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强奇异 Calderón-Zygmund 多线性交换子在 Campanato 空间上的估计

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摘要:研究多线性强奇异 Calderón-Zygmund 算子与 Lipschitz 函数生成的多线性交换子的映射的性质。证明在一定条件下,强奇异 Calderón-Zygmund 多线性交换子 $T_{\vec{b}}$ 是从 Lebesgue 空间 $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ 到 Campanato 空间 $C^{p,\beta}(\mathbf{R}^n)$ 上有界的。

关键词:多线性强奇异 Calderón-Zygmund 算子;交换子;Lipschitz 空间;Campanato 空间
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Estimates of multilinear commutators of strongly singular Calderón-Zygmund operators on Campanato Spaces

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Abstract: This paper studies the mappings' properties of the multilinear commutators generated by the multilinear strongly singular Calderón-Zygmund operators and Lipschitz functions. It is proved that the multilinear commutators $T_{\vec{b}}$ of the strongly singular Calderón-Zygmund operators is bounded from Lebesgue spaces to Campanato spaces under some certain conditions.

Key words: multilinear strongly singular Calderón-Zygmund operators; commutator; Lipschitz spaces; Campanato spaces

1 引言及主要结果

由算子 T 和局部可积函数 b 生成的交换子定义为

$$T_b f(x)=b(x)Tf(x)-T(bf)(x)。$$

当 $b\in\mathrm{BMO}(\mathbf{R}^n)$,$T$ 为经典的 Calderón-Zygmund 算子时,Coifman 等[1]证明了 T_b 是在 $L^p(\mathbf{R}^n)$($1<p<\infty$)上有界的,并且也给出了 $\mathrm{BMO}(\mathbf{R}^n)$空间的刻画。随后,Janson[2]研究了当 b 属于齐次 Lipschitz 空间 $\dot{\Lambda}_\gamma(\mathbf{R}^n)$($0<\gamma<1$)的有界性,其中

$$\|b\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}=\sup_{\substack{x,y\in\mathbf{R}^n\\x\neq y}}\frac{|b(x)-b(y)|}{|x-y|^\gamma}<\infty,$$

当 $1<p<q<\infty$, $\gamma=n(1/p-1/q)$时,Janson 证明了 T_b 是从 $L^p(\mathbf{R}^n)$到 $L^q(\mathbf{R}^n)$有界的当且仅当 $b\in\dot{\Lambda}_\gamma(\mathbf{R}^n)$。1995 年,Paluszyński[3]证明了 T_b 是从 $L^p(\mathbf{R}^n)$($1<p<\infty$)到 Triebel-Lizorkin 空间 $\dot{F}_p^{r,\infty}$ 上有界的,当且仅当

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$b\in\dot{\Lambda}_{\gamma}(\mathbf{R}^n)(0<\gamma<1)$。2015 年,Zhang 等[4]得到了 T_b 是从 $L^p(\mathbf{R}^n)$ 到 $C^{p,\beta}(\mathbf{R}^n)$ 有界的,当且仅当 $b\in\dot{\Lambda}_{\gamma}(\mathbf{R}^n)$,这里,$1<p<\infty$,$-n/p\leqslant\beta<0$,$0<\gamma=\beta+n/p<1$,其中,$C^{p,\beta}(\mathbf{R}^n)$ 表示 Campanato 空间(见定义 3)。

最近,Zhang 等[5]研究了当 T 为强奇异 Calderón-Zygmund 算子的情形,并证明了当 $1<p<\infty$,$-n/p\leqslant\beta<0$,$0<\gamma=\beta+n/p<1$,$b\in\dot{\Lambda}_{\gamma}(\mathbf{R}^n)$ 时,T_b 是从 $L^p(\mathbf{R}^n)$ 到 $C^{p,\beta}(\mathbf{R}^n)$ 有界的。一些学者做了更多的相关研究[6-13]。受上面研究结果的启发,本文主要是证明多线性强奇异 Calderón-Zygmund 算子与 Lipschitz 函数生成的多线性交换子 $T_{\vec{b}}$ 是从 Lebesgue 空间 $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ 到 Campanato 空间 $C^{p,\beta}(\mathbf{R}^n)$ 上有界的。为此,首先回顾一些定义和记号。

定义 1[14]　设 $T:S\to S'$ 为有界线性算子。称 T 是强奇异 Calderón-Zygmund 算子,如果满足以下条件:

(1) T 是 $L^2(\mathbf{R}^n)$ 上的有界线性算子。

(2) 存在 $\{(x,y):x\neq y\}$ 上的连续函数 $K(x,y)$,使得对某个 $0<\delta\leqslant1$,$0<\alpha<1$,当 $2|y-z|^{\alpha}\leqslant|x-z|$ 时,有

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\leqslant\frac{|y-z|^{\delta}}{|x-z|^{n+\delta/\alpha}},$$

且当 $f,g\in S$ 无公共支集时,有

$$\langle Tf,g\rangle=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}K(x,y)f(y)g(x)\,\mathrm{d}y\mathrm{d}x。$$

(3) 对某个 $\frac{(1-\alpha)n}{2}\leqslant\beta<\frac{n}{2}$,$T$ 和它的共轭算子 T^* 从 $L^q(\mathbf{R}^n)$ 到 $L^2(\mathbf{R}^n)$ 有界,其中 $\frac{1}{q}=\frac{1}{2}+\frac{\beta}{n}$。

设 $m\in\mathbf{N}_+$ 且 $K(y_0,y_1,\cdots,y_m)$ 是定义在 $(\mathbf{R}^n)^{m+1}\setminus\{y_0=y_1=\cdots=y_m\}$ 上的函数。相应于核函数 K 的 m 线性算子 T 定义为

$$T(f_1,f_2,\cdots,f_m)=\int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n}K(x,y_1,\cdots,y_m)\prod_{j=1}^{m}f_j(y_j)\,\mathrm{d}y_1\cdots\mathrm{d}y_m,\tag{1}$$

其中,$f_j\in C_c^{\infty}$,$j=1,2\cdots,m$,$x\notin\bigcap_{j=1}^{m}\operatorname{supp}f_j$。

若核函数 K 满足尺寸和光滑条件:对某个 $C>0$ 和所有 $(y_0,y_1,\cdots,y_m)\in(\mathbf{R}^n)^{m+1}\setminus\{y_0=y_1=\cdots=y_m\}$,有

$$K(y_0,y_1,\cdots,y_m)\leqslant\frac{C}{\left(\sum_{k,l=0}^{m}|y_k-y_l|\right)^{mn}}。\tag{2}$$

对某个 $\varepsilon>0$,当 $0\leqslant j\leqslant m$ 且 $|y_j-y_{j'}|\leqslant\frac{1}{2}\max_{0\leqslant k\leqslant m}|y_j-y_k|$,有

$$|K(y_0,\cdots,y_j,\cdots,y_m)-K(y_0,\cdots,y_{j'},\cdots,y_m)|\leqslant C\frac{|y_j-y_{j'}|^{\varepsilon}}{\left(\sum_{k,l=0}^{m}|y_k-y_l|\right)^{mn+\varepsilon}},\tag{3}$$

这时 K 被称为标准的 m 线性 Calderón-Zygmund 算子核。

设 T 是形如式(1)的 m 线性算子,$\vec{b}=(b_1,\cdots,b_m)$,其中 b_j 为局部可积函数,$j=1,2,\cdots,m$。由 T 和 $\vec{b}$ 生成的 m 线性交换子定义为

$$T_{\vec{b}}(f_1,f_2,\cdots,f_m)=\sum_{j=1}^{m}T_{\vec{b}}^{j}(\vec{f}),$$

其中

$$T_{\vec{b}}^{j}(\vec{f})=b_jT(f_1,\cdots,f_m)-T(f_1,\cdots,f_{j-1},b_jf_j,f_{j+1},\cdots,f_m)。$$

记 $\vec{b}\in\dot{\Lambda}_{\gamma}^{m}$ 表示 $b_j\in\dot{\Lambda}_{\gamma}(\mathbf{R}^n)$,其中 $j=1,2\cdots,m$,且 $\|\vec{b}\|_{\dot{\Lambda}_{\gamma}^{m}}=\max_{1\leqslant j\leqslant m}\|b_j\|_{\dot{\Lambda}_{\gamma}(\mathbf{R}^n)}$。

定义 2[10]　设 T 是形如式(1)的 m 线性算子,则称 T 是 m 线性强奇异 Calderón-Zygmund 算子,如果满足以下条件:

(1) 对某个 $\varepsilon>0$ 和 $0<\alpha\leqslant1$,当 $|x-x'|^{\alpha}\leqslant\frac{1}{2}\max_{1\leqslant j\leqslant m}|x-y_j|$ 时,有

$$|K(x,y_1,\cdots,y_m)-K(x',y_1,\cdots,y_m)|\leqslant\frac{C|x-x'|^{\varepsilon}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon/\alpha}}。\tag{4}$$

（2）对于给定的 $1\leqslant r_1,\cdots,r_m<\infty$，使得 $1/r=1/r_1+\cdots+1/r_m$，T 从 $L^{r_1}\times\cdots\times L^{r_m}$ 到 $L^{r,\infty}$ 有界。

（3）对于给定的 $1\leqslant l_1,\cdots,l_m<\infty$，使得 $1/l=1/l_1+\cdots+1/l_m$，T 从 $L^{l_1}\times\cdots\times L^{l_m}$ 到 $L^{q,\infty}$ 有界，其中 $0<\dfrac{l}{q}\leqslant\alpha$。

通常用 $B=B(x_0,r)$ 表示以 x_0 为中心，半径为 r 的球，对任意的 $a>0$，aB 表示与球 B 同中心且半径为 a 倍的球，即 $aB=B(x_0,ar)$。$|B|$ 表示球 B 的 Lebesgue 测度，χ_B 是它的特征函数。对于任意的 $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$，记 $f_B=\dfrac{1}{|B|}\displaystyle\int_B f(x)\mathrm{d}x$。

定义 3[5] 设 $1\leqslant p<\infty$，$-\dfrac{n}{p}\leqslant\beta<1$，Campanato 空间 $C^{p,\beta}(\mathbf{R}^n)$ 定义为

$$C^{p,\beta}(\mathbf{R}^n)=\{f\in L^p_{\mathrm{loc}}(\mathbf{R}^n),\ \|f\|_{C^{p,\beta}(\mathbf{R}^n)}<\infty\},$$

其中

$$\|f\|_{C^{p,\beta}(\mathbf{R}^n)}=\sup_B\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|f(x)-f_B|^p\mathrm{d}x\right)^{1/p},$$

上确界取自所有 $\mathbf{R}^n$ 中的球 B。

本文的主要结果如下：

定理 设 T 为定义 2 中的 m 线性强奇异 Calderón-Zygmund 算子，$\vec{b}\in\dot{\Lambda}^m_\gamma$，$s_0=\max\{r_1,\cdots,r_m,l_1,\cdots,l_m\}$。那么对于任意的 $s_0<p_1,\cdots,p_m<\infty$，满足 $1/p=1/p_1+\cdots+1/p_m$，$-\dfrac{n}{p}\leqslant\beta<0$，$0<\gamma=\beta+\dfrac{n}{p}<1$，则 $T_{\vec{b}}$ 是从 $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ 到 $C^{p,\beta}(\mathbf{R}^n)$ 有界的，即

$$\|T_{\vec{b}}(\vec{f})\|_{C^{p,\beta}(\mathbf{R}^n)}\leqslant C\|\vec{b}\|_{\dot{\Lambda}^m_\gamma}\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbf{R}^n)}。$$

注 当 $m=1$ 时，定理中的结果和文献[5]的主要结果一致，即单线性强奇异 Calderón-Zygmund 交换子 T_b 是从 $L^p(\mathbf{R}^n)$ 到 $C^{p,\beta}(\mathbf{R}^n)$ 有界的，其中 $b\in\dot{\Lambda}_\gamma(\mathbf{R}^n)$。

2 定理的证明

为证明定理 1，本文需要一些引理。

引理 1[15-16] 设 $0<\gamma<1$，$b\in\dot{\Lambda}_\gamma(\mathbf{R}^n)$，则对于所有的 $1<p<\infty$，

$$\|b\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\approx\sup_B\frac{1}{|B|^{\gamma/n}}\left(\frac{1}{|B|}\int_B|b(x)-b_B|^p\mathrm{d}x\right)^{1/p}\approx\sup_B\frac{1}{|B|^{\gamma/n}}\|b-b_B\|_{L^\infty(B)}。$$

引理 2[15] 设 $0<\gamma<1$，$b\in\dot{\Lambda}_\gamma(\mathbf{R}^n)$，对于 $\mathbf{R}^n$ 中的球 B 和 B'，如果 $B'\subset B$，则

$$|b'_B-b_B|\leqslant C\|b\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}|B|^{\gamma/n}。$$

引理 3[17-18] 设 $0<p<q<\infty$，则存在一个正常数 $C=C_{p,q}$ 使得对于任意的可测函数 f，有

$$|Q|^{-1/p}\|f\|_{L^p(Q)}\leqslant C|Q|^{-1/q}\|f\|_{L^{q,\infty}(Q)}。$$

引理 4[19] 设 T 是 1 个 m 线性强奇异 Calderón-Zygmund 算子，对定义 2 中的 r_j 和 l_j，$j=1,2,\cdots,m$，设 $s=\max\{r_1,\cdots,r_m,l_1,\cdots,l_m\}$，若对任意的 $s<p_1,\cdots,p_m<\infty$，有 $1/p=1/p_1+\cdots+1/p_m$，则 T 是从 $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ 到 $L^p(\omega)$ 的有界算子，其中 $(\omega_1,\cdots,\omega_m)\in(A_{p_1/s},\cdots,A_{p_m/s})$，$\omega=\displaystyle\prod_{j=1}^m\omega_j^{p/p_j}$。

定理 1 的证明 不妨设 $m=2$，$m>2$ 的证明是类似的。设 f_1,f_2 是有紧支集的可测函数，对于任意的球 $B=B(x_0,r)$，$r>0$，先估计 $T^1_{\vec{b}}(f_1,f_2)(z)$，$z\in B$。只需证明

$$\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T^1_{\vec{b}}(f_1,f_2)(z)-(T^1_{\vec{b}}(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\prod_{i=1}^2\|f_i\|_{L^{p_i}(\mathbf{R}^n)}。$$

下面分别考虑 2 种情况。

情况 1 $r\geqslant1$ 时，设 $B^*=2B=B(x_0,2r)$，记

$$f_1=f_1\chi_{B^*}+f_1\chi_{(B^*)^c}=f_1^1+f_1^2,\quad f_2=f_2\chi_{B^*}+f_2\chi_{(B^*)^c}=f_2^1+f_2^2,$$

有

$$T_{\vec b}^1(f_1,f_2)(z)=(b_1(z)-b_B^1)T(f_1,f_2)(z)-T((b_1-b_B^1)f_1^1,f_2^1)(z)-T((b_1-b_B^1)f_1^1,f_2^2)(z)$$
$$-T((b_1-b_B^1)f_1^2,f_2^1)(z)-T((b_1-b_B^1)f_1^2,f_2^2)(z)。$$

对任意的常数 c_1,根据 Hölder 不等式,有

$$\begin{aligned}
&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-(T_{\vec b}^1(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\\
\leqslant&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-c_1|^p\mathrm{d}z\right)^{1/p}+\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|(T_{\vec b}^1(f_1,f_2))_B-c_1|^p\mathrm{d}z\right)^{1/p}\\
\leqslant&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-c_1|^p\mathrm{d}z\right)^{1/p}+\frac{1}{|B|^{\beta/n}}|(T_{\vec b}^1(f_1,f_2))_B-c_1|\\
\leqslant&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-c_1|^p\mathrm{d}z\right)^{1/p}+\frac{1}{|B|^{\beta/n}}\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(\omega)-c_1|\mathrm{d}\omega\\
\leqslant&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-c_1|^p\mathrm{d}z\right)^{1/p}+\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(\omega)-c_1|^p\mathrm{d}\omega\right)^{1/p}\\
\leqslant&\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-c_1|^p\mathrm{d}z\right)^{1/p}。
\end{aligned}$$

令 $c_1=-(T((b_1-b_B^1)f_1^1,f_2^2)(x_0)+T((b_1-b_B^1)f_1^2,f_2^1)(x_0)+T((b_1-b_B^1)f_1^2,f_2^2)(x_0))$,则

$$\begin{aligned}
&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-(T_{\vec b}^1(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\\
\leqslant&\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|(b_1(z)-b_B^1)T(f_1,f_2)(z)|^p\mathrm{d}z\right)^{1/p}\\
&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)f_1^1,f_2^1)(z)|^p\mathrm{d}z\right)^{1/p}\\
&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)f_1^1,f_2^2)(z)-T((b_1-b_B^1)f_1^1,f_2^2)(x_0)|^p\mathrm{d}z\right)^{1/p}\\
&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)f_1^2,f_2^1)(z)-T((b_1-b_B^1)f_1^2,f_2^1)(x_0)|^p\mathrm{d}z\right)^{1/p}\\
&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)f_1^2,f_2^2)(z)-T((b_1-b_B^1)f_1^2,f_2^2)(x_0)|^p\mathrm{d}z\right)^{1/p}\\
:=&\sum_{i=1}^{5}I_i。
\end{aligned}$$

因为 $1<p<\infty$, $0<\gamma=\beta+n/p<1$, $s_0=\max\{r_1,r_2,l_1,l_2\}$, $s_0<p_1$, $p_2<\infty$,且 $1/p=1/p_1+1/p_2$,根据引理 1 和引理 4,有

$$\begin{aligned}
I_1&\leqslant\frac{2}{|B|^{\gamma/n}}\|b_1-b_B^1\|_{L^\infty(B)}\|T(f_1,f_2)\|_{L^p(\mathbf{R}^n)}\\
&\leqslant C\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。
\end{aligned}$$

根据引理 1、引理 2 和引理 4,有

$$\begin{aligned}
I_2&=\frac{2}{|B|^{\gamma/n}}\|T((b_1-b_B^1)f_1^1,f_2^1)\|_{L^p(B)}\\
&\leqslant\frac{C}{|B|^{\gamma/n}}\left(\int_{B^*}|(b_1(z)-b_B^1)f_1(y_1)|_1^p\mathrm{d}y_1\right)^{1/p_1}\left(\int_{B^*}|f_2(y_2)|^{p_2}\mathrm{d}y_2\right)^{1/p_2}\\
&\leqslant\frac{C}{|B|^{\gamma/n}}(\|b_1-b_{B^*}^1\|_{L^\infty(B^*)}+|b_{B^*}^1-b_B^1|)\|f_1\|_{L^{p_1}(B^*)}\|f_2\|_{L^{p_2}(B^*)}
\end{aligned}$$

$$\leqslant \frac{C}{|B|^{\gamma/n}}(|B^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}+|B^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)})\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

接下来估计 I_3,对任意的 $z\in B=B(x_0,r)$, $y_1\in B^*$, $y_2\in(B^*)^c$,则 $|z-x_0|^\alpha\leqslant r^\alpha\leqslant r<\frac{1}{2}|y_2-x_0|$。根据式(1)和式(4),有

$$|T((b_1-b_B^1)f_1^1,f_2^2)(z)-T((b_1-b_B^1)f_1^1,f_2^2)(x_0)|$$

$$\leqslant \int_{B^*}\int_{(B^*)^c}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\int_{B^*}\int_{(B^*)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^\infty\int_{B^*}\int_{2^kB^*\backslash 2^{k-1}B^*}\frac{|z-x_0|^\varepsilon}{|x_0-y_2|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^\infty\frac{r^\varepsilon}{(2^kr)^{2n+\varepsilon/\alpha}}\int_{B^*}\int_{2^kB^*\backslash 2^{k-1}B^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^\infty\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{B^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1,$$

因此,

$$I_3\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B\left|C\sum_{k=1}^\infty\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{B^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\right|^p\mathrm{d}z\right)^{1/p}$$

$$\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{B^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{B^*}\int_{2^kB^*}|b_1(y_1)-b_{B^*}^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$+C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{B^*}\int_{2^kB^*}|b_{B^*}^1-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$:=I_{3,1}+I_{3,2}。$$

设 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,则 $1<p_1'$, $p_2'<\infty$,又 $0<\gamma=\beta+\frac{n}{p}<1$, $1/p=1/p_1+1/p_2$,则根据 Hölder 不等式和引理1,有

$$I_{3,1}\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\|b_1-b_{B^*}^1\|_{L^\infty(B^*)}\int_{B^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^kB^*}|f_2(y_2)|\mathrm{d}y_2$$

$$\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|B^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(B^*)}|B^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^kB^*)}|2^kB^*|^{1/p_2'}$$

$$\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)+kn/p_2'}|B^*|^{2+\beta/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$$

$$\leqslant Cr^{\varepsilon(1-1/\alpha)}2^{2n}\sum_{k=1}^\infty 2^{kn(1/p_2'-2)-k\varepsilon/\alpha+\beta}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

根据 Hölder 不等式和引理2,有

$$I_{3,2}=C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|b_{B^*}^1-b_B^1|\int_{B^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^kB^*}|f_2(y_2)|\mathrm{d}y_2$$

$$\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|B^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(B^*)}|B^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^kB^*)}|2^kB^*|^{1/p_2'}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

结合 $I_{3,1}$ 与 $I_{3,2}$,有 $I_3 \leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

接下来估计 I_4,对任意的 $z\in B=B(x_0,r)$, $y_2\in B^*$, $y_1\in(B^*)^c$,则$|z-x_0|^\alpha\leqslant r^\alpha\leqslant r<\frac{1}{2}|y_1-x_0|$。根据式(1)和式(4),有

$$
\begin{aligned}
&|T((b_1-b_B^1)f_1^2,f_2^1)(z)-T((b_1-b_B^1)f_1^2,f_2^1)(x_0)|\\
&\leqslant\int_{(B^*)^c}\int_{B^*}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\leqslant C\int_{(B^*)^c}\int_{B^*}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\leqslant C\sum_{k=1}^{\infty}\int_{2^kB^*\backslash 2^{k-1}B^*}\int_{B^*}\frac{|z-x_0|^\varepsilon}{|x_0-y_1|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\leqslant C\sum_{k=1}^{\infty}\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^kB^*}\int_{B^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1,
\end{aligned}
$$

因此,

$$
\begin{aligned}
I_4&\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B\left|C\sum_{k=1}^{\infty}\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^kB^*}\int_{B^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\right|^p\mathrm{d}z\right)^{1/p}\\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{B^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{B^*}|b_1(y_1)-b_{2^kB^*}^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\quad+C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{B^*}|b_{2^kB^*}^1-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&:=I_{4,1}+I_{4,2}。
\end{aligned}
$$

根据 Hölder 不等式和引理 1,因为 $0<\gamma=\beta+\frac{n}{p}<1$, $1/p=1/p_1+1/p_2$,且 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,所以有

$$
\begin{aligned}
I_{4,1}&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\|b_1-b_{2^kB^*}^1\|_{L^\infty(2^kB^*)}\int_{2^kB^*}|f_1(y_1)|\mathrm{d}y_1\int_{B^*}|f_2(y_2)|\mathrm{d}y_2\\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^kB^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^kB^*)}|2^kB^*|^{1/p_1'}\|f_2\|_{L^{p_2}(B^*)}|B^*|^{1/p_2'}\\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)+kn(\gamma/n+1/p_1')}|B^*|^{2+\beta/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\
&\leqslant Cr^{\varepsilon(1-1/\alpha)}2^{2n}\sum_{k=1}^{\infty}2^{kn(1/p_1'-1)+kn(1/p-1)+(k+1)\beta-k\varepsilon/\alpha}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\
&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。
\end{aligned}
$$

根据 Hölder 不等式和引理 2,有

$$
\begin{aligned}
I_{4,2}&=C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|b_{2^kB^*}^1-b_B^1|\int_{2^kB^*}|f_1(y_1)|\mathrm{d}y_1\int_{B^*}|f_2(y_2)|\mathrm{d}y_2\\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^kB^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^kB^*)}|2^kB^*|^{1/p_1'}\|f_2\|_{L^{p_2}(B^*)}|B^*|^{1/p_2'}\\
&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。
\end{aligned}
$$

结合 $I_{4,1}$ 与 $I_{4,2}$,有 $I_4\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

同理估计 I_5,对任意的 $z\in B=B(x_0,r)$, $y_1\in(B^*)^c$, $y_2\in(B^*)^c$,则$|z-x_0|^\alpha\leqslant r^\alpha\leqslant r<\frac{1}{2}|y_1-x_0|$, $|z-x_0|^\alpha\leqslant$

$r^\alpha \leqslant r<\frac{1}{2}|y_2-x_0|$。根据式(4),有

$$
\begin{aligned}
&|T((b_1-b_B^1)f_1^2,f_2^2)(z)-T((b_1-b_B^1)f_1^2,f_2^2)(x_0)| \\
\leqslant &\int_{(B^*)^c}\int_{(B^*)^c}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
\leqslant &C\int_{(B^*)^c}\int_{(B^*)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
\leqslant &C\sum_{k=1}^\infty\int_{2^kB^*\backslash 2^{k-1}B^*}\int_{2^kB^*\backslash 2^{k-1}B^*}\frac{|z-x_0|^\varepsilon}{|x_0-y_2|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
\leqslant &C\sum_{k=1}^\infty\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^kB^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1,
\end{aligned}
$$

因此

$$
\begin{aligned}
I_5 &\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B\left|C\sum_{k=1}^\infty\frac{r^{\varepsilon(1-1/\alpha)-2n}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^kB^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\right|^p\mathrm{d}z\right)^{1/p} \\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{2^kB^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{2^kB^*}|b_1(y_1)-b_{2^kB^*}^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
&\quad+C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\int_{2^kB^*}\int_{2^kB^*}|b_{2^kB^*}^1-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1 \\
&:=I_{5,1}+I_{5,2}。
\end{aligned}
$$

因为 $0<\gamma=\beta+\frac{n}{p}<1$, $1/p=1/p_1+1/p_2$,且 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,根据 Hölder 不等式和引理 1,有

$$
\begin{aligned}
I_{5,1}&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}\|b_1-b_{2^kB^*}^1\|_{L^\infty(2^kB^*)}\int_{2^kB^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^kB^*}|f_2(y_2)|\mathrm{d}y_2 \\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|2^kB^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^kB^*)}|2^kB^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^kB^*)}|2^kB^*|^{1/p_2'} \\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)+kn(2+\beta/n)}|B^*|^{2+\beta/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)} \\
&\leqslant Cr^{\varepsilon(1-1/\alpha)}2^{2n}\sum_{k=1}^\infty 2^{(k+1)\beta-k\varepsilon/\alpha}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)} \\
&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。
\end{aligned}
$$

根据 Hölder 不等式和引理 2,有

$$
\begin{aligned}
I_{5,2}&=C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|b_{2^kB^*}^1-b_B^1|\int_{2^kB^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^kB^*}|f_2(y_2)|\mathrm{d}y_2 \\
&\leqslant C\frac{r^{\varepsilon(1-1/\alpha)-2n}}{|B|^{\beta/n}}\sum_{k=1}^\infty 2^{-k(2n+\varepsilon/\alpha)}|2^kB^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^kB^*)}|2^kB^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^kB^*)}|2^kB^*|^{1/p_2'} \\
&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。
\end{aligned}
$$

结合 $I_{5,1}$ 与 $I_{5,2}$,有 $I_5\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

情况 2 $0<r<1$ 时,设 $\tilde{B}=B(x_0,r^\alpha)$, $\tilde{B}^*=2\tilde{B}=B(x_0,2r^\alpha)$,记

$$f_1=f_1\chi_{\tilde{B}^*}+f_1\chi_{(\tilde{B}^*)^c}=\tilde{f}_1^1+\tilde{f}_1^2,\quad f_2=f_2\chi_{\tilde{B}^*}+f_2\chi_{(\tilde{B}^*)^c}=\tilde{f}_2^1+\tilde{f}_2^2,$$

有

$$
\begin{aligned}
T_{\vec{b}}^1(f_1,f_2)(z)=&(b_1(z)-b_B^1)T(f_1,f_2)(z)-T((b_1-b_B^1)\tilde{f}_1^1,\tilde{f}_2^1)(z)-T((b_1-b_B^1)\tilde{f}_1^1,\tilde{f}_2^2)(z) \\
&-T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^1)(z)-T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^2)(z)。
\end{aligned}
$$

类似于情况 1,有

$$\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-(T_{\vec b}^1(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-\tilde c_1|^p\mathrm{d}z\right)^{1/p}。$$

令 $\tilde c_1=-(T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^2)(x_0)+T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^1)(x_0)+T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^2)(x_0))$,则

$$\begin{aligned}&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec b}^1(f_1,f_2)(z)-(T_{\vec b}^1(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\\&\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|(b_1(z)-b_B^1)T(f_1,f_2)(z)|^p\mathrm{d}z\right)^{1/p}\\&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^1)(z)|^p\mathrm{d}z\right)^{1/p}\\&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^2)(z)-T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^2)(x_0)|^p\mathrm{d}z\right)^{1/p}\\&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^1)(z)-T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^1)(x_0)|^p\mathrm{d}z\right)^{1/p}\\&+\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^2)(z)-T((b_1-b_B^1)\tilde f_1^2,\tilde f_2^2)(x_0)|^p\mathrm{d}z\right)^{1/p}\\&:=\sum_{i=1}^5\tilde I_i。\end{aligned}$$

设存在 q,使得 $0<\frac{p}{q}\leqslant\alpha$,又 $1<p<\infty$, $0<\gamma=\beta+\frac{n}{p}<1$, $1/p=1/p_1+1/p_2$,根据引理 1、引理 3 和定义 2,有

$$\begin{aligned}\tilde I_1&\leqslant\frac{2}{|B|^{\beta/n}}\|b_1-b_B^1\|_{L^\infty(B)}|B|^{-1/p}\|T(f_1,f_2)\|_{L^p(B)}\\&\leqslant\frac{C}{|B|^{\beta/n}}\|b_1-b_B^1\|_{L^\infty(B)}|B|^{-1/q}\|T(f_1,f_2)\|_{L^{q,\infty}(B)}\\&\leqslant\frac{C}{|B|^{\gamma/n}}|B|^{1/p-1/q}\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}|B|^{\gamma/n}\|f_1\|_{L^{p_1}(B)}\|f_2\|_{L^{p_2}(B)}\\&\leqslant Cr^{n(1/p-1/q)}\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\&\leqslant C\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。\end{aligned}$$

根据引理 1、引理 2、引理 3 和定义 2,有

$$\begin{aligned}\tilde I_2&=\frac{2}{|B|^{\beta/n}}|B|^{-1/p}\|T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^1)\|_{L^p(B)}\\&\leqslant\frac{C}{|B|^{\beta/n}}|B|^{-1/q}\|T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^1)\|_{L^{q,\infty}(B)}\\&\leqslant\frac{C}{|B|^{\beta/n+1/q}}\left(\int_{\tilde B^*}|(b_1(z)-b_B^1)f_1(y_1)|^{p_1}\mathrm{d}y_1\right)^{1/p_1}\left(\int_{\tilde B^*}|f_2(y_2)|^{p_2}\mathrm{d}y_2\right)^{1/p_2}\\&\leqslant\frac{C}{|B|^{\beta/n+1/q}}(\|b_1-b_{\tilde B^*}^1\|_{L^\infty(\tilde B^*)}+|b_{\tilde B^*}^1-b_B^1|)\|f_1\|_{L^{p_1}(\tilde B^*)}\|f_2\|_{L^{p_2}(\tilde B^*)}\\&\leqslant\frac{C}{|B|^{\beta/n+1/q}}(|\tilde B^*|^{\gamma/n}\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}+|\tilde B^*|^{\gamma/n}\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)})\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\&\leqslant Cr^{(\alpha-1)\beta+n(\alpha/p-1/q)}\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\&\leqslant C\|b_1\|_{\dot\Lambda_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。\end{aligned}$$

对任意的 $z\in B=B(x_0,r)$, $y_1\in\tilde B^*$, $y_2\in(\tilde B^*)^c$,则 $|z-x_0|^\alpha\leqslant r^\alpha<\frac{1}{2}|y_2-x_0|$。根据式(4),有

$$\begin{aligned}&|T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^2)(z)-T((b_1-b_B^1)\tilde f_1^1,\tilde f_2^2)(x_0)|\\&\leqslant\int_{\tilde B^*}\int_{(\tilde B^*)^c}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\end{aligned}$$

$$\leqslant C\int_{\tilde{B}^*}\int_{(\tilde{B}^*)^c}\frac{|z-x_0|^{\varepsilon}}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^{\infty}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*\setminus 2^{k-1}\tilde{B}^*}\frac{|z-x_0|^{\varepsilon}}{|x_0-y_2|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1,$$

因此,

$$\tilde{I}_3\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B\left|C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\right|^p\mathrm{d}z\right)^{1/p}$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_{\tilde{B}^*}^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$+C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_{\tilde{B}^*}^1-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$:=\tilde{I}_{3,1}+\tilde{I}_{3,2}。$$

因为 $0<\gamma=\beta+\dfrac{n}{p}<1$, $1/p=1/p_1+1/p_2$,且 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,根据 Hölder 不等式和引理 1,有

$$\tilde{I}_{3,1}\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\|b_1-b_{\tilde{B}^*}^1\|_{L^\infty(\tilde{B}^*)}\int_{\tilde{B}^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^k\tilde{B}^*}|f_2(y_2)|\mathrm{d}y_2$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\tilde{B}^*)}|\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_2'}$$

$$\leqslant Cr^{\beta(\alpha-1)}2^{2n}\sum_{k=1}^{\infty}2^{kn(1/p_2'-2)-k\varepsilon/\alpha+\beta}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

根据 Hölder 不等式和引理 2,有

$$\tilde{I}_{3,2}=C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|b_{\tilde{B}^*}^1-b_B^1|\int_{\tilde{B}^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^k\tilde{B}^*}|f_2(y_2)|\mathrm{d}y_2$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\tilde{B}^*)}|\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_2'}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

结合 $\tilde{I}_{3,1}$ 与 $\tilde{I}_{3,2}$,有 $\tilde{I}_3\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

对任意的 $z\in B=B(x_0,r)$, $y_2\in\tilde{B}^*$, $y_1\in(\tilde{B}^*)^c$,则 $|z-x_0|^{\alpha}\leqslant r^{\alpha}<\dfrac{1}{2}|y_1-x_0|$。根据式(4),有

$$|T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^1)(z)-T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^1)(x_0)|$$

$$\leqslant\int_{(\tilde{B}^*)^c}\int_{\tilde{B}^*}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\int_{(\tilde{B}^*)^c}\int_{\tilde{B}^*}\frac{|z-x_0|^{\varepsilon}}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^{\infty}\int_{2^k\tilde{B}^*\setminus 2^{k-1}\tilde{B}^*}\int_{\tilde{B}^*}\frac{|z-x_0|^{\varepsilon}}{|x_0-y_1|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$\leqslant C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^k\tilde{B}^*}\int_{\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1,$$

因此,

$$\tilde{I}_4 \leqslant \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B \left| C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^k\tilde{B}^*}\int_{\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\,dy_2dy_1\right|^p dz\right)^{1/p}$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{\tilde{B}^*}|b_1(y_1)-b_{2^k\tilde{B}^*}^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$+C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{\tilde{B}^*}|b_{2^k\tilde{B}^*}^1-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$:=\tilde{I}_{4,1}+\tilde{I}_{4,2}。$$

因为 $0<\gamma=\beta+\dfrac{n}{p}<1$, $1/p=1/p_1+1/p_2$,且 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,根据 Hölder 不等式和引理 1,有

$$\tilde{I}_{4,1}\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\|b_1-b_{2^k\tilde{B}^*}^1\|_{L^\infty(2^k\tilde{B}^*)}\int_{2^k\tilde{B}^*}|f_1(y_1)|dy_1\int_{\tilde{B}^*}|f_2(y_2)|dy_2$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^k\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(\tilde{B}^*)}|\tilde{B}^*|^{1/p_2'}$$

$$\leqslant Cr^{\beta(\alpha-1)}2^{2n}\sum_{k=1}^{\infty}2^{(k+1)\beta+kn(1/p-1)+kn(1/p_1'-1)-k\varepsilon/\alpha}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

根据 Hölder 不等式和引理 2,有

$$\tilde{I}_{4,2}=C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|b_{2^k\tilde{B}^*}^1-b_B^1|\int_{2^k\tilde{B}^*}|f_1(y_1)|dy_1\int_{\tilde{B}^*}|f_2(y_2)|dy_2$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^k\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(\tilde{B}^*)}|\tilde{B}^*|^{1/p_2'}$$

$$\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。$$

结合 $\tilde{I}_{4,1}$ 与 $\tilde{I}_{4,2}$,有 $\tilde{I}_4\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

对任意的 $z\in B=B(x_0,r)$, $y_1\in(\tilde{B}^*)^c$, $y_2\in(\tilde{B}^*)^c$,则 $|z-x_0|^\alpha\leqslant r^\alpha<\dfrac{1}{2}|y_1-x_0|$, $|z-x_0|^\alpha\leqslant r^\alpha<\dfrac{1}{2}|y_2-x_0|$。根据式(4),有

$$|T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^2)(z)-T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^2)(x_0)|$$

$$\leqslant\int_{(\tilde{B}^*)^c}\int_{(\tilde{B}^*)^c}|K(z,y_1,y_2)-K(x_0,y_1,y_2)||b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$\leqslant C\int_{(\tilde{B}^*)^c}\int_{(\tilde{B}^*)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$\leqslant C\sum_{k=1}^{\infty}\int_{2^k\tilde{B}^*\backslash 2^{k-1}\tilde{B}^*}\int_{2^k\tilde{B}^*\backslash 2^{k-1}\tilde{B}^*}\frac{|z-x_0|^\varepsilon}{|x_0-y_2|^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$\leqslant C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^k\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1,$$

因此,

$$\tilde{I}_5\leqslant\frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B\left|C\sum_{k=1}^{\infty}\frac{r^{-2n\alpha}}{2^{k(2n+\varepsilon/\alpha)}}\int_{2^k\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\,dy_2dy_1\right|^p dz\right)^{1/p}$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{2^k\tilde{B}^*}|b_1(y_1)-b_{2^k\tilde{B}^*}^1||f_1(y_1)||f_2(y_2)|dy_2dy_1$$

$$+C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\int_{2^k\tilde{B}^*}\int_{2^k\tilde{B}^*}|b^1_{2^k\tilde{B}^*}-b^1_B||f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1$$

$$:=\tilde{I}_{5,1}+\tilde{I}_{5,2}。$$

因为 $0<\gamma=\beta+\frac{n}{p}<1$, $1/p=1/p_1+1/p_2$,且 $1/p_1+1/p_1'=1$, $1/p_2+1/p_2'=1$,根据 Hölder 不等式和引理 1,有

$$\begin{aligned}\tilde{I}_{5,1}&\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}\|b_1-b^1_{2^k\tilde{B}^*}\|_{L^\infty(2^k\tilde{B}^*)}\int_{2^k\tilde{B}^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^k\tilde{B}^*}|f_2(y_2)|\mathrm{d}y_2\\&\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^k\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_2'}\\&\leqslant Cr^{\beta(\alpha-1)}2^{2n}\sum_{k=1}^{\infty}2^{(k+1)\beta-k\varepsilon/\alpha}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}\\&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。\end{aligned}$$

根据 Hölder 不等式和引理 2,有

$$\begin{aligned}\tilde{I}_{5,2}&=C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|b^1_{2^k\tilde{B}^*}-b^1_B|\int_{2^k\tilde{B}^*}|f_1(y_1)|\mathrm{d}y_1\int_{2^k\tilde{B}^*}|f_2(y_2)|\mathrm{d}y_2\\&\leqslant C\frac{r^{-2n\alpha}}{|B|^{\beta/n}}\sum_{k=1}^{\infty}2^{-k(2n+\varepsilon/\alpha)}|2^k\tilde{B}^*|^{\gamma/n}\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_1'}\|f_2\|_{L^{p_2}(2^k\tilde{B}^*)}|2^k\tilde{B}^*|^{1/p_2'}\\&\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}。\end{aligned}$$

结合 $\tilde{I}_{5,1}$ 与 $\tilde{I}_{5,2}$,有 $\tilde{I}_5\leqslant C\|b_1\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\|f_2\|_{L^{p_2}(\mathbf{R}^n)}$。

关于 $T^2_{\vec{b}}(\vec{f})(z)$,类似于 $T^1_{\vec{b}}(\vec{f})(z)$ 的证明,有

$$\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T^2_{\vec{b}}(f_1,f_2)(z)-(T^2_{\vec{b}}(f_1,f_2))_B|^p\mathrm{d}z\right)^{1/p}\leqslant C\|b_2\|_{\dot{\Lambda}_\gamma(\mathbf{R}^n)}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}。$$

因此,

$$\|T_{\vec{b}}(\vec{f})\|_{C^{p,\beta}(\mathbf{R}^n)}\approx\sup_B\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B|T_{\vec{b}}(f_1,f_2)(z)-(T_{\vec{b}}(f_1,f_2)))_B|^p\mathrm{d}z\right)^{1/p}\leqslant C\|\vec{b}\|_{\dot{\Lambda}^2_\gamma}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}。$$

从而完成定理的证明。

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