boldsymbol{p}(0)\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\gamma_{n_0}-\frac{\bar{g}_{02}}{3\mathrm{i}\omega_{n_0}\tilde{\tau}}\bar{\boldsymbol{p}}(0)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\gamma_{n_0},$$

$$\boldsymbol{W}_{11}(\theta)=\boldsymbol{E}_2+\frac{g_{11}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}\boldsymbol{p}(0)\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\gamma_{n_0}-\frac{\bar{g}_{11}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}\bar{\boldsymbol{p}}(0)\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\gamma_{n_0}。$$

由式(25),当$\theta=0$时,

$$\boldsymbol{H}_{20}(0)=-[g_{20}\boldsymbol{p}(0)+\bar{g}_{02}\bar{\boldsymbol{p}}(0)]\gamma_{n_0}+2\tilde{\tau}\gamma_{n_0}^2\begin{pmatrix}\alpha_1+\alpha_2p_1+\alpha_3\delta(n_0)\\ \alpha_6p_1^2\,\delta(n_0)+\alpha_7p_1\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+\alpha_8\mathrm{e}^{-2\mathrm{i}\omega_{n_0}\tilde{\tau}}\end{pmatrix},$$

$$\boldsymbol{H}_{11}(0)=-[g_{11}\boldsymbol{p}(0)+\bar{g}_{11}\bar{\boldsymbol{p}}(0)]\gamma_{n_0}+\tilde{\tau}\gamma_{n_0}^2\begin{pmatrix}2\alpha_1+\alpha_2(p_1+\bar{p}_1)+2\alpha_3\delta(n_0)\\ 2\alpha_6p_1\bar{p}_1\delta(n_0)+\alpha_7(p_1\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}+\bar{p}_1\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}})+2\alpha_8\end{pmatrix},$$

得到

$$\boldsymbol{E}_1=\left[2\mathrm{i}\omega_{n_0}\tilde{\tau}\boldsymbol{I}-\int_{-1}^{0}\mathrm{e}^{2\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\mathrm{d}\boldsymbol{\eta}_{n_0}(\theta,\tilde{\tau})\right]^{-1}\begin{pmatrix}2\alpha_1+2\alpha_2p_1+2\alpha_3\delta(n_0)\\ 2\alpha_6p_1^2\,\delta(n_0)+2\alpha_7p_1\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}}+2\alpha_8\mathrm{e}^{-2\mathrm{i}\omega_{n_0}\tilde{\tau}}\end{pmatrix}\tilde{\tau}\gamma_{n_0}^2,$$

$$\boldsymbol{E}_2=-\left[\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}_{n_0}(\theta,\tilde{\tau})\right]^{-1}\begin{pmatrix}2\alpha_1+\alpha_2(p_1+\bar{p}_1)+2\alpha_3\delta(n_0)\\ 2\alpha_6p_1\bar{p}_1\delta(n_0)+\alpha_7(p_1\mathrm{e}^{\mathrm{i}\omega_{n_0}\tilde{\tau}}+\bar{p}_1\mathrm{e}^{-\mathrm{i}\omega_{n_0}\tilde{\tau}})+2\alpha_8\end{pmatrix}\tilde{\tau}\gamma_{n_0}^2。$$

从而,可以计算 g_{21},进而得到

$$c_1(0)=\frac{\mathrm{i}}{2\omega_{n_0}\tilde{\tau}}\left[g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right]+\frac{g_{21}}{2},\quad \mu_2=-\frac{\operatorname{Re}c_1(0)}{\operatorname{Re}\lambda'(\tilde{\tau})},$$

$$T_2=-\frac{\operatorname{Im}c_1(0)+\mu_2\operatorname{Im}\lambda'(\tilde{\tau})}{\omega_{n_0}\tilde{\tau}},\quad \beta_2=2\operatorname{Re}(c_1(0))。$$

作为文献[22]中结果的直接应用,得到以下定理。

定理 4　对于系统(3),有

(1) μ_2 决定 Hopf 分支的方向,当$\mu_2>0(\mu_2<0)$时,分支方向是超临界的(亚临界的);

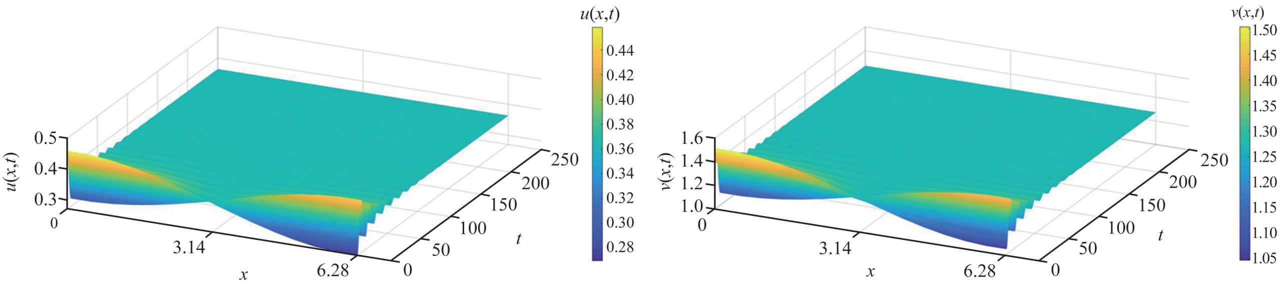
(2) β_2 决定分支周期解的稳定性,当$\beta_2<0(\beta_2>0)$时,分支周期解是渐近稳定的(不稳定的);

(3) T_2 决定分支周期解的周期,当$T_2>0(T_2<0)$时,周期增大(减小)。

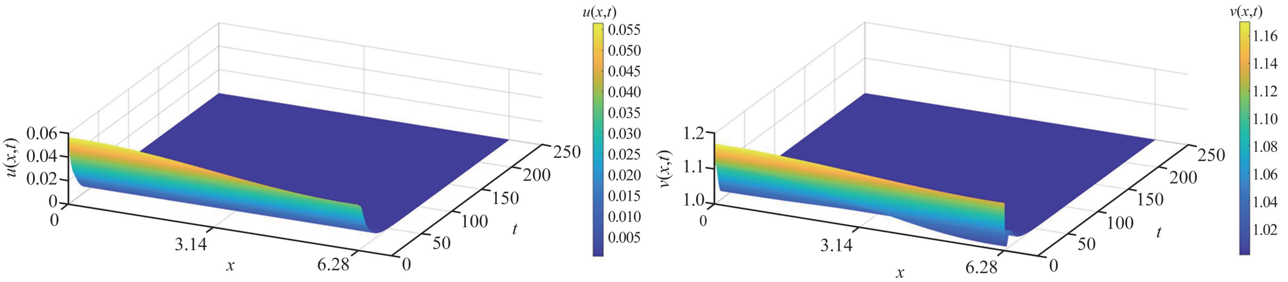
3　数值模拟

取参数$E=1.1$、$h=3$、$\alpha=1.5$、$r=3$、$d_1=0.8$、$d_2=0.8$、$l=2$,可验证(H_0)成立,且$E_*=(0.363\,6,1.272\,7)$。由引理1、2,可得$\tilde{\tau}=\tau_1^0\approx0.223\,8$。由第2章规范型公式,得到$c_1(0)\approx-2.106\,3+0.076\,0i$,$\mu_2\approx17.996\,6$,$\beta_2\approx-4.212\,6$,因此,当$\tau$经过$\tilde{\tau}$时,系统发生 Hopf 分支,其方向是超临界的,分支周期解是渐近稳定的。

由定理3可知,当$\tau\in[0,\tilde{\tau})=[0,0.223\,8)$时,正平衡点$E_*$是局部渐近稳定的。由定理1可知,平衡点$E_2$是局部渐近稳定的,因此,当$\tau\in[0,\tilde{\tau})$时,无食饵边界平衡点$E_2(0,1)$与正平衡点$E_*$都局部渐近稳定,即双稳。当选取初值为$u_0(x,t)=0.363\,6+0.1\cos0.5x$,$v_0(x,t)=1.272\,7+0.1\cos0.5x$时,解趋向于$E_*$,此时食饵与捕食者共存,如图1(a)、(b)所示;当选取初值为$u_0'(x,t)=0.05+0.01\cos0.5x$,$v_0'(x,t)=1.272\,7+0.01\cos0.5x$时,解趋向于$E_2(0,1)$,如图1(c)、(d)所示,此时食饵(即入侵物种)被清除。



(a) 初值为$u_0(x,t)$、$v_0(x,t)$时, $u(x,t)$的解曲面　　(b) 初值为$u_0(x,t)$、$v_0(x,t)$时, $v(x,t)$的解曲面

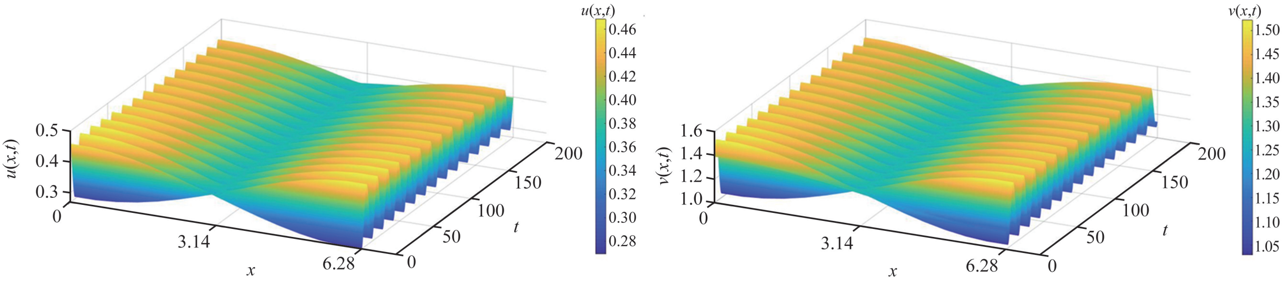


(c) 初值为$u'_0(x,t)$、$v'_0(x,t)$时, $u(x,t)$的解曲面　　(d) 初值为$u'_0(x,t)$、$v'_0(x,t)$时, $v(x,t)$的解曲面

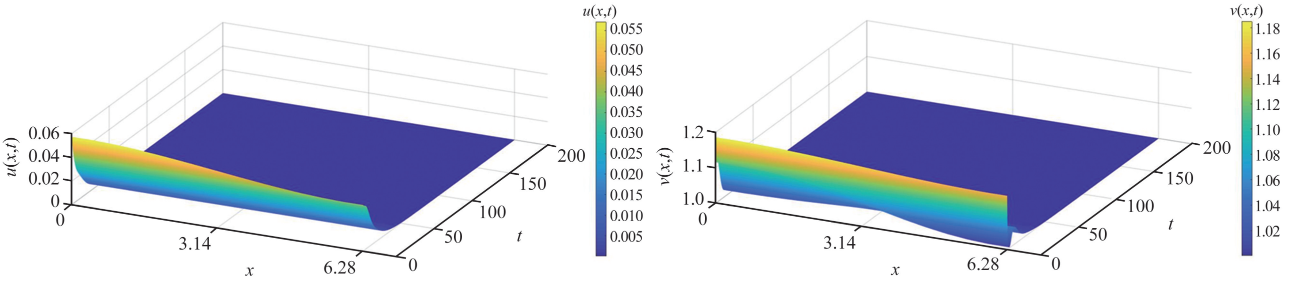
图 1　当 $\tau=0.15<\tilde{\tau}$ 时,系统(3)存在双稳态:无食饵边界平衡点 $E_2(0,1)$ 与正平衡点 E_* 双稳

Fig.1　System (3) has bistable states: the prey-free equilibrium $E_2(0,1)$ and the positive equilibrium E_* are both stable when $\tau=0.15<\tilde{\tau}$

当 $\tau=0.5>\tilde{\tau}$ 时,正平衡点不稳定,空间非齐次周期解轨道渐近稳定。此时系统(3)存在双稳态:无食饵边界平衡点 $E_2(0,1)$ 与一个周期解双稳。当选取初值为 $u_0(x,t)=0.3636+0.1\cos 0.5x$, $v_0(x,t)=1.2727+0.1\cos 0.5x$时,解趋于分支周期解,如图 2(a)、(b)所示,这意味着食饵与捕食者以振荡的方式共存;当选取初值为 $u'_0(x,t)=0.05+0.01\cos 0.5x$, $v'_0(x,t)=1.2727+0.01\cos 0.5x$ 时,解趋向于 $E_2(0,1)$,如图 2(c)、(d)所示,此时食饵(即入侵物种)被清除。



(a) 初值为$u_0(x,t)$、$v_0(x,t)$时, $u(x,t)$的解曲面　　(b) 初值为$u_0(x,t)$、$v_0(x,t)$时, $v(x,t)$的解曲面



(c) 初值为$u'_0(x,t)$、$v'_0(x,t)$时, $u(x,t)$的解曲面　　(d) 初值为$u'_0(x,t)$、$v'_0(x,t)$时, $v(x,t)$的解曲面

图 2　当 $\tau=0.5>\tilde{\tau}$ 时,系统(3)存在双稳态:无食饵边界平衡点 $E_2(0,1)$ 与一个周期解双稳

Fig.2　System (3) has bistable states: the prey-free equilibrium $E_2(0,1)$ and a periodic solution are both stable when $\tau=0.5>\tilde{\tau}$

4 结语

通过对具有非局部竞争和妊娠时滞的广食性捕食者-食饵模型的研究,得到了正平衡点的稳定性和Hopf分支的存在性条件,并得到了系统在发生Hopf分支时的时滞临界值。利用中心流形定理和规范型理论,确定了分支方向和分支周期解的稳定性,最后,通过数值模拟探究了$E>1$时系统在不同时滞下的动力学行为。$E_2(0,1)$的稳定性不受非局部竞争及时滞的影响,总是局部渐近稳定。时滞会对正平衡点的稳定性产生影响。当时滞值小于临界值时,正平衡点是稳定的,食饵和捕食者共存。而当时滞值大于临界值时,正平衡点不稳定,产生了稳定的周期解,此时捕食者和食饵以振荡的形式共存。

引入非局部竞争后,时滞所产生的分支周期解为空间非齐次的,这与局部模型中,一般只能出现空间齐次周期解的情况有明显区别。当时滞小于临界值时,系统存在双稳态,即无食饵边界平衡点$E_2(0,1)$和正平衡点E_*双稳,入侵物种被清除或入侵成功,它取决于入侵物种的初始数量;当时滞大于临界值时,系统存在双稳态,即无食饵边界平衡点$E_2(0,1)$和一个周期解双稳,入侵物种被消除或以振荡的方式存活,这也取决于其初始数量。

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