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区间犹豫梯形模糊 Choquet Muirhead 平均算子及其在多属性决策中的应用

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摘要:为了保证决策信息的完整性,将 Muirhead 平均算子运用到区间犹豫梯形模糊环境中,提出了区间犹豫梯形模糊 Muirhead 平均算子、区间犹豫梯形模糊几何 Muirhead 平均算子的概念。给出了它们的性质及定理,并介绍了算子的几种退化形式。考虑到 Choquet 积分可以客观的计算属性权重,将 Choquet 积分引入上述算子,提出了区间犹豫梯形模糊 Choquet Muirhead 平均算子和区间犹豫梯形模糊几何 Choquet Muirhead 平均算子的概念。同时构造了基于区间犹豫梯形模糊 Choquet Muirhead 平均算子的多属性决策模型。最后,将其应用到绿色电池供应商选择问题中,分析其可行性和有效性。

关键词:区间犹豫梯形模糊集; Choquet 积分; Muirhead 平均算子;区间犹豫梯形模糊 Choquet Muirhead 平均算子;多属性决策

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Interval hesitant trapezoidal fuzzy Choquet Muirhead mean operator and its application in multi-attribute decision making

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Abstract: In order to ensure the integrity of decision information, we extend the Muirhead mean operator in the environment of interval hesitant trapezoidal fuzzy sets. The concepts of interval hesitant trapezoidal fuzzy Muirhead mean operator and interval hesitant trapezoidal fuzzy geometric Muirhead mean operator have been proposed. Their properties and theorems are given, and several degenerate forms of the operators are introduced. Considering that the Choquet integral can objectively calculate the weights of attributes, this paper introduces the Choquet integral into the above operators, and proposes the concepts of interval hesitant trapezoidal fuzzy Choquet Muirhead mean operator and interval hesitant trapezoidal fuzzy geometric Choquet Muirhead mean operator. Meanwhile, a multi-attribute decision making model based on interval hesitant trapezoidal fuzzy Choquet Muirhead mean operator is constructed. Finally, it was applied to the green battery supplier selection problem to analyze its feasibility and effectiveness.

Key words: interval hesitant trapezoidal fuzzy set; Choquet integral; Muirhead mean operator; interval hesitant trapezoidal fuzzy Choquet Muirhead mean operator; multi-attribute decision making

0　引言

多属性决策存在于日常社会经济生活各个层面,实际生活中由于决策问题的复杂性以及决策者自身能

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力的局限性,单纯使用实数信息已不能满足决策过程中的信息表述,为解决这一问题 Zadeh[1] 首次提出了模糊集理论,并用它来处理模糊决策问题。其后,模糊理论得到学者们的广泛关注,相继提出了直觉模糊集、区间模糊集、二型模糊集、语言模糊集等概念。在决策过程中决策者可能会在多个评价信息中犹豫不决,为解决这类问题,Torrra[2] 提出了犹豫模糊集的概念,因为其采用多个隶属度来描述不确定信息,保证了决策信息的完整性。考虑到实际决策过程中,[0,1]区间数比精确数在描述信息时更加灵活,因此 Chen[3] 将区间数与犹豫模糊数相结合提出了区间犹豫模糊集的概念。Zhang 等[4] 在此基础上将犹豫模糊数和区间梯形模糊数相结合,提出了区间犹豫梯形模糊集的概念,进一步扩展了决策信息的完整性与灵活性。

处理多属性决策问题往往会涉及到信息的集结,因此,信息集结算子的研究也受到了广大学者的关注,先后提出有序加权平均算子、诱导有序加权平均算子、连续区间有序加权集结算子、不确定广义集结算子、梯形直觉模糊幂平均算子、梯形直觉模糊幂几何平均算子等[5-11]。上述算子都是建立在各属性间相互独立的基本假设之上,没有考虑到各属性之间的相关关系,而实际决策过程中属性之间一般不满足相互独立。考虑到属性间的相关关系可能会影响决策结果,Bonferroni 算子[12] 和 Heronian 算子[13] 被先后提出,它们虽然能够处理属性间相互关联的情况,但是仅仅考虑了两两属性间的相关关系,仍然具有一定的局限性。在实际情况中多个属性间的相关关系也十分重要, Muirhead 平均算子[14] 恰好可以满足这一需求,它能够处理任意多个变量之间的关联关系。此外,Muirhead 算子还可以退化为几何平均算子、Bonferroni 平均算子、Maclaurin 对称平均算子等经典算子,其一般性更强,成为学者们近期关注的热点。Zang 等[15] 提出了 q 阶正交犹豫模糊幂 Muirhead 平均算子,并将其应用于购买社会工作服务项目的选择问题中。Yang 等[16] 定义了区间 q 阶正交模糊加权幂 Muirhead 平均算子的概念,并将其应用于垃圾处理厂选址方案评估问题中。任耀军等[17] 定义了毕达哥拉斯三角犹豫模糊 Muirhead 平均算子的概念,并将其应用于军事领域的目标威胁评估中。Punetha 等[18] 定义了图模糊幂 Muirhead 平均算子的概念,并将其应用于多属性决策问题中。Raiha 等[19] 定义了中智优先 Muirhead 平均算子和中智优先对偶 Muirhead 平均算子的概念,并将其应用于房屋建筑材料公司选择问题中。上述文献将 Muirhead 算子运用到犹豫模糊环境中,与 q 阶正交犹豫模糊数、毕达哥拉斯三角犹豫模糊数、图模糊数、中智数等模糊信息相结合提出了新的 Muirhead 算子,但未考虑到区间梯形犹豫模糊数的情形。

实际决策问题中属性之间往往存在优先关系。为得到合理的决策结果,此时的信息融合方式就需要能够反映属性间的优先关系。由于存在这种交互关系,造成决策属性间的权重可加性不再合理,而 Choquet 积分[20] 能够较好地反映属性之间的优先关系,在含有相互依赖关系的决策问题中得到广泛的应用,一些专家将 Choquet 积分与现有算子相结合,提出了新的集结算子。如罗世华等[21] 将 Choquet 积分与 Bonferroni 平均算子相融合,提出梯形直觉模糊 Choquet Bonferroni 调和均值算子的概念,刘超等[22] 将 Choquet 积分与 Banzhaf 函数相结合,提出了区间对偶犹豫不确定语言广义 Banzhaf Choquet 积分算子的概念。由于模糊集理论的快速发展,一部分学者还将 Choquet 积分的研究拓展到模糊环境中,提出了基于三角直觉模糊[23-24]、诱导直觉模糊[25]、单值中智犹豫模糊[26]、区间值直觉犹豫模糊[27] 和毕达哥拉斯模糊[28] 等模糊背景下的 Choquet 积分算子,并将其应用于高等教育评估[29]、互联网项目投资[30]、能源存储策略[31]、共享单车投资[32] 等各类涉及到决策目标属性存在相互依赖关系的实际问题中。

考虑到 Muirhead 算子能同时处理多个属性之间的关系,并且 Choquet 积分能够充分体现属性间的优先性,因此本文将 Muirhead 算子与 Choquet 积分相融合,提出了区间犹豫梯形模糊 Choquet Muirhead 平均算子的概念,并基于该算子构建了决策模型,同时进行了实证分析。

1　基本理论

1.1　区间犹豫梯形模糊集

定义 1[4]　设 X 为非空集合,$D[0,1]$ 是区间 $[0,1]$ 的幂集,X 上的映射 $h_A(x_i):X\to D[0,1]$,若 $h_A(x_i)=\{\rho\mid\rho\in h_A(x_i)\}$ 表示 $x_i\in X$ 的所有可能的区间值隶属度,则称 $A=\{\langle x_i,h_A(x_i)\rangle\mid x_i\in X,\ i=1,2,\cdots,n\}$ 为区间犹豫梯形模糊集,记为 IVHTrFSs。$\rho=[\rho^-,\rho^+]$ 表示一个区间数,$\rho^-=(\alpha^L,\beta^L,\gamma^L,\mu^L)$,$\rho^+=(\alpha^U,\beta^U,\gamma^U,\mu^U)$ 分别表示 ρ 的左右边界。$0\leqslant\alpha^L\leqslant\beta^L\leqslant\gamma^L\leqslant\mu^L\leqslant1$,$0\leqslant\alpha^U\leqslant\beta^U\leqslant\gamma^U\leqslant\mu^U\leqslant1$ 且 $\alpha^L\leqslant\alpha^U$,$\beta^L\leqslant\beta^U$,$\gamma^L\leqslant\gamma^U$,

$\mu^L \leqslant \mu^U$。

为方便起见,将 $f = h_A(x_i) = \{[(\alpha^L, \beta^L, \gamma^L, \mu^L), (\alpha^U, \beta^U, \gamma^U, \mu^U)]\}$ 称为区间梯形犹豫模糊数元,表示由 $[0,1]$ 上所有可能的区间梯形模糊数构成的集合,记为 IVHTrFE。

定义 2[4] 区间犹豫梯形模糊元的运算法则。

设 $f=\{[(\alpha^L,\beta^L,\gamma^L,\mu^L),(\alpha^U,\beta^U,\gamma^U,\mu^U)]\}$, $f_1=\{[(\alpha_1^L,\beta_1^L,\gamma_1^L,\mu_1^L),(\alpha_1^U,\beta_1^U,\gamma_1^U,\mu_1^U)]\}$, $f_2=\{[(\alpha_2^L,\beta_2^L,\gamma_2^L,\mu_2^L),(\alpha_2^U,\beta_2^U,\gamma_2^U,\mu_2^U)]\}$ 是 3 个区间犹豫梯形模糊元,其运算法则为

$$f_1 \oplus f_2 = \{[(\alpha_1^L+\alpha_2^L-\alpha_1^L\alpha_2^L, \beta_1^L+\beta_2^L-\beta_1^L\beta_2^L, \gamma_1^L+\gamma_2^L-\gamma_1^L\gamma_2^L, \mu_1^L+\mu_2^L-\mu_1^L\mu_2^L), (\alpha_1^U+\alpha_2^U-\alpha_1^U\alpha_2^U, \beta_1^U+\beta_2^U-\beta_1^U\beta_2^U, \gamma_1^U+\gamma_2^U-\gamma_1^U\gamma_2^U, \mu_1^U+\mu_2^U-\mu_1^U\mu_2^U)]\};$$

$$f_1 \otimes f_2 = \{[(\alpha_1^L\alpha_2^L, \beta_1^L\beta_2^L, \gamma_1^L\gamma_2^L, \mu_1^L\mu_2^L), (\alpha_1^U\alpha_2^U, \beta_1^U\beta_2^U, \gamma_1^U\gamma_2^U, \mu_1^U\mu_2^U)]\};$$

$$f^k = \{[((\alpha^L)^k, (\beta^L)^k, (\gamma^L)^k, (\mu^L)^k, ((\alpha^U)^k, (\beta^U)^k, (\gamma^U)^k, (\mu^U)^k)]\};$$

$$kf = \{[(1-(1-\alpha^L)^k, 1-(1-\beta^L)^k, 1-(1-\gamma^L)^k, 1-(1-\mu^U)^k), (1-(1-\alpha^U)^k, 1-(1-\beta^U)^k, 1-(1-\gamma^U)^k, 1-(1-\mu^U)^k)]\}。$$

定义 3[33] 设 $f=\{\rho|\rho\in f\}$ 是一个区间犹豫梯形模糊元, $\rho=[(\alpha^L,\beta^L,\gamma^L,\mu^L),(\alpha^U,\beta^U,\gamma^U,\mu^U)]$, l_f 为区间犹豫梯形模糊元中 ρ 的个数。令 $\rho^L=\dfrac{\alpha^L+\beta^L+\gamma^L+\mu^L}{4}$, $\rho^U=\dfrac{\alpha^U+\beta^U+\gamma^U+\mu^U}{4}$,则 f 的得分函数和偏差函数分别为

$$S(f) = \frac{\sum_{\rho\in f}(\rho^L+\rho^U)}{2l_f}, \tag{1}$$

$$D(f) = \left(\frac{1}{l_f}\sum_{\rho\in f}\left(\frac{(\rho^L+\rho^U)}{2} - S(f)\right)^2\right)^{\frac{1}{2}}。 \tag{2}$$

给出 2 个区间犹豫梯形模糊元 f_1、f_2。

当 $S(f_1)>S(f_2)$ 时, $f_1>f_2$;反之, $f_1<f_2$。

当 $S(f_1)=S(f_2)$ 时,若 $D(f_1)>D(f_2)$,则 $f_1<f_2$;若 $D(f_1)<D(f_2)$,则 $f_1>f_2$;若 $D(f_1)=D(f_2)$,则 $f_1=f_2$。

1.2 模糊测度

定义 4[34] 设 $\zeta(Y)$ 为 $Y=\{y_1,y_2,\cdots,y_n\}$ 的幂集,映射 $\upsilon:\zeta(Y)\to[0,1]$,若满足以下条件:

(1) $\upsilon(\varnothing)=0$, $\upsilon(Y)=1$;

(2) 若 $\beta\subseteq\gamma$ 且 β, $\gamma\in\zeta(Y)$,则 $\upsilon(\beta)\leqslant\upsilon(\gamma)$;

则称 υ 为定义在 Y 上的模糊测度。

为了提高模糊测度计算的可行性,Sugeno 提出了 τ 模糊测度。

定义 5[34] 给定 $\tau\in(-1,+\infty)$,若在满足定义 4 条件的基础上还满足 $\forall\beta$, $\gamma\in\zeta(Y)$, $\beta\cap\gamma=\varnothing$,有 $\upsilon_\tau(\beta\cup\gamma)=\upsilon_\tau(\beta)+\upsilon_\tau(\gamma)+\tau\upsilon_\tau(\beta)\upsilon_\tau(\gamma)$,则称 υ_τ 为 Y 上的 τ 模糊测度。

若 Y 是某个多属性决策的指标集合,则 $\forall\beta$, $\gamma\in\zeta(Y)$, $\upsilon_\tau(\beta)$, $\upsilon_\tau(\gamma)$ 可认为是指标子集 β,γ 的权重。同时,对于有限集合 Y,有

$$\upsilon_\tau(Y) = \frac{1}{\tau}\left(\prod_{i=1}^{n}(1+\tau\upsilon_\tau(y_i))-1\right)。 \tag{3}$$

特别地,当 $\upsilon_\tau(Y)=1$ 时,可以根据公式(3)求得

$$\tau = \prod_{i=1}^{n}(1+\tau\upsilon_\tau(y_i))-1。 \tag{4}$$

1.3 Choquet 积分

定义 6[35] 设 f 为定义在 $Y=\{y_1,y_2,\cdots,y_n\}$ 上的非负函数,υ 是 Y 上的模糊测度,则 f 关于 υ 的离散 Choquet 积分为

$$\int f\mathrm{d}\upsilon = \sum_{i=1}^{n} f(y_{(i)})[\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})], \tag{5}$$

其中, $f(y_{(i)})$ 是 $f(y_i)$ 的置换,满足 $f(y_{(1)})\leqslant f(y_{(2)})\leqslant\cdots\leqslant f(y_{(n)})$, $Y_{(i)}=\{y_{(i)},y_{(i+1)},\cdots,y_{(n)}\}$ 且 $Y_{(n+1)}=0$。

Choquet 积分对 $f(y_i)$ 升序排列后再加权关联集结,集结时的位置权重为

$$\omega_{(i)}=\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})。\tag{6}$$

1.4 MM^K 算子

定义 7[14]　设 $b_i(i=1,2\cdots,n)$ 是一组非负实数,$K=(k_1,k_2,\cdots,k_n)$ 是一组参数向量,若

$$\mathrm{MM}^{\mathrm{K}}(b_1,b_2,\cdots,b_n)=\left(\frac{1}{n!}\sum_{\sigma\in s_n}\prod_{i=1}^{n}b_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\tag{7}$$

则称 MM^{K} 为 Muirhead 平均算子,简记为 MM^{K} 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。

1.5 GMM^K 算子

定义 8[36]　设 $c_i(1,2,\cdots,n)$ 是一组非负实数,$\boldsymbol{K}=(k_1,k_2,\cdots,k_n)$ 是一组参数向量,若

$$\mathrm{GMM}^{K}(c_1,c_2,\cdots,c_n)=\frac{1}{\sum_{i=1}^{n}k_i}\left(\prod_{\sigma\in s_n}\sum_{i=1}^{n}k_i\times c_{\sigma(i)}\right)^{\frac{1}{n!}}\tag{8}$$

则称 GMM^K 为几何 Muirhead 算子,简记为 GMM^K 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。

2 区间犹豫梯形模糊 Muirhead 平均算子

2.1 $IVHTrFMM^K$ 算子

定义 9　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$ 为一组区间犹豫梯形模糊元,$\boldsymbol{K}=(k_1,k_2,\cdots,k_n)$ 是一组参数向量。设 $\mathrm{IVHTrFMM}^K:\Theta^n\to\Theta$,若

$$\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n}f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\tag{9}$$

则称 $\mathrm{IVHTrFMM}^K$ 为区间犹豫梯形模糊 Muirhead 算术平均算子,简记为 $\mathrm{IVHTrFMM}^K$ 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。

定理 1　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$ 为一组区间犹豫梯形模糊元,$\boldsymbol{K}=(k_1,k_2,\cdots,k_n)$ 为一组参数向量,则

$$\begin{aligned}&\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)\\=&\left\{\left[\begin{array}{l}\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\\ \left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right),\\ \left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\\ \left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right)\end{array}\right]\right\},\end{aligned}$$

其中,$\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。

证明　因为 $f_{\sigma(i)}^{k_i}=\left\{[((\alpha_{\sigma(i)}^L)^{k_i},(\beta_{\sigma(i)}^L)^{k_i},(\gamma_{\sigma(i)}^L)^{k_i},(\mu_{\sigma(i)}^L)^{k_i}),((\alpha_{\sigma(i)}^U)^{k_i},(\beta_{\sigma(i)}^U)^{k_i},(\gamma_{\sigma(i)}^U)^{k_i},(\mu_{\sigma(i)}^U)^{k_i})]\right\}$,由定义 2 可知

$$\begin{aligned}\bigotimes_{i=1}^{n}f_{\sigma(i)}^{k_i}=\Bigg\{\Bigg[&\left(\prod_{i=1}^{n}(\alpha_{\sigma(i)}^L)^{k_i},\prod_{i=1}^{n}(\beta_{\sigma(i)}^L)_i^k,\prod_{i=1}^{n}(\gamma_{\sigma(i)}^L)^{k_i},\prod_{i=1}^{n}(\mu_{\sigma(i)}^L)^{k_i}\right),\\&\left(\prod_{i=1}^{n}(\alpha_{\sigma(i)}^U)^{k_i},\prod_{i=1}^{n}(\beta_{\sigma(i)}^U)^{k_i},\prod_{i=1}^{n}(\gamma_{\sigma(i)}^U)^{k_i},\prod_{i=1}^{n}(\mu_{\sigma(i)}^U)^{k_i}\right)\Bigg]\Bigg\},\end{aligned}$$

进一步可得

$$\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n} f_{\sigma(i)}^{k_i}=\left\{\left[\begin{array}{l}
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{L})^{k_i}\right)\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{L})^{k_i}\right)\right),\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{L})^{k_i}\right)\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{L})^{k_i}\right)\right)\right),\\
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{U})^{k_i}\right)\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{U})^{k_i}\right)\right),\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{U})^{k_i}\right)\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{U})^{k_i}\right)\right)\right)
\end{array}\right]\right\},$$

所以

$$\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n} f_{\sigma(i)}^{k_i}$$

$$=\left\{\left[\begin{array}{l}
\left(\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{L})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{L})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\right.\\
\left.\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{L})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{L})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right)\right),\\
\left(\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{U})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{U})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\right.\\
\left.\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{U})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right),\left(1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{U})^{k_i}\right)\right)\right)^{\frac{1}{n!}}\right)\right)
\end{array}\right]\right\}$$

$$=\left\{\left[\begin{array}{l}
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right),\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)\right),\\
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right),\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right),\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)\right)
\end{array}\right]\right\}。$$

故

$$\mathrm{IVHTrFMM}^{K}(f_1,f_2,\cdots,f_n)=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n} f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}$$

$$=\left\{\left[\begin{array}{l}
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right),\\
\left(\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\\
\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right)
\end{array}\right]\right\}。$$

证毕。

2.1.1 IVHTrFMMk 算子的相关性质

（1）幂等性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元，如果有 $f_i=f_0=\{[(\alpha_0^L,\beta_0^L,\gamma_0^L,\mu_0^L),(\alpha_0^U,\beta_0^U,\gamma_0^U,\mu_0^U)]\}$，则 $\mathrm{IVHTrFMM}^{K}(f_1,f_2,\cdots,f_n)=f_0$。

证明　由定理 1 可知

$$\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n}f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}}=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n}f_0^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}}$$

$$=\left\{\left[\begin{pmatrix}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_0^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_0^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_0^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_0^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{pmatrix},\begin{pmatrix}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_0^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\beta_0^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\gamma_0^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\mu_0^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{pmatrix}\right]\right\}$$

$$=\left\{\left[\begin{pmatrix}\left(1-\left(1-\prod_{i=1}^{n}(\alpha_0^L)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\left(1-\prod_{i=1}^{n}(\beta_0^L)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\left(1-\prod_{i=1}^{n}(\gamma_0^L)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\left(1-\prod_{i=1}^{n}(\mu_0^L)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{pmatrix},\begin{pmatrix}\left(1-\left(1-\prod_{i=1}^{n}(\alpha_0^U)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\left(1-\prod_{i=1}^{n}(\beta_0^U)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\left(1-\prod_{i=1}^{n}(\gamma_0^U)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(1-\left(1-\prod_{i=1}^{n}(\mu_0^U)^{k_i}\right)\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{pmatrix}\right]\right\}$$

$$=\left\{\left[\left(\left(\prod_{i=1}^{n}(\alpha_0^L)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\beta_0^L)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\gamma_0^L)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\mu_0^L)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\right),\left(\left(\prod_{i=1}^{n}(\alpha_0^U)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\beta_0^U)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\gamma_0^U)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\left(\prod_{i=1}^{n}(\mu_0^U)^{k_i}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\right)\right]\right\}$$

$$=\{[(\alpha_0^L,\beta_0^L,\gamma_0^L,\mu_0^L),(\alpha_0^U,\beta_0^U,\gamma_0^U,\mu_0^U)]\}=f_0。$$

证毕。

(2) 有界性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元,若记

$$f_i^+=\{[(\max\alpha_i^L,\ \max\beta_i^L,\ \max\gamma_i^L,\ \max\mu_i^L),(\max\alpha_i^U,\ \max\beta_i{}^U,\ \gamma_i^U,\ \mu_i^U)]\},$$

$$f_i^-=\{[(\min\alpha_i^L,\ \min\beta_i^L,\ \min\gamma_i^L,\ \min\mu_i^L),\ (\min\alpha_i^U,\ \min\beta_i^U,\ \min\gamma_i^U,\ \min\mu_i^U)]\},$$

则 $\mathrm{IVHTrFMM}^K(f_1^-,f_2^-,\cdots,f_n^-)\leqslant\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)\leqslant\mathrm{IVHTrFMM}^K(f_1^+,f_2^+,\cdots,f_n^+)$。

证明　因为 $0\leqslant\min\alpha_i^L\leqslant\alpha_i^L\leqslant\max\alpha_i^L\leqslant1$,所以 $0\leqslant\prod_{i=1}^{n}(\min\alpha_{\sigma(i)}^L)^{k_i}\leqslant\prod_{i=1}^{n}(\alpha_{\sigma(i)}^L)^{k_i}\leqslant\prod_{i=1}^{n}(\max\alpha_{\sigma(i)}^L)^{k_i}\leqslant1$,

进一步有

$$0\leqslant\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\min\alpha_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\leqslant\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\alpha_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}$$

$$\leqslant\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(\max\alpha_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\leqslant1。$$

类似可以得到区间梯形模糊数其他几个端点也满足上述不等式,因此

$S(\mathrm{IVHTrFMM}^K(f_1^-,f_2^-,\cdots,f_n^-))\leqslant S(\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n))\leqslant S(\mathrm{IVHTrFMM}^K(f_1^+,f_2^+,\cdots,f_n^+))$,

由定义 3 可知

$$\mathrm{IVHTrFMM}^K(f_1^-, f_2^-, \cdots, f_n^-) \leqslant \mathrm{IVHTrFMM}^K(f_1, f_2, \cdots, f_n) \leqslant \mathrm{IVHTrFMM}^K(f_1^+, f_2^+, \cdots, f_n^+)。$$

证毕。

(3) 单调性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}$，$f'_i=\{[(\alpha_i^{L'},\beta_i^{L'},\gamma_i^{L'},\mu_i^{L'}),(\alpha_i^{U'},\beta_i^{U'},\gamma_i^{U'},\mu_i^{U'})]\}$ 是2组区间犹豫梯形模糊元。若 $\forall i\in 1,2,\cdots,n$，有

$$\alpha_i^L \geqslant \alpha_i^{L'},\ \beta_i^L \geqslant \beta_i^{L'},\ \gamma_i^L \geqslant \gamma_i^{L'},\ \mu_i^L \geqslant \mu_i^{L'};\ \alpha_i^U \geqslant \alpha_i^{U'},\ \beta_i^U \geqslant \beta_i^{U'},\ \gamma_i^U \geqslant \gamma_i^{U'},\ \mu_i^U \geqslant \mu_i^{U'} \text{ 成立，则}$$

$$\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n) \geqslant \mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n)。$$

证明 对于 $\forall i\in 1,2,\cdots,n$，因为 $0\leqslant \alpha_i^{L'} \leqslant \alpha_i^L \leqslant 1$，所以 $0\leqslant \prod_{i=1}^n (\alpha_{\sigma(i)}^{L'})^{k_i} \leqslant \prod_{i=1}^n (\alpha_{\sigma(i)}^{L})^{k_i} \leqslant 1$，进一步有

$$0 \leqslant \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\alpha_{\sigma(i)}^{L'})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}} \leqslant \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}} \leqslant 1;$$

类似可以得到区间梯形模糊数其他几个端点也满足上述不等式，因此

$$S(\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)) \geqslant S(\mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n))。$$

由定义3可知 $\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n) \geqslant \mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n)$。

证毕。

(4) 置换不变性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}$，$f'_i=\{[(\alpha'^L_i,\beta'^L_i,\gamma'^L_i,\mu'^L_i),(\alpha'^U_i,\beta'^U_i,\gamma'^U_i,\mu'^U_i)]\}$ 是2组区间犹豫梯形模糊元。若 f'_i 是 f_i 的任意置换，则 $\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)=\mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n)$。

证明 由定理1可知

$$\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n) = \left\{\left[\begin{array}{l}\left(\begin{array}{l}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\beta_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\gamma_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\mu_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{array}\right),\\ \left(\begin{array}{l}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\alpha_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\beta_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\gamma_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^n(\mu_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\end{array}\right)\end{array}\right]\right\}。$$

因为 $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列，并且 f'_i 是 f_i 的任意置换，因此 $\mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n)$ 展开式中的 $\alpha'^L_{\sigma(i)}$ 一定能找到一个的 $\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)$ 展开式中 $\alpha^L_{\sigma(i)}$ 与之一一对应，据此可知 $\mathrm{IVHTrFMM}^K(f_1,f_2,\cdots,f_n)=\mathrm{IVHTrFMM}^K(f'_1,f'_2,\cdots,f'_n)$。

证毕。

2.1.2 $\mathrm{IVHTrFMM}^K$ 算子的几种退化形式

$\mathrm{IVHTrFMM}^K$ 算子在 $K=(k_1,k_2,\cdots,k_n)$ 取不同数值时可转化为其他类型的算子，

(1) 当 $K=(1,0,\cdots,0)$ 时，$\mathrm{IVHTrFMM}^K$ 变为区间犹豫梯形模糊算数平均算子

$$\mathrm{IVHTrFMM}^{(1,0,\cdots,0)}(f_1,f_2,\cdots,f_n)=\frac{1}{n}\bigoplus_{i=1}^n f_i,$$

(2) 当 $K=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$ 时，$\mathrm{IVHTrFMM}^K$ 变为区间犹豫梯形模糊几何平均算子

$$\mathrm{IVHTrFMM}^{\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)}(f_1,f_2,\cdots,f_n)=\bigotimes_{i=1}^n f_i^{\frac{1}{n}},$$

(3) 当 $K=(1,1,\cdots,0)$ 时，$\mathrm{IVHTrFMM}^K$ 变为区间犹豫梯形模糊 Bonferroni 算子

$$\mathrm{IVHTrFMM}^{(11\cdots0)}(f_1,f_2,\cdots,f_n)=\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^n (f_i\otimes f_j)\right)^{\frac{1}{2}},$$

(4) 当 $K=(\underbrace{1,1,\cdots,1}_{t},\underbrace{0,0,\cdots,0}_{n-t})$ 时，$\mathrm{IVHTrFMM}^K$ 变为区间犹豫梯形模糊 Maclaurin 对称平均算子

$$\mathrm{IVHTrFMM}^{\overbrace{(1,1,\cdots,1}^{t},\overbrace{0,0,\cdots,0)}^{n-t}}(f_1,f_2,\cdots,f_n)=\frac{1}{C_n^t}\left(\bigoplus_{\sigma'\in s'_n}\left(\bigotimes_{i=1}^{n}f_{\sigma'(i)}\right)\right)^{\frac{1}{t}},$$

其中 $\sigma'=\{\sigma'(1),\sigma'(2),\cdots,\sigma'(t)\}$ 是 $(1,2,\cdots,n)$ 的任意 t 元组,且满足 $1\leqslant\sigma'(1)\leqslant\sigma'(2)\leqslant\cdots\sigma'(n)\leqslant n$。$S'_n$ 是所有 t 元组的集合,$C_n^t=\dfrac{n!}{t!\,(n-t)!}$ 是二项式系数。

2.2　区间犹豫梯形模糊几何 Muirhead 平均算子

定义 10　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元,$K=(k_1,k_2,\cdots,k_n)$ 是一组参数向量。设 $\mathrm{IVHTrFGMM}^K:\Theta^n\to\Theta$,若

$$\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)=\frac{1}{\sum_{i=1}^{n}k_i}\left(\bigotimes_{\sigma\in s_n}\bigoplus_{i=1}^{n}k_i f_{\sigma(i)}\right)^{\frac{1}{n!}},\tag{10}$$

则称 $\mathrm{IVHTrFGMM}^K$ 为区间犹豫梯形模糊几何 Muirhead 平均算子,简记 $\mathrm{IVHTrFGMM}^K$ 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。

定理 2　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元,$K=(k_1,k_2,\cdots,k_n)$ 为一组参数向量,则

$$\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)=\frac{1}{\sum_{i=1}^{n}k_i}\left(\bigotimes_{\sigma\in s_n}\bigoplus_{i=1}^{n}k_i f_{\sigma(i)}\right)^{\frac{1}{n!}}$$

$$=\left\{\left[\begin{array}{l}\left(\begin{array}{l}1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^L)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right),\\ \left(\begin{array}{l}1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\ 1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^U)^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right)\end{array}\right]\right\}。$$

证明　因为 $k_i f_{\sigma(i)}=\left\{\left[\begin{array}{l}(1-(1-\alpha_{\sigma(i)}^L)^{k_i},\ 1-(1-\beta_{\sigma(i)}^L)^{k_i},\ 1-(1-\gamma_{\sigma(i)}^L)^{k_i},\ 1-(1-\mu_{\sigma(i)}^L)^{k_i}),\\ (1-(1-\alpha_{\sigma(i)}^U)^{k_i},\ 1-(1-\beta_{\sigma(i)}^U)^{k_i},\ 1-(1-\gamma_{\sigma(i)}^U)^{k_i},\ 1-(1-\mu_{\sigma(i)}^U)^{k_i})\end{array}\right]\right\}$,

由定义 2 可知

$$\bigoplus_{i=1}^{n}k_i f_{\sigma(i)}=\left\{\left[\begin{array}{l}\left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^L)^{k_i},\ 1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^L)^{k_i},\ 1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^L)^{k_i},\ 1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^L)^{k_i}\right),\\ \left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^U)^{k_i},\ 1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^U)^{k_i},\ 1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^U)^{k_i},\ 1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^U)^{k_i}\right)\end{array}\right]\right\},$$

所以

$$\bigotimes_{\sigma\in s_n}\bigoplus_{i=1}^{n}k_i f_{\sigma(i)}=\left\{\left[\begin{array}{l}\left(\begin{array}{l}\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^L)^{k_i}\right),\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^L)^{k_i}\right),\\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^L)^{k_i}\right),\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^L)^{k_i}\right)\end{array}\right),\\ \left(\begin{array}{l}\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\alpha_{\sigma(i)}^U)^{k_i}\right),\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\beta_{\sigma(i)}^U)^{k_i}\right),\\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^U)^{k_i}\right),\ \prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-\mu_{\sigma(i)}^U)^{k_i}\right)\end{array}\right)\end{array}\right]\right\},$$

进一步可得

$$\left(\underset{\sigma\in s_n}{\otimes}\overset{n}{\underset{i=1}{\oplus}}k_i f_{\sigma(i)}\right)^{\frac{1}{n!}}=\left\{\left[\begin{array}{l}\left(\begin{array}{l}\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}},\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\beta_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}},\\ \prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\gamma_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}},\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\mu_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\end{array}\right),\\ \left(\begin{array}{l}\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\alpha_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}},\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\beta_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}},\\ \prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\gamma_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}},\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\mu_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\end{array}\right)\end{array}\right]\right\}。$$

因此

$$\frac{1}{\sum\limits_{i=1}^{n}k_i}\left(\underset{\sigma\in s_n}{\otimes}\overset{n}{\underset{i=1}{\oplus}}k_i f_{\sigma(i)}\right)^{\frac{1}{n!}}$$

$$=\left\{\left[\begin{array}{l}\left(\begin{array}{l}1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\alpha_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\beta_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ 1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\gamma_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\mu_{\sigma(i)}^{L})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right),\\ \left(\begin{array}{l}1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\alpha_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\beta_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ 1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\gamma_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},1-\left(1-\prod\limits_{\sigma\in s_n}\left(1-\prod\limits_{i=1}^{n}(1-\mu_{\sigma(i)}^{U})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right)\end{array}\right]\right\}。$$

证毕。

与 $\mathrm{IVHTrFMM}^K$ 算子类似,$\mathrm{IVHTrFGMM}^K$ 算子也可以得到如下性质,证明过程与 $\mathrm{IVHTrFMM}^K$ 算子的证明类似。

2.2.1 $\mathrm{IVHTrFGMM}^K$ 算子的相关性质

(1) 幂等性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元,如果有 $f_i=f_0=\{[(\alpha_0^L,\beta_0^L,\gamma_0^L,\mu_0{}^L),(\alpha_0{}^U,\beta_0^U,\gamma_0^U,\mu_0^U)]\}$,则 $\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)=f_0$。

(2) 有界性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2,\cdots,n)$ 为一组区间犹豫梯形模糊元,若记

$$f_i^+=\{[(\max\alpha_i^L,\max\beta_i^L,\max\gamma_i^L,\max\mu_i^L),(\max\alpha_i^U,\max\beta_i^U,\max\gamma_i^U,\max\mu_i^U)]\},$$

$$f_i^-=\{[(\min\alpha_i^L,\min\beta_i^L,\min\gamma_i^L,\min\mu_i^L),(\min\alpha_i^U,\min\beta_i^U,\min\gamma_i^U,\min\mu_i^U)]\},$$

则 $\mathrm{IVHTrFGMM}^K(f_1^-,f_2^-,\cdots,f_n^-)\leqslant\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)\leqslant\mathrm{IVHTrFGMM}^K(f_1^+,f_2^+,\cdots,f_n^+)$。

(3) 单调性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}$, $f_i'=\{[(\alpha_i^{L'},\beta_i^{L'},\gamma_i^{L'},\mu_i^{L'}),(\alpha_i^{U'},\beta_i^{U'},\gamma_i^{U'},\mu_i^{U'})]\}$ 是2组区间犹豫梯形模糊元。若 $\forall i\in1,2,\cdots,n$,都有 $\alpha_i^L\geqslant\alpha_i^{L'}$, $\beta_i^L\geqslant\beta_i^{L'}$, $\gamma_i^L\geqslant\gamma_i^{L'}$, $\mu_i^L\geqslant\mu_i^{L'}$; $\alpha_i^U\geqslant\alpha_i^{U'}$, $\beta_i^U\geqslant\beta_i^{U'}$, $\gamma_i^U\geqslant\gamma_i^{U'}$, $\mu_i^U\geqslant\mu_i^{U'}$ 成立,则

$$\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)\geqslant\mathrm{IVHTrFGMM}^K(f_1',f_2',\cdots,f_n')。$$

(4) 置换不变性

设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}$, $f_i'=\{[(\alpha'^L_i,\beta'^L_i,\gamma'^L_i,\mu'^L_i),(\alpha'^U_i,\beta'^U_i,\gamma'^U_i,\mu'^U_i)]\}$ 是2组区间犹豫梯形模糊元。若 f_i' 是 f_i 的任意置换,则

$$\mathrm{IVHTrFGMM}^K(f_1,f_2,\cdots,f_n)=\mathrm{IVHTrFGMM}^K(f_1',f_2',\cdots,f_n')。$$

2.2.2 IVHTrFGMMK 算子的几种退化形式

$\mathrm{IVHTrFGMM}^K$ 算子在 $K=(k_1,k_2,\cdots,k_n)$ 取不同数值时可转化为其他类型的算子,

(1) 当 $K=(1,0,\cdots,0)$ 时,$\mathrm{IVHTrFGMM}^k$ 变为区间犹豫梯形模糊几何平均算子

$$\mathrm{IVHTrFGMM}^{(1,0,\cdots,0)}(f_1,f_2,\cdots,f_n)=\overset{n}{\underset{i=1}{\otimes}}f_i^{\frac{1}{n}},$$

(2) 当 $K=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$时,IVHTrFGMM^k 变为区间犹豫梯形模糊算术平均算子

$$\mathrm{IVHTrFGMM}^{\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)}(f_1,f_2,\cdots,f_n)=\frac{1}{n}\bigoplus_{i=1}^{n}f_i,$$

(3) 当 $K=(1,1,\cdots,0)$时,IVHTrFGMM^k 变为区间犹豫梯形模糊几何 Bonferroni 平均算子

$$\mathrm{IVHTrFGMM}^{(1,1,\cdots,0)}(f_1,f_2,\cdots,f_n)=\frac{1}{2}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}(f_i\oplus f_j)\right)^{\frac{1}{n(n-1)}},$$

(4) 当 $K=(\underbrace{1,1,\cdots,1}_{t},\underbrace{0,0,\cdots,0}_{n-t})$时,IVHTrFGMM^k 变为区间犹豫梯形模糊双 Maclaurin 对称平均算子

$$\mathrm{IVHTrFGMM}^{(\underbrace{1,1,\cdots,1}_{t},\underbrace{0,0,\cdots,0}_{n-t})}(f_1,f_2,\cdots,f_n)=\frac{1}{t}\left(\bigotimes_{\sigma'\in s'_n}\left(\bigoplus_{i=1}^{n}f'_{\sigma(i)}\right)\right)^{\frac{1}{C_n^t}}。$$

其中 $\sigma'=\{\sigma'(1),\sigma'(2),\cdots,\sigma'(t)\}$是$(1,2,\cdots,n)$的任意 t 元组,且满足 $1\leqslant\sigma'(1)\leqslant\sigma'(2)\leqslant\cdots\sigma'(n)\leqslant n$。$S'_n$是所有 t 元组的集合,$C_n^t=\frac{n!}{t!\,(n-t)!}$是二项式系数。

2.3 区间犹豫梯形模糊 Choquet Muirhead 平均算子

定义 11　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$为一组区间犹豫梯形模糊元,$\boldsymbol{K}=(k_1,k_2,\cdots,k_n)$是一组参数向量。$\upsilon_\tau$ 为 $Y=\{y_1,\ y_2,\cdots,\ y_n\}$上 τ 的模糊测度。设$\mathrm{IVHTrFCMM}^K:\Theta^n\rightarrow\Theta$,若

$$\mathrm{IVHTrFCMM}^K(f_1,f_2,\cdots,f_n)=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n}n[\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})]f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\tag{11}$$

则称 IVHTrFCMM^K 为区间犹豫梯形模糊 Choquet Muirhead 平均算子,简记作 IVHTrFCMM^K 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$为$(1,2,\cdots,n)$的任意排列,$S_n$ 为$(1,2,\cdots,n)$所有排列的集合。$f_{(i)}$是 f_i 的置换,满足 $f_{(1)}\leqslant f_{(2)}\leqslant\cdots\leqslant f_{(n)}$,$Y_{(i)}=\{y_{(i)},y_{(i+1)},\cdots,y_{(n)}\}$,$Y_{(n+1)}=0$。

定理 3　设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$为一组区间犹豫梯形模糊元,$K=(k_1,k_2,\cdots,k_n)$为一组参数向量,$\upsilon_\tau$ 为 $Y=\{y_1,y_2,\cdots,y_n\}$上 τ 的模糊测度。则

$$\mathrm{IVHTrFCMM}^K(f_1,f_2,\cdots,f_n)=\left(\frac{1}{n!}\bigoplus_{\sigma\in s_n}\bigotimes_{i=1}^{n}n[\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})]f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}$$

$$=\left\{\begin{bmatrix}\left[\begin{array}{l}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\alpha_{\sigma(i)}^L)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\beta_{\sigma(i)}^L)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\gamma_{\sigma(i)}^L)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\mu_{\sigma(i)}^L)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right],\\ \left[\begin{array}{l}\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\alpha_{\sigma(i)}^U)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\beta_{\sigma(i)}^U)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\gamma_{\sigma(i)}^U)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\ \left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\mu_{\sigma(i)}^L)^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\end{array}\right]\end{bmatrix}\right\}$$

证明　为简化证明步骤,设 $\rho_{\sigma(i)}=[(\alpha_{\sigma(i)}^L,\beta_{\sigma(i)}^L,\gamma_{\sigma(i)}^L,\mu_{\sigma(i)}^L),\alpha_{\sigma(i)}^U,\beta_{\sigma(i)}^U,\gamma_{\sigma(i)}^U,\mu_{\sigma(i)}^U)]$,$f_{\sigma(i)}=\{\rho_{\sigma(i)}\mid\rho_{\sigma(i)}\in f_{\sigma(i)}$,由定义 2 可得

$$n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]f_{\sigma(i)}^{k_i}=\left\{1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]}\right\},$$

所以

$$\underset{i=1}{\overset{n}{\otimes}} n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]f_{\sigma(i)}^{k_i}=\left\{\prod_{i=1}^{n}(1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_i+1)]})\right\},$$

进一步可得

$$\underset{\sigma\in s_n}{\oplus}\underset{i=1}{\overset{n}{\otimes}} n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]f_{\sigma(i)}^{k_i}=\left\{1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)\right\},$$

因此

$$\frac{1}{n!}\underset{\sigma\in s_n}{\oplus}\underset{i=1}{\overset{n}{\otimes}} n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]f_{\sigma(i)}^{k_i}=\left\{1-\left(1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)\right)\right)^{\frac{1}{n!}}\right\}$$

$$=\left\{1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right\}。$$

故

$$\begin{aligned}
\mathrm{IVHTrFCMM}^K(f_1,f_2,\cdots,f_n)&=\left(\frac{1}{n!}\underset{\sigma\in s_n}{\oplus}\underset{i=1}{\overset{n}{\otimes}} n[\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})]f_{\sigma(i)}^{k_i}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\\
&=\left\{\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-\rho_{\sigma(i)}^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right\}\\
&=\left\{\left[\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\alpha_{\sigma(i)}^{L})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\right.\\
&\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\beta_{\sigma(i)}^{L})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\
&\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\gamma_{\sigma(i)}^{L})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\
&\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\mu_{\sigma(i)}^{L})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right],\\
&\left[\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\alpha_{\sigma(i)}^{U})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\right.\\
&\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\beta_{\sigma(i)}^{U})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\
&\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\gamma_{\sigma(i)}^{U})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}},\\
&\left.\left.\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(1-(\mu_{\sigma(i)}^{U})^{k_i})^{n[\upsilon_\tau(Y_i)-\upsilon_\tau(Y_{i+1})]})\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^{n}k_i}}\right]\right\}。
\end{aligned}$$

证毕。

2.4 区间犹豫梯形模糊几何 Choquet Muirhead 平均算子

定义 12 设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$ 为一组区间犹豫梯形模糊元,$K=(k_1,k_2,\cdots,k_n)$ 是一组参数向量。υ_τ 为 $Y=\{y_1,y_2,\cdots,y_n\}$ 上 τ 的模糊测度。设 $\mathrm{IVHTrFGCMM}^K:\Theta^n\rightarrow\Theta$,若

$$\mathrm{IVHTrFGCMM}^K(f_1,f_2,\cdots,f_n)=\frac{1}{\sum_{i=1}^{n}k_i}\left(\underset{\sigma\in s_n}{\otimes}\underset{i=1}{\overset{n}{\oplus}}k_i\, f_{\sigma(i)}^{n[\upsilon_\tau(Y_{(i)})-\upsilon_\tau(Y_{(i+1)})]}\right)^{\frac{1}{n!}},\tag{12}$$

则称 $\mathrm{IVHTrFGCMM}^K$ 为区间犹豫梯形模糊几何 Choquet Muirhead 平均算子,简记为 $\mathrm{IVHTrFGCMM}^K$ 算子,其中 $\sigma=\{\sigma(1),\sigma(2),\cdots\sigma(n)\}$ 为 $(1,2,\cdots,n)$ 的任意排列,S_n 为 $(1,2,\cdots,n)$ 所有排列的集合。$f_{(i)}$ 是 f_i 的置换,满足 $f_{(1)}\leqslant f_{(2)}\leqslant\cdots\leqslant f_{(n)}$, $Y_{(i)}=\{y_{(i)},y_{(i+1)},\cdots,y_{(n)}\}$, $Y_{(n+1)}=0$。

定理 4 设 $f_i=\{[(\alpha_i^L,\beta_i^L,\gamma_i^L,\mu_i^L),(\alpha_i^U,\beta_i^U,\gamma_i^U,\mu_i^U)]\}(i=1,2\cdots,n)$ 为一组区间犹豫梯形模糊元,$K=$

$(k_1,k_2,\cdots,k_n)$为一组参数向量,则

$$\mathrm{IVHTrFGCMM}^K(f_1,f_2,\cdots,f_n)=\frac{1}{\sum_{i=1}^{n}k_i}\left(\bigotimes_{\sigma\in s_n}\bigoplus_{i=1}^{n}k_i\, f_{\sigma(i)}^{n[v_\tau(Y_{(i)})-v_\tau(Y_{(i+1)})]}\right)^{\frac{1}{n!}}$$

$$=\left\{\left[1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\alpha_{\sigma(i)}^L)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\right.\right.$$

$$1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\beta_{\sigma(i)}^L)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},$$

$$1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\gamma_{\sigma(i)}^L)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},$$

$$\left.1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\mu_{\sigma(i)}^L)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\right],$$

$$\left[1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\alpha_{\sigma(i)}^U)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},\right.$$

$$1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\beta_{\sigma(i)}^U)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},$$

$$1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\gamma_{\sigma(i)}^U)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}},$$

$$\left.\left.1-\left(1-\prod_{\sigma\in s_n}\left(1-\prod_{i=1}^{n}(1-(\mu_{\sigma(i)}^U)^{n[v_\tau(Y_i)-v_\tau(Y_{i+1})]})^{k_i}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{i=1}^n k_i}}\right]\right\}。$$

其中,$\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(n)\}$为$(1,2,\cdots,n)$的任意排列,$S_n$ 为$(1,2,\cdots,n)$所有排列的集合。$f_{(i)}$是f_i 的置换,满足$f_{(1)}\leqslant f_{(2)}\leqslant\cdots\leqslant f_{(n)}$, $Y_{(i)}=\{y_{(i)},y_{(i+1)},\cdots,y_{(n)}\}$, $Y_{(n+1)}=0$。

证明与定理 3 类似,此处省略。

与 $\mathrm{IVHTrFMM}^K$ 算子和 $\mathrm{IVHTrFGMM}^K$ 算子类似,$\mathrm{IVHTrFCMM}^K$ 算子和 $\mathrm{IVHTrFGCMM}^K$ 算子也具有类似的性质和退化形式,这里不再赘述。

3　区间犹豫梯形模糊 Choquet Muirhead 平均算子的多属性决策模型

针对某一多属性决策问题,假设方案集合为 $A=\{A_1,A_2,\cdots,A_l\}$,属性集合为 $C=\{C_1,C_2,\cdots,C_t\}$,属性集合权重向量为 $\boldsymbol{\Omega}=(\omega_1,\ \omega_2,\cdots,\omega_t)^{\mathrm{T}}$, $\omega_i\in[0,1]$, $\sum_{i=1}^{t}\omega_i=1$,参数向量为 $\boldsymbol{K}=(k_1,k_2,\cdots\cdots,k_t)$,属性 C_i 的 τ 模糊测度为 $v_\tau(C_i)$。区间犹豫梯形模糊决策矩阵记作 $\boldsymbol{F}=(f_{ij})_{l\times t}$,其中 $f_{ij}=\{[(\alpha_{ij}^L,\beta_{ij}^L,\ \gamma_{ij}^L,\ \mu_{ij}^L),\ (\alpha_{ij}^U,\ \beta_{ij}^U,\gamma_{ij}^U,\mu_{ij}^U)]\}$。

模糊测度如果采用主观假设,很有可能会导致决策出现偏差。参考文献[20],本文采用线性规划模型求得各个属性集合的 τ 模糊测度。令

$$S(A_i)=\sum_{j=1}^{t}v_\tau(C_j)S(f_{ij}),\tag{13}$$

$$D(A_i)=\sum_{j=1}^{t}v_\tau(C_j)D(f_{ij}),\tag{14}$$

$$\psi(A_i)=S(A_i)+\frac{1-D(A_i)}{2}。\tag{15}$$

$\psi(A_i)$表示方案 A_i 的综合得分,若 $\psi(A_i)$越大,表明其越符合决策者需求。

构造下列线性规划模型求得各属性的 τ 模糊测度

$$\max \sum_{i=1}^{l} \psi(A_i) = \max\left\{ \sum_{i=1}^{l} \left(S(A_i) + \frac{1-D(A_i)}{2} \right) \right\}, \tag{16}$$

$$\text{s.t.} \begin{cases} \upsilon_\tau(C_j^L) \leqslant \upsilon_\tau(C_j) \leqslant \upsilon_\tau(C_j^U), \\ \upsilon_\tau(C_{ij}^L) \leqslant \upsilon_\tau(C_i) + \upsilon_\tau(C_j) \leqslant \upsilon_\tau(C_{ij}^U), \\ i,j=1,2,\cdots,t。 \end{cases}$$

其中,$\upsilon_\tau(C_j^L)$,$\upsilon_\tau(C_j^U)$分别是$\{C_j\}$的 τ 模糊测度下界和上界,$\upsilon_\tau(C_{ij}^L)$, $\upsilon_\tau(C_{ij}^U)$分别是集合$\{C_i, C_j\}$ τ 模糊测度的下界和上界。

利用区间犹豫梯形模糊 Choquet Muirhead 平均算子构建多属性决策模型,步骤如下:

Step 1 规范化区间犹豫梯形模糊决策矩阵 $\boldsymbol{F}=(\boldsymbol{f}_{ij})_{l\times t}$,使得

$$\tilde{\boldsymbol{f}}_{ij} = \begin{cases} \boldsymbol{f}_{ij}, & C_j \text{ 是效益型指标}, \\ \boldsymbol{f}_{ij}^c, & C_j \text{ 是成本型指标}, \end{cases} \quad i=1,2,\cdots,l;\ j=1,2,\cdots,t。 \tag{17}$$

Step 2 利用式(16)计算各个属性的 τ 模糊测度,确定各属性集的 τ 模糊测度及各供应商的属性权重。

Step 3 通过区间犹豫梯形模糊 Choquet Muirhead 平均算子求得方案 A_i 的综合区间犹豫梯形模糊评价值 $f_i(i=1,2,\cdots,l)$。

Step 4 计算综合区间犹豫梯形模糊评价值 f_i 的得分函数 $S(f_i)$,偏差函数 $D(f_i)$。通过比较 $S(f_i)$和 $D(f_i)$的大小,选择最优决策方案。

4 案例分析

某绿色电池生产企业为响应国家“双碳”目标,面临绿色电池原料供应商选择问题。绿色电池材料选择属性 $C_t(t=1,2,3,4)$包括:产品竞争力(C_1),服务态度(C_2),供货能力(C_3),绿色绩效(C_4)。经过专家评定,各属性的 τ 模糊测度满足

$0.2 \leqslant \upsilon_\tau(C_1) \leqslant 0.4$, $0.3 \leqslant \upsilon_\tau(C_2) \leqslant 0.4$, $0.1 \leqslant \upsilon_\tau(C_3) \leqslant 0.3$, $0.2 \leqslant \upsilon_\tau(C_4) \leqslant 0.4$, $0.3 \leqslant \upsilon_\tau(C_1)+\upsilon_\tau(C_3) \leqslant 0.4$。

属性权重向量为 $\boldsymbol{\omega}_i=(\omega_{i1},\omega_{i2},\omega_{i3},\omega_{i4})^{\mathrm{T}}$,该企业现有 5 个绿色电池原料供应商 $A_i(i=1,2,3,4,5)$可供选择。决策数据如表 1 所示[4]。

Step 1 由于上述 4 个属性值均为效益型指标,故原决策矩阵 $\boldsymbol{F}$ 无需标准化。

表 1 IVHTrF 决策矩阵 $\boldsymbol{F}$

Table 1 IVHTrF decision matrix $\boldsymbol{F}$

供应商	选择属性			
	C_1	C_2	C_3	C_4
A_1	{[(0.1, 0.2, 0.4, 0.5), (0.2,0.3,0.5,0.6)], [(0.2,0.4,0.5,0.7), (0.3,0.5,0.6,0.8)]}	{[(0.1,0.3,0.4,0.5), (0.2,0.4,0.5,0.6)]}	{[(0.3,0.4,0.5,0.6), (0.4,0.5,0.6,0.7)], [(0.2,0.3,0.4,0.6), (0.3,0.4,0.6,0.7)]}	{[(0.4,0.5,0.6,0.7), (0.5,0.6,0.7,0.8)]}
A_2	{[(0.4, 0.5, 0.6, 0.7), (0.5,0.6,0.7,0.8)]}	{[(0.1,0.3,0.4,0.5),(0.2,0 .4, 0.5,0.6)],[(0.2,0.3,0.4,0.5), (0.3,0.4,0.5,0.6)]}	{[(0.1,0.3,0.5,0.6), (0.2,0.4,0.6,0.7)]}	{[(0.3,0.5,0.6,0.7), (0.4,0.6,0.7,0.8)], [(0.2,0.3,0.4,0.8), (0.3,0.4,0.5,0.9)]}
A_3	{[(0.3,0.4,0.6,0.7), (0.4,0.5,0.7,0.8)], [(0.2,0.3,0.4,0.6), (0.3,0.4,0.5,0.7)]}	{[(0.1,0.4,0.6,0.8),(0.2,0.5, 0.7,0.9)],[(0.1,0.5,0.6,0.7), (0.2,0.6,0.7,0.8)],[(0.4,0.6, 0.7,0.8),(0.5,0.7,0.8,0.9)]}	{[(0.4,0.5,0.6,0.7), (0.5,0.6,0.7,0.8)]}	{[(0.1,0.3,0.5,0.6), (0.2,0.4,0.6,0.7)]}
A_4	{[(0.1,0.2,0.3,0.4), (0.2,0.3,0.4,0.5)]}	{[(0.1,0.2,0.5,0.6), (0.2,0.3, 0.6,0.7)], [(0.1,0.4,0.6,0.7), (0.2,0.5,0.7,0.8)]}	{[(0.3,0.4,0.5,0.7), (0.4,0.5,0.6,0.8)]}	{[(0.3,0.4,0.5,0.6), (0.4,0.5,0.6,0.7)], [(0.4,0.5,0.6,0.7), (0.5,0.6,0.7,0.8)]}

表 1(续)

供应商	选择属性			
	C_1	C_2	C_3	C_4
A_5	{[(0.2,0.3,0.4,0.5),(0.3,0.4,0.5,0.6)],[(0.3,0.4,0.7,0.8)],(0.4,0.5,0.8,0.9)]}	{[(0.1,0.3,0.4,0.7),(0.2,0.4,0.5,0.8)]}	{[(0.1,0.4,0.5,0.7),(0.2,0.5,0.6,0.8)]}	{[(0.3,0.4,0.6,0.8),(0.4,0.5,0.7,0.9)]}

Step 2　利用式(16)计算各个属性的 τ 模糊测度,确定各属性集的 τ 模糊测度及各供应商的属性权重。由式(13)—(15)计算可得

$$\begin{aligned}\sum_{i=1}^{l}\psi(A_i)&=\sum_{i=1}^{l}\left[S(A_i)+\frac{1-D(A_i)}{2}\right]\\&=\sum_{i=1}^{l}\left[\sum_{j=1}^{t}\upsilon_\tau(C_j)S(A_{ij})+\frac{1}{2}(1-\sum_{j=1}^{t}\upsilon_\tau(C_j)D(A_{ij}))\right]\\&=\sum_{i=1}^{l}\sum_{j=1}^{t}\left[\left(S(A_{ij})-\frac{D(A_{ij})}{2}\right)\times\upsilon_\tau(C_j)\right]+\frac{5}{2}\\&=\sum_{j=1}^{t}\left[\sum_{i=1}^{l}\left(S(A_{ij})-\frac{D(A_{ij})}{2}\right)\times\upsilon_\tau(C_j)\right]+\frac{5}{2}\\&=\sum_{j=1}^{t}\left(S(A_j)-\frac{D(A_j)}{2}\right)\times\upsilon_\tau(C_j)+\frac{5}{2}\\&=0.912\ 22\times\upsilon_\tau(C_1)+0.919\ 9\times\upsilon_\tau(C_2)+0.964\ 5\upsilon_\tau\times(C_3)+1.003\ 8\times\upsilon_\tau(C_4)+\frac{5}{2}。\end{aligned}$$

构造下列线性规划模型

$$\max\sum_{i=1}^{l}\psi(A_i)=0.912\ 22\times\upsilon_\tau(C_1)+0.919\ 9\times\upsilon_\tau(C_2)+0.964\ 5\times\upsilon_\tau(C_3)+1.003\ 8\times\upsilon_\tau(C_4)+\frac{5}{2},$$

$$\text{s.t.}\begin{cases}0.2\leqslant\upsilon_\tau(C_1)\leqslant0.4,\\0.3\leqslant\upsilon_\tau(C_2)\leqslant0.4,\\0.1\leqslant\upsilon_\tau(C_3)\leqslant0.3,\\0.2\leqslant\upsilon_\tau(C_4)\leqslant0.4,\\0.3\leqslant\upsilon_\tau(C_1)+\upsilon_\tau(C_3)\leqslant0.4。\end{cases}$$

利用 Lingo 软件求解得 $\upsilon_\tau(C_1)=0.3$, $\upsilon_\tau(C_2)=0.4$, $\upsilon_\tau(C_3)=0.1$, $\upsilon_\tau(C_4)=0.4$。

根据式(4)可得 $\tau=-0.422\ 2$;根据式(3)可得 $\upsilon_\tau(C_1,C_2)=\frac{1}{-0.422\ 2}[(1+(-0.422\ 2)\times0.3)\times(1+(-0.422\ 2)\times0.4)-1]=0.649\ 3$,同理,可以利用式(3)求得各属性集合的 τ 模糊测度:

$\upsilon_\tau(C_1,C_3)=0.387\ 3$,　$\upsilon_\tau(C_1,C_4)=0.649\ 3$,　$\upsilon_\tau(C_2,C_3)=\upsilon_\tau(C_3,C_4)=0.483\ 1$,

$\upsilon_\tau(C_2,C_4)=0.732\ 4$,　$\upsilon_\tau(C_1,C_2,C_3)=\upsilon_\tau(C_1,C_3,C_4)=0.721\ 9$,　$\upsilon_\tau(C_1,C_2,C_4)=0.939\ 6$,

$\upsilon_\tau(C_2,C_3,C_4)=0.801\ 5$,　$\upsilon_\tau(C_1,C_2,C_3,C_4)=1$。

因为 $S(f_{11})=0.425$, $S(f_{12})=0.375$, $S(f_{13})=0.468\ 8$, $S(f_{14})=0.6$,所以 $S(f_{12})\leqslant S(f_{11})\leqslant S(f_{13})\leqslant S(f_{14})$,故 $f_{12}\leqslant f_{11}\leqslant f_{13}\leqslant f_{14}$,有

$$\begin{aligned}\omega_{12}&=\upsilon_\tau(C_{(1)})-\upsilon_\tau(C_{(2)})=\upsilon_\tau(C_1,C_2,C_3,C_4)-\upsilon_\tau(C_2,C_3,C_4)=0.198\ 5,\\\omega_{11}&=\upsilon_\tau(C_{(2)})-\upsilon_\tau(C_{(3)})=\upsilon_\tau(C_2,C_3,C_4)-\upsilon_\tau(C_3,C_4)=0.318\ 4,\\\omega_{13}&=\upsilon_\tau(C_{(3)})-\upsilon_\tau(C_{(4)})=\upsilon_\tau(C_3,C_4)-\upsilon_\tau(C_4)=0.083\ 1,\\\omega_{14}&=\upsilon_\tau(C_{(4)})-\upsilon_\tau(C_{(5)})=\upsilon_\tau(C_4)-\upsilon_\tau(C_5)=0.4。\end{aligned}$$

即供应商 A_1 的属性权重为 $\omega_1=(0.318\ 4,0.198\ 5,0.083\ 1,0.4)^{\mathrm{T}}$,其他供应商属性权重求解步骤同上。

Step 3　令 $K=(1,1,1,1)$,利用公式(11)将 f_{ij} 转为综合区间犹豫梯形模糊评价值 $f_i(i=1,2,\cdots,l)$。各备选方案方案综合评价值具体结果如表 2。

表 2　备选方案综合评价值 f_i

Table 2　Combined evaluation values of options f_i

	IVHTrFCMM
f_1	{[(0.158 4, 0.281 7, 0.397 3, 0.484 4), (0.254 6, 0.371 3, 0.484 4, 0.572 4)], [(0.141 6, 0.259 1, 0.370 9,0.484 4), (0.234 2, 0.346 6, 0.484 4, 0.572 4)], [(0.187 6, 0.332 3, 0.418 2, 0.521 2), (0.280 6, 0.418 2, 0.504 4, 0.607 6)], [(0.167 8, 0.305 6, 0.390 3, 0.521 2), (0.258 2, 0.390 3, 0.504 4, 0.607 6)]}
f_2	{[0.158 4, 0.332 5, 0.444 3, 0.533 8), (0.254 6, 0.422 2, 0.533 8, 0.626 2)], [(0.141 6, 0.285 5, 0.390 1, 0.565 1), (0.234 2, 0.370 8, 0.474 4, 0.667 5)], [(0.188 9, 0.332 5, 0.444 3, 0.533 8), (0.2827, 0.422 2, 0.533 8, 0.626 2)], [(0.168 9, 0.285 5, 0.390 1, 0.565 1), (0.260 0, 0.370 8, 0.474 4, 0.667 5)]}
f_3	{[(0.154 0, 0.332 3, 0.491 3, 0.614 1), (0.247 0, 0.418 2, 0.580 1, 0.716 2)], [(0.154 0, 0.356 0, 0.491 3, 0.580 1), (0.247 0, 0.444 5, 0.580 1, 0.671 9)], [(0.224 8, 0.378 4, 0.520 1, 0.614 1), (0.321 7, 0.470 6, 0.614 1, 0.716 2)], [(0.139 8, 0.310 5, 0.448 3, 0.594 3), (0.230 8, 0.397 3, 0.539 1, 0.697 3)], [(0.139 8, 0.332 7, 0.448 3, 0.561 3), (0.230 8, 0.422 3, 0.539 1, 0.654 2], [(0.204 0, 0.353 6, 0.474 6, 0.594 3), (0.300 7, 0.447 1, 0.570 7, 0.697 3)]}
f_4	{[(0.148 8, 0.243 3, 0.375 8, 0.491 3), (0.243 3, 0.333 8, 0.464 1, 0.585 5)], [(0.158 4, 0.254 6, 0.389 0, 0.504 4), (0.254 6, 0.345 5, 0.476 4, 0.597 1)], [(0.148 8, 0.286 9, 0.391 2, 0.507 5), (0.243 3, 0.375 8, 0.479 4, 0.601 1)], [(0.158 4, 0.300 3, 0.405 0, 0.521 0), (0.254 6, 0.389 0, 0.492 1, 0.613 0]}
f_5	{[(0.133 0, 0.296 6, 0.396 1, 0.556 8), (0.223 8, 0.382 7, 0.481 1, 0.640 3)], [(0.148 8, 0.322 2, 0.476 3, 0.661 1), (0.243 3, 0.409 7, 0.571 3, 0.761 8)]}

Step 4　计算各备选方案的得分函数 $S(f_i)$,偏差函数 $D(f_i)$,选择最优方案

$$S(f_1)=0.384\,8,\quad S(f_2)=0.405\,7,\quad S(f_3)=0.443\,1,\quad S(f_4)=0.376\,0,\quad S(f_5)=0.419\,1。$$

因为 $S(f_3)>S(f_5)>S(f_2)>S(f_1)>S(f_4)$,所以备选方案排序为 $A_3>A_5>A_2>A_1>A_4$,故 A_3 为最佳绿色电池原料供应商。

为了保证决策的一般性和准确性,分别求出 $K=(k_1,k_2,\cdots,k_t)$ 取不同值时的得分函数 $S(f_i)$,偏差函数 $D(f_i)$, 并进行排序得出最优选择。

表 3　参数不同时的排序结果

Table 3　Sorting results when parameters are different

参数	排序
$K=(1,0,0,0)$	$A_3>A_5>A_2>A_1>A_4$
$K=(1,1,1,1)$	$A_3>A_5>A_2>A_1>A_4$
$K=(1,1,0,0)$	$A_3>A_5>A_2>A_1>A_4$
$K=(0.25,0.25,0.25,0.25)$	$A_3>A_5>A_2>A_1>A_4$

如表 3 所示,当参数 $K=(k_1,k_2,k_3,k_4)$ 取不同值时,绿色电池原料供应商的最优选择都是 A_3,且随着参数向量 K 的变化,排序结果较为稳定。

接下来,利用文献[4]中提出的 IVHTrFWAM,IVHTrFOWA 和 IVHTrFWBM 算子进行计算,并和本文提出的算子进行结果对比分析,其得分函数如表 4 所示。

表 4　不同决策方法结果对比

Table 4　Comparison of results of different decision-making methods

算子	得分函数	排序	刻画属性间关联关系的数量	灵活性
IVHTrFWAM	$S(f_1)=0.503\,1$, $S(f_2)=0.496\,3$, $S(f_3)=0.505\,0$, $S(f_4)=0.482\,5$, $S(f_5)=0.515\,0$	$A_5>A_3>A_1>A_2>A_4$	1	弱
IVHTrFOWA	$S(f_1)=0.473\,8$, $S(f_2)=0.417\,5$, $S(f_3)=0.433\,1$, $S(f_4)=0.475\,6$, $S(f_5)=0.444\,4$	$A_5>A_3>A_1>A_2>A_4$	1	弱
IVHTrFWBM	$S(f_1)=0.152\,9$, $S(f_2)=0.159\,9$, $S(f_3)=0.167\,8$, $S(f_4)=0.148\,3$, $S(f_5)=0.170\,6$	$A_5>A_3>A_2>A_1>A_4$	2	较强
$\text{IVHTrFCMM}^{(1,0,0,0)}$ (IVHTrFCAM)	$S(f_1)=0.495\,1$, $S(f_2)=0.495\,3$, $S(f_3)=0.530\,7$, $S(f_4)=0.478\,0$, $S(f_5)=0.518\,5$	$A_3>A_5>A_2>A_1>A_4$	多个	弱
$\text{IVHTrFCMM}^{(1,1,0,0)}$ (IVHTrFCBM)	$S(f_1)=0.432\,3$, $S(f_2)=0.439\,1$, $S(f_3)=0.479\,8$, $S(f_4)=0.419\,7$, $S(f_5)=0.466\,2$	$A_3>A_5>A_2>A_1>A_4$	多个	较强
$\text{IVHTrFCMM}^{(1,1,1,1)}$	$S(f_1)=0.384\,8$, $S(f_2)=0.405\,7$, $S(f_3)=0.443\,1$, $S(f_4)=0.376\,0$, $S(f_5)=0.419\,1$	$A_3>A_5>A_2>A_1>A_4$	多个	强

通过表 3、4 的决策结果分析可得出以下结论:

(1) 在表 3 中,当 K 取不同参数时,决策结果排序都相同,说明了本文构造的基于区间犹豫梯形模糊 Choquet Muirhead 平均算子决策模型的稳定性和可靠性较好,同时说明了该模型决策的准确性较高。

(2) 对比表 3 和表 4 中数据可以发现,IVHTrFCMM^K 系列算子决策最优选择方案都是 A_3,而 IVHTrF-WAM、IVHTrFOWA、IVHTrFWBM 算子最优选择方案都是 A_5,这与它们刻画属性间关联性程度有关系,IVHTrFCMMK 系列算子可以刻画多个属性间的关联性,而 IVHTrFWAM、IVHTrFOWA、IVHTrFWBM 算子无法或者只能刻画 2 个属性间的关联性,因此利用 IVHTrFCMM^K 系列算子可以最大程度的避免信息的丢失,得到更为准确、客观的决策结果。

(3) IVHTrFCMMK 系列算子可以让决策者者根据自身的决策偏好以及风险态度来调整参数 K 的取值,使得决策更为灵活和便捷,同时当 K 取值不断变化时,决策结果变化不大,说明该决策模型具有较强的稳定性。

5 结束语

算子理论研究是决策理论研究的一个重要组成部分。在犹豫模糊环境下,决策数据类型多样、情况复杂,目前是学者们研究的热点。传统集结算子没有关注到属性间的关联性,而实际问题中属性间往往存在着关联性,而且对决策结果会产生影响。Muirhead 算子能够处理多个变量间的关联关系,相较于 Bonferroni 算子和 Heronian 算子这类仅能考虑 2 个变量之间关联关系的算子来说,适用范围更广。此外,Muirhead 算子还可以退化为几何平均算子、Bonferroni 平均算子、Maclaurin 对称平均算子等一系列经典集结算子,更具一般性。基于这种考虑,本文对 Muirhead 算子展开研究,引入能够反映属性优先关系的 Choquet 积分,以区间犹豫梯形模糊数为信息载体,定义了一系列新的 Muirhead 算子,并对其性质和定理进行了讨论,对算子理论进行了补充。此外,结合测度理论构建了基于区间犹豫梯形模糊 Choquet Muirhead 算子的多属性决策模型,并进行实证分析,说明了决策方法的可行性和有效性。

结合属性值为犹豫模糊数以及现有集结算子,兼顾 Muirhead 算子的特点,系统研究犹豫模糊环境下 Muirhead 算子的相关理论,定义新的犹豫模糊环境下 Muirhead 集成算子,从而丰富 Muirhead 算子理论。同时,结合 Muirhead 算子提出新的决策方法,在交通问题、聚类分析、人工智能、时间序列等新的场景下开展应用拓展,这些都是未来可持续研究的内容。

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