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E-mail:xblxb@sdu.edu.cn
http://lxbwk.njournal.sdu.edu.cn
Tel:+86-531-88366917

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广义 Petersen 图 $P(n,k)$ 的等全着色

苏亚男,仝春灵*,李勇,苏森原
(山东交通学院信息科学与电气工程学院, 山东 济南 250357)

摘要:广义 Petersen 图 $P(n,k)$ 是着色问题中研究得最广泛的一类图,但是当 $k(\mathrm{mod}\ 4)=0$ 时 $P(n,k)$ 的全着色还有待进一步研究。采用计算机搜索和数学证明相结合的方法,求得 $k(\mathrm{mod}\ 16)=4,8,12$ 以及 $k(\mathrm{mod}\ 16)=0 \wedge n(\mathrm{mod}\ 2k)=0,1,2,4$ 时 $P(n,k)$ 的等全色数。

关键词:广义 Petersen 图;等全着色;等全色数
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Equitable total coloring of generalized Petersen graphs $P(n,k)$

SU Yanan, TONG Chunling*, LI Yong, SU Senyuan
(College of lnformation Science and Electrical Engineering, Shandong Jiaotong University, Jinan 250357, Shandong, China)

Abstract: Generalized Petersen graphs $P(n,k)$ are the most widely studied in coloring problems. However, the total chromatic number of $P(n,k)$ for $k(\mathrm{mod}\ 4)=0$ needs to be further studied. By combining computer searching and mathematics techniques, the equitable total chromatic number of $P(n,k)$ for $k(\mathrm{mod}\ 16)=4,8,12$ and $k(\mathrm{mod}\ 16)=0 \wedge n(\mathrm{mod}\ 2k)=0,1,2,4$ are obtained.

Key words: generalized Petersen graph; equitable total coloring; equitable total chromatic number

0　引言

图着色是图论主要的研究方向,不仅在理论上具有重要的研究意义,而且在现实生活中也有广泛的应用[1]。学者们在点着色和边着色的基础上,又提出了全着色和等全着色的概念[2-3],该问题得到大量关注和研究。

定义 1　给定图 $G=(V,E)$, 对它的一个映射 $\sigma: V\cup E\rightarrow\{1,2,\cdots,k\}$, 若满足下列条件:

(1) $\forall u,v\in V, uv\in E$, 有 $\sigma(u)\neq\sigma(v)$, $\sigma(uv)\neq\sigma(u)$, $\sigma(uv)\neq\sigma(v)$;

(2) $\forall uv', uv''\in E$, 有 $\sigma(uv')\neq\sigma(uv'')$,

则称 σ 是图 G 的一个 k-全着色。使图 G 可 k-全着色的最小 k 值称为 G 的全色数,记为 $\chi''(G)$。

Sánchez-Arroyo[4] 证明了确定一个任意图的全色数是不确定多项式(non-deterministic polynomial, NP)困难的;Mcdiarmid 和 Sánchez-Arroyo[5] 证明了对于 $k\geqslant 3$,确定一个 k-正则二部图的全色数也是 NP 困难的。

对图 $G=(V,E)$ 的一个映射 $\sigma: V\cup E\rightarrow\{1,2,\cdots,k\}$,若除了满足定义 1 的(1)、(2)外,还满足 $\forall i,j\in\{1,2,\cdots,k\}$,有 $||\sigma^{-1}(i)|-|\sigma^{-1}(j)||\leqslant 1$,则称 σ 是图 G 的一个 k-等全着色。使图 G 可 k-等全着色的最小 k 值称为 G 的等全色数, 记为 $\chi''_{=}(G)$。Fu[3] 最早提出了等全着色的概念, 并猜想: 对于图 G, 若 $k\geqslant$

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第一作者简介:苏亚男(1998—),女,硕士研究生,研究方向为图论与组合优化. E-mail:1846925207@qq.com
* **通信作者简介**:仝春灵(1971—),女,教授,博士,研究方向为算法设计与分析、组合最优化. E-mail:tongcl@sdjtu.edu.cn

$\max(\chi''(G),\Delta(G)+2)$ ($\Delta(G)$ 为图 G 的最大度),则图 G 存在 k-等全着色。Wang[6] 进一步猜想:对于图 G,有$\chi''_{=}(G)\leqslant\Delta(G)+2$。这些猜想对树、完全图、多部图、3-退化图和 $\Delta(G)\leqslant 3$ 的图等成立[3,6-12]。也有一些特殊的图类,如笛卡尔积图 $C_m\square C_n$、Knödel 图、snarks 图、平面图、联图等[13-22],求得了等全色数的确切值。

在与着色相关的图当中,有一类非常重要的三正则图——广义 Petersen 图,人们对其着色进行了广泛而深入的研究。Dantas 等[23] 给出了 $n\geqslant(2k-1)(2k-2)$ 时 $P(n,k)$ 的全色数,Tong 等[24] 找到了 $k(\bmod 4)=1,2,3$ 时 $P(n,k)$ 的等全色数, 但是 $k(\bmod 4)=0$ 时 $P(n,k)$ 的全色数还有很多未知结果。本文研究了 $k(\bmod 4)=0$ 时 $P(n,k)$ 的等全着色,证明了 $k(\bmod 16)=4,8,12$ 以及 $k(\bmod 16)=0\wedge n(\bmod 2k)=0,1,2,4$ 时 $P(n,k)$ 的等全色数是 4,并猜想 $k(\bmod 16)=0$ 时 $P(n,k)$ 的等全色数也是 4。

1 引理及证明

定义 2 广义 Petersen 图 $P(n,k)$[25] 是由顶点集 $V(P(n,k))$ 和边集 $E(P(n,k))$ 构成的具有 $2n$ 个顶点的图, 其中顶点集 $V(P(n,k))=\{v_0,v_1,\cdots,v_{n-1},u_0,u_1,\cdots,u_{n-1}\}$, 边集 $E(P(n,k))=\{v_iu_i,v_iv_{i+1},u_iu_{i+k}\mid i=0,1,\cdots,n-1\}$, 所有顶点下标对 n 取模。通常,v_iv_{i+1}称为外圈的边,u_iu_{i+k}称为内圈的边,v_iu_i 称为弦。图 1 给出了 Petersen 图 $P(5,2)$。

令导出子图 $G_i=P(n,k)[v_{i-1},v_i,v_{i+1},u_{i-k},u_i,u_{i+k}]$ $(0\leqslant i\leqslant n-1)$,如图 2 所示。

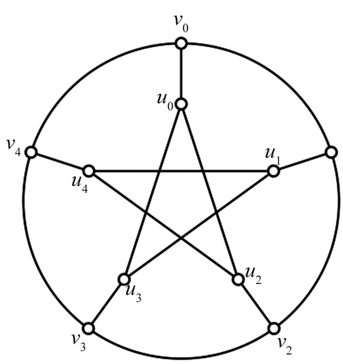


图 1 Petersen 图 $P(5,2)$
Fig.1 Petersen graph $P(5,2)$

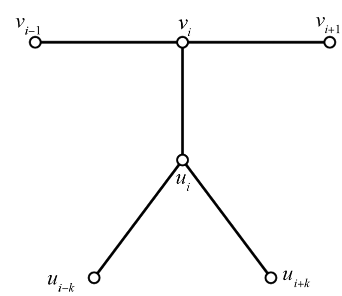


图 2 导出子图 G_i
Fig.2 Induced subgraph G_i

引理 1[24] 如果 σ 是 $P(n,k)$的 4-全着色,则:

$$\sigma(v_iu_i)=\{1,2,3,4\}-\sigma(E(G_i)-v_iu_i);$$
$$\sigma(v_i)=\{1,2,3,4\}-\sigma(\{v_{i-1}v_i,v_iv_{i+1},v_iu_i\});$$
$$\sigma(u_i)=\{1,2,3,4\}-\sigma(\{u_{i-k}u_i,u_iu_{i+k},u_iv_i\})。$$

令

$$E_1=\{v_iv_{i+1}:0\leqslant i\leqslant n-1\};$$
$$E_2=\{u_iu_{i+k}:0\leqslant i\leqslant n-1\};$$
$$E_3=\{v_iu_i:0\leqslant i\leqslant n-1\};$$
$$V_1=\{v_i:0\leqslant i\leqslant n-1\};$$
$$V_2=\{u_i:0\leqslant i\leqslant n-1\}。$$

根据引理 1,如果 σ 是 $P(n,k)$ 的 4-全着色,则 σ 可由 $\sigma(E_1\cup E_2)$决定,因此,在后面的证明过程中只给出 $\sigma(E_1\cup E_2)$。

引理 2 当 $k(\bmod 8)=4$ 时,$P(n,k)$可 4-等全着色。

证明 用 $(i_1i_2i_3i_4)^m$ 表示 $\underbrace{i_1i_2i_3i_4\cdots i_1i_2i_3i_4}_{m}$ $(i_1, i_2, i_3, i_4\in\{1,2,3,4\})$, 例如:$23(2431)^2312=2324312431312$。令 $n_8=n(\bmod 8)$, 根据 n_8 的取值,构造如下的 $\sigma(E_1\cup E_2)$:

$$\sigma(E_1)=\begin{cases}(12123434)^{\frac{n}{8}}, & n_8=0;\\ (34342121)^{\frac{n-k-5}{8}}(43421213)^{\frac{k-4}{8}}434242121, & n_8=1;\\ (23234141)^{\frac{n-2k-2}{8}}23234141(23234141)^{\frac{k-4}{8}}(21234341)^{\frac{k-4}{8}}21, & n_8=2;\\ (32341412)^{\frac{n-11}{8}}32343434132, & k=4,\ n_8=3;\\ (23234231)^{\frac{k-4}{8}}23234241(41341431)^{\frac{k-12}{8}}41341413214, & k\geqslant 12,\ n=2k+3,\ n_8=3;\\ (32341412)^{\frac{n-2k-3}{8}}323(23414123)^{\frac{k-4}{8}}1434(21324314)^{\frac{k-12}{8}}213243134132, & k\geqslant 12,\ n\geqslant 2k+11,\ n_8=3;\\ (12323414)^{\frac{n-2k-4}{8}}123234(24324214)^{\frac{k-4}{8}}243213(21314313)^{\frac{k-4}{8}}, & n_8=4;\\ (23141423)^{\frac{n-2k-5}{8}}231412(32314142)^{\frac{k-4}{8}}3234341(24321341)^{\frac{k-4}{8}}, & n_8=5;\\ (12123434)^{\frac{n-k-2}{8}}142343(41212343)^{\frac{k-4}{8}}, & n_8=6;\\ (12123434)^{\frac{n-7}{8}}1212343, & n_8=7。\end{cases}$$

$$\sigma(E_2)=\begin{cases}(12123434)^{\frac{n}{8}}, & n_8=0;\\ (12134342)^{\frac{n-k-5}{8}}(21213434)^{\frac{k-4}{8}}212134342, & n_8=1;\\ (12123434)^{\frac{n-2k-2}{8}}12123134(13123134)^{\frac{k-4}{8}}(24124234)^{\frac{k-4}{8}}24, & n_8=2;\\ (12123434)^{\frac{n-11}{8}}12123333424, & k=4,n_8=3;\\ (12123434)^{\frac{k-4}{8}}12123324(42423133)^{\frac{k-12}{8}}42423133424, & k\geqslant 12,\ n=2k+3,\ n_8=3;\\ (12123434)^{\frac{n-2k-3}{8}}121(22334411)^{\frac{k-4}{8}}1333(34241242)^{\frac{k-12}{8}}342412423424, & k\geqslant 12,\ n\geqslant 2k+11,\ n_8=3;\\ (12123434)^{\frac{n-2k-4}{8}}121234(23234141)^{\frac{k-4}{8}}232334(12123434)^{\frac{k-4}{8}}, & n_8=4;\\ (13142423)^{\frac{n-2k-5}{8}}131422(33114422)^{\frac{k-4}{8}}3314123(43143423)^{\frac{k-4}{8}}, & n_8=5;\\ (12123434)^{\frac{n-k-2}{8}}124343(41212343)^{\frac{k-4}{8}}, & n_8=6;\\ (12123434)^{\frac{n-7}{8}}1212343, & n_8=7。\end{cases}$$

由引理 1,可得到:

$$\sigma(E_3)=\begin{cases}(23434121)^{\frac{n}{8}}, & n_8=0;\\ (21213434)^{\frac{n-k-5}{8}}3213(43421213)^{\frac{k-4}{8}}13434, & n_8=1;\\ (41412323)^{\frac{n-2k-2}{8}}41412323(41412323)^{\frac{k-4}{8}}(43412123)^{\frac{k-4}{8}}43, & n_8=2;\\ (41412323)^{\frac{n-11}{8}}41412121241, & k=4,\ n_8=3;\\ 31411142(41411142)^{\frac{k-12}{8}}41412133(23212324)^{\frac{k-4}{8}}143, & k\geqslant 12,\ n=2k+3,\ n_8=3;\\ (41412323)^{\frac{n-2k-3}{8}}41(41412323)^{\frac{k-4}{8}}42212(13413123)^{\frac{k-12}{8}}134131242241, & k\geqslant 12,\ n\geqslant 2k+11,\ n_8=3;\\ 44414(12323414)^{\frac{n-2k-4}{8}}13111(13333111)^{\frac{k-4}{8}}(42442422)^{\frac{k-4}{8}}42, & n_8=4;\\ 3442(32314142)^{\frac{n-2k-5}{8}}34414(23231414)^{\frac{k-4}{8}}1212(31214212)^{\frac{k-4}{8}}, & n_8=5;\\ 2434(12123434)^{\frac{k-4}{8}}4(12123434)^{\frac{n-2k-6}{8}}121231121(23434121)^{\frac{k-4}{8}}, & n_8=6;\\ 4434(12123434)^{\frac{k-4}{8}}4(12123434)^{\frac{n-k-3}{8}}12, & n_8=7。\end{cases}$$

$$\sigma(V_1)=\begin{cases}(34341212)^{\frac{n}{8}}, & n_8=0;\\ (42121343)^{\frac{n-k-5}{8}}(21213434)^{\frac{k-4}{8}}212131343, & n_8=1;\\ (34141232)^{\frac{n-2k-2}{8}}34141232(34141232)^{\frac{k-4}{8}}(34341212)^{\frac{k-4}{8}}34, & n_8=2;\\ (14123234)^{\frac{n-11}{8}}14121212324, & k=4,\ n_8=3;\\ 1414(23143414)^{\frac{k-4}{8}}13(12324232)^{\frac{k-12}{8}}1232423232432, & k\geqslant 12,\ n=2k+3,\ n_8=3;\\ (14123234)^{\frac{n-2k-3}{8}}1(41412323)^{\frac{k-4}{8}}414321(34241242)^{\frac{k-12}{8}}342412421324, & k\geqslant 12,\ n\geqslant 2k+11,\ n_8=3;\\ 2314(12323414)^{\frac{n-2k-4}{8}}121324(31421324)^{\frac{k-4}{8}}34(13423124)^{\frac{k-4}{8}}, & n_8=4;\\ 412323(14142323)^{\frac{n-2k-5}{8}}(14142323)^{\frac{k-4}{8}}1412123(43143423)^{\frac{k-4}{8}}, & n_8=5;\\ 4343(41212343)^{\frac{k-4}{8}}(12123434)^{\frac{n-2k-6}{8}}1212323412(12343412)^{\frac{k-4}{8}}, & n_8=6;\\ 2343(41212343)^{\frac{k-4}{8}}(12123434)^{\frac{n-k-3}{8}}121, & n_8=7。\end{cases}$$

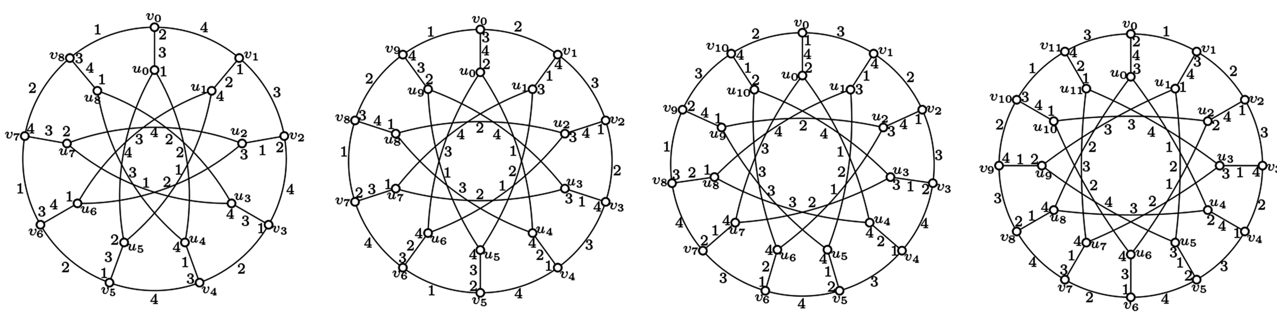
$$\sigma(V_2)=\begin{cases}(41212343)^{\frac{n}{8}}, & n_8=0;\\ (34342121)^{\frac{n-k-5}{8}}143(42121343)^{\frac{k-4}{8}}442121, & n_8=1;\\ 23334(14123234)^{\frac{n-10}{8}}44112, & k=4,\ n_8=2;\\ 23334(11123334)^{\frac{n-2k-2}{8}}(41122334)^{\frac{3k-n-2}{8}}(44122234)^{\frac{n-2k-2}{8}}(44112233)^{\frac{k-4}{8}}44112, & k\geqslant 12,\ 2k+2\leqslant n\leqslant 3k-10,\ n_8=2;\\ 23334(11123334)^{\frac{k-4}{8}}(14123234)^{\frac{n-3k+2}{8}}(44122234)^{\frac{k-4}{8}}(44112233)^{\frac{k-4}{8}}44112, & k\geqslant 12,\ n\geqslant 3k-2,\ n_8=2;\\ 2333(41412323)^{\frac{n-11}{8}}4444112, & k=4,\ n_8=3;\\ 2333(42212333)^{\frac{k-4}{8}}(44411113)^{\frac{k-4}{8}}4441211, & k\geqslant 12,\ n=2k+3,\ n_8=3;\\ 2333(41112333)^{\frac{n-2k-3}{8}}3(41112333)^{\frac{3k-n-1}{8}}4411223(34411223)^{\frac{n-2k-11}{8}}4444112(23344112)^{\frac{k-12}{8}}23334112, & k\geqslant 20,\ 2k+11\leqslant n\leqslant 3k-9,\ n_8=3;\\ 2333(41112333)^{\frac{k-4}{8}}(41412323)^{\frac{n-3k+1}{8}}(34411223)^{\frac{k-4}{8}}4444112(23344112)^{\frac{k-12}{8}}23334112, & k\geqslant 12,\ n\geqslant 3k-1,\ n_8=3;\\ 312323(41412323)^{\frac{n-2k-4}{8}}444(22224444)^{\frac{k-4}{8}}211(31331311)^{\frac{k-4}{8}}, & n_8=4;\\ 223(14142323)^{\frac{n-13}{8}}1412233444, & k=4,\ n_8=5;\\ 223(11142223)^{\frac{n-2k-5}{8}}1112223(11142223)^{\frac{3k-n-7}{8}}(11442233)^{\frac{n-2k+3}{8}}444(22331144)^{\frac{k-4}{8}}, & k\geqslant 12,\ 2k+5\leqslant n\leqslant 3k-7,\ n_8=5;\\ 223(11142223)^{\frac{k-4}{8}}(14142323)^{\frac{n-3k-1}{8}}1412233(11442233)^{\frac{k-4}{8}}444(22331144)^{\frac{k-4}{8}}, & k\geqslant 12,\ n\geqslant 3k+1,\ n_8=5;\\ 3(12123434)^{\frac{n-k-2}{8}}12234(34121234)^{\frac{k-4}{8}}, & n_8=6;\\ 3(12123434)^{\frac{n-7}{8}}121234, & n_8=7。\end{cases}$$

根据全着色的定义,可以证明 σ 是 $P(n,k)$ 的 4-全着色。由

$$
\begin{aligned}
&|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=\frac{5n}{4}, && n_8=0,4;\\
&|\sigma^{-1}(1)|=\frac{5n+3}{4},\quad |\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=\frac{5n-1}{4}, && n_8=1;\\
&|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=\frac{5n-2}{4},\quad |\sigma^{-1}(3)|=|\sigma^{-1}(4)|=\frac{5n+2}{4}, && n_8=2;\\
&|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(4)|=\frac{5n+1}{4},\quad |\sigma^{-1}(3)|=\frac{5n-3}{4}, && n_8=3;\\
&|\sigma^{-1}(1)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=\frac{5n-1}{4},\quad |\sigma^{-1}(2)|=\frac{5n+3}{4}, && n_8=5;\\
&|\sigma^{-1}(1)|=|\sigma^{-1}(4)|=\frac{5n-2}{4},\quad |\sigma^{-1}(2)|=|\sigma^{-1}(3)|=\frac{5n+2}{4}, && n_8=6;\\
&|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=\frac{5n+1}{4},\quad |\sigma^{-1}(4)|=\frac{5n-3}{4}, && n_8=7,
\end{aligned}
$$

得 $||\sigma^{-1}(i)|-|\sigma^{-1}(j)||\leqslant 1(1\leqslant i\leqslant j\leqslant 4)$, 因此,当 $k(\mathrm{mod}\ 8)=4$ 时,σ 是 $P(n,k)$ 的 4-等全着色。

图 3 给出了 $n=9,10,11,12$ 时 $P(n,4)$ 的 4-等全着色。



(a) $\sigma(P(9,4))$　(b) $\sigma(P(10,4))$　(c) $\sigma(P(11,4))$　(d) $\sigma(P(12,4))$

图 3　$n=9,10,11,12$ 时 $\sigma(P(n,4))$

Fig.3　$\sigma(P(n,4))$ for $n=9,10,11,12$

引理 3　当 $k(\mathrm{mod}\ 16)=8$ 时, $P(n,k)$ 可 4-等全着色。

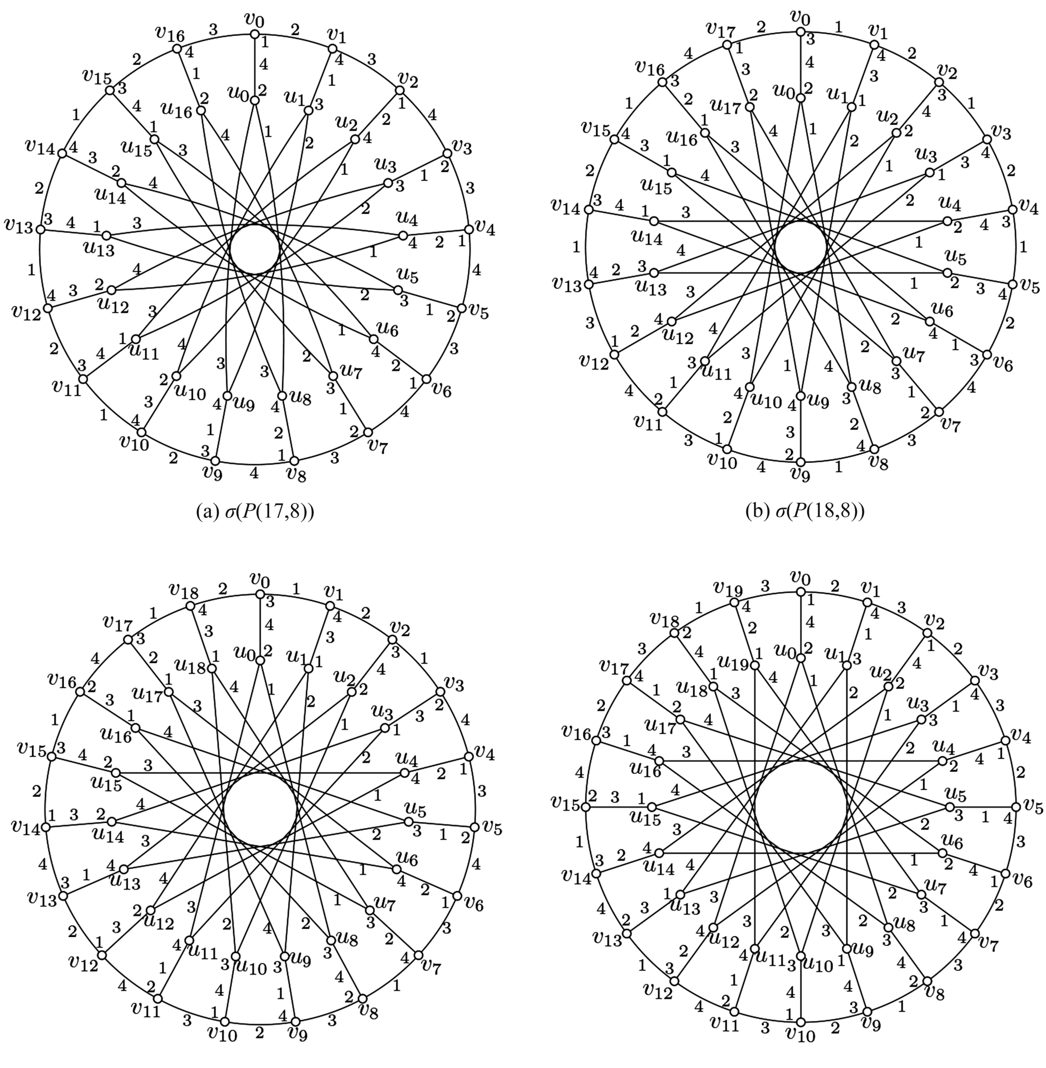
证明　令 $n_{16}=n(\mathrm{mod}\ 16)$, 构造如下的 $\sigma(E_1\cup E_2)$:

$$
\sigma(E_1)=\begin{cases}
(1212121234343434)^{\frac{n}{16}}, & n_{16}=0;\\
(23434343\ 41212121)^{\frac{n-k-9}{16}}23434343421212123(4343434121212123)^{\frac{k-8}{16}}, & n_{16}=1;\\
(1212343434341212)^{\frac{n-2k-2}{16}}12121243(1434342132121243)^{\frac{k-8}{16}}1434312142(1213432434312142)^{\frac{k-8}{16}}, & n_{16}=2;\\
(12143434\ 34321212)^{\frac{n-2k-3}{16}}12143431(3232121214143431)^{\frac{k-8}{16}}32342421412(4213142342421412)^{\frac{k-8}{16}}, & n_{16}=3;\\
(2323232341414141)^{\frac{n-2k-4}{16}}23232323(1234131424132323)^{\frac{k-8}{16}}123414142313(2423412414142313)^{\frac{k-8}{16}}, & n_{16}=4;\\
(12323434\ 34141212)^{\frac{n-2k-5}{16}}12323432(3414121412323432)^{\frac{k-8}{16}}3414131214232(4243142431214232)^{\frac{k-8}{16}}, & n_{16}=5;\\
(2323232341414141)^{\frac{n-2k-6}{16}}23232323(2412341421421323)^{\frac{k-8}{16}}24123413413213(2341231413413213)^{\frac{k-8}{16}}, & n_{16}=6;\\
(12434121\ 43123243)^{\frac{n-2k-7}{16}}12434124(1314242432434124)^{\frac{k-8}{16}}131424241242323(2142132141242323)^{\frac{k-8}{16}}, & n_{16}=7;
\end{cases}
$$

$$\sigma(E_1)=\begin{cases}(1414141232323234)^{\frac{n-2k-8}{16}}1414141241231341(2341312341231341)^{\frac{k-8}{16}}34124234(1234241234124234)^{\frac{k-8}{16}}, & n_{16}=8;\\ (12123412~34341234)^{\frac{n-2k-9}{16}}1212341232434134(1241234232434134)^{\frac{k-8}{16}}234124134(1231234234124134)^{\frac{k-8}{16}}, & n_{16}=9;\\ (41414141~23232323)^{\frac{n-2k-10}{16}}4141414123412314(1421231323412314)^{\frac{k-8}{16}}1421423423(2412341421423423)^{\frac{k-8}{16}}, & n_{16}=10;\\ (4343212121214343)^{\frac{n-2k-11}{16}}4343212121241324(3132314212141324)^{\frac{k-8}{16}}31323212343(4143121323212343)^{\frac{k-8}{16}}, & n_{16}=11;\\ (41414141~23232323)^{\frac{n-2k-12}{16}}4141414123232343(4141412123232343)^{\frac{k-8}{16}}414124232323(4241414124232323)^{\frac{k-8}{16}}, & n_{16}=12;\\ (43432121~21214343)^{\frac{n-2k-13}{16}}4343212121214142(4313132421243142)^{\frac{k-8}{16}}4313123214343(4143212123214343)^{\frac{k-8}{16}}, & n_{16}=13;\\ (1212121234343434)^{\frac{n-k-6}{16}}12121423434343(4121212123434343)^{\frac{k-8}{16}}, & n_{16}=14;\\ (12121212~34343434)^{\frac{n-15}{16}}121212123434343, & n_{16}=15。\end{cases}$$

$$\sigma(E_2)=\begin{cases}(1212121234343434)^{\frac{n}{16}}, & n_{16}=0;\\ (12121212~34343434)^{\frac{n-k-9}{16}}12121212334343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=1;\\ (12121212~34343434)^{\frac{n-2k-2}{16}}12121122(4134334423121122)^{\frac{k-8}{16}}4134343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=2;\\ (12121212~34343434)^{\frac{n-2k-3}{16}}12121211(2424343412421211)^{\frac{k-8}{16}}24234343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=3;\\ (12121212~34343434)^{\frac{n-2k-4}{16}}12121212(2323334441411212)^{\frac{k-8}{16}}232334343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=4;\\ (12121212~34343434)^{\frac{n-2k-5}{16}}12121212(2431343441131212)^{\frac{k-8}{16}}2431334342434(1212422234342434)^{\frac{k-8}{16}}, & n_{16}=5;\\ (12121212~34343434)^{\frac{n-2k-6}{16}}12121213(2323233441411113)^{\frac{k-8}{16}}23232334343434(1212124234343434)^{\frac{k-8}{16}}, & n_{16}=6;\\ (1132343124431224)^{\frac{n-2k-7}{16}}11323431(3244412243123131)^{\frac{k-8}{16}}324441224431224(1112343124431224)^{\frac{k-8}{16}}, & n_{16}=7;\\ (1212121234343434)^{\frac{n-2k-8}{16}}1212121241414141(2323232341414141)^{\frac{k-8}{16}}34343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=8;\\ (1212341234341234)^{\frac{n-2k-9}{16}}1212341234241244(1242312334241244)^{\frac{k-8}{16}}234341234(1212341234341234)^{\frac{k-8}{16}}, & n_{16}=9;\\ (12121212~34343434)^{\frac{n-2k-10}{16}}1212121234414141(4222232334414141)^{\frac{k-8}{16}}4234343434(1312121234343434)^{\frac{k-8}{16}}, & n_{16}=10;\\ (1212121234343434)^{\frac{n-2k-11}{16}}1212121234313123(2123134134313123)^{\frac{k-8}{16}}21234342434(1212421234342434)^{\frac{k-8}{16}}, & n_{16}=11;\\ (1212121234343434)^{\frac{n-2k-12}{16}}1212121234342443(1212422134342443)^{\frac{k-8}{16}}121231343434(1312121231343434)^{\frac{k-8}{16}}, & n_{16}=12;\\ (12121212~34343434)^{\frac{n-2k-13}{16}}1212121234343131(2121231343434131)^{\frac{k-8}{16}}2121234343434(1212121234343434)^{\frac{k-8}{16}}, & n_{16}=13;\\ (12121212~34343434)^{\frac{n-k-6}{16}}12121243434343(4121212123434343)^{\frac{k-8}{16}}, & n_{16}=14;\\ (12121212~34343434)^{\frac{n-15}{16}}121212123434343, & n_{16}=15。\end{cases}$$

用与引理 2 类似的方法可证明，当 $k(\bmod 16)=8$ 时，该着色 σ 是 $P(n,k)$的 4-等全着色，此处不再赘述。图 4 给出了 $n=17,18,19,20$ 时 $P(n,8)$的 4-等全着色。



(a) $\sigma(P(17,8))$　(b) $\sigma(P(18,8))$

(c) $\sigma(P(19,8))$　(d) $\sigma(P(20,8))$

图 4　$n=17,18,19,20$ 时 $\sigma(P(n,8))$
Fig.4　$\sigma(P(n,8))$ for $n=17,18,19,20$

引理 4　当 $k(\bmod 16)=0 \wedge n(\bmod 2k)=0,1,2,4$ 时，$P(n,k)$ 可 4-等全着色。

证明　令 $n_{2k}=n(\bmod 2k)$，构造如下的 $\sigma(E_1 \cup E_2)$：

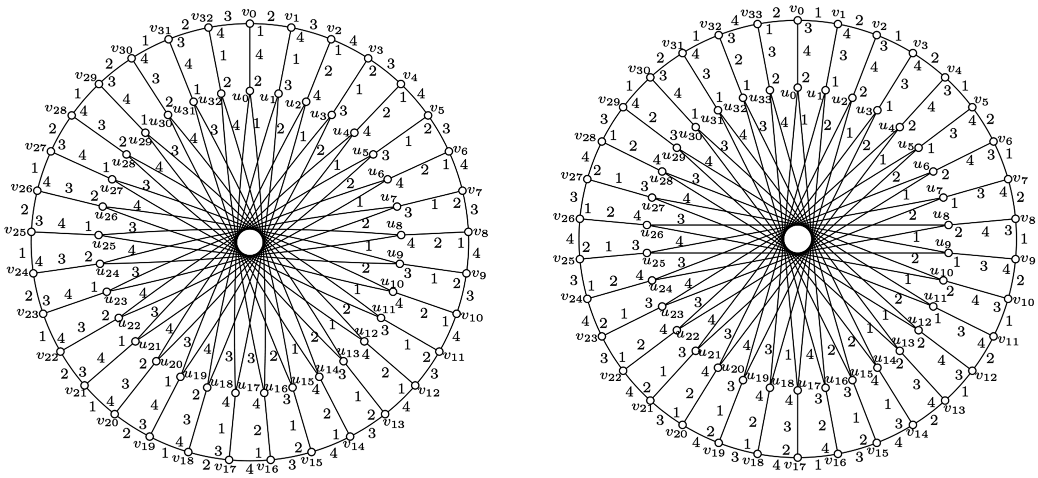
$$\sigma(E_1)=\begin{cases}((12)^{\frac{k}{2}}(34)^{\frac{k}{2}})^{\frac{n}{2k}}, & n_{2k}=0;\\ (23(43)^{\frac{k-2}{2}}41(21)^{\frac{k-2}{2}})^{\frac{n-2k-1}{2k}}23(43)^{\frac{k-2}{2}}42(12)^{\frac{k-2}{2}}3, & n_{2k}=1;\\ ((12)^{\frac{k-4}{2}}3434(34)^{\frac{k-4}{2}}1212)^{\frac{n-2k-2}{2k}}(12)^{\frac{k-2}{2}}431(43)^{\frac{k-4}{2}}12142, & n_{2k}=2;\\ ((23)^{\frac{k}{2}}(41)^{\frac{k}{2}})^{\frac{n-2k-4}{2k}}(23)^{\frac{k}{2}}1234(14)^{\frac{k-4}{2}}2313, & n_{2k}=4。\end{cases}$$

$$\sigma(E_2)=\begin{cases}((12)^{\frac{k}{2}}(34)^{\frac{k}{2}})^{\frac{n}{2k}}, & n_{2k}=0;\\ ((12)^{\frac{k}{2}}(34)^{\frac{k}{2}})^{\frac{n-2k-1}{2k}}(12)^{\frac{k}{2}}3(34)^{\frac{k}{2}}, & n_{2k}=1;\\ ((12)^{\frac{k}{2}}(34)^{\frac{k}{2}})^{\frac{n-2k-2}{2k}}(12)^{\frac{k-4}{2}}112241(34)^{\frac{k}{2}}, & n_{2k}=2;\\ ((12)^{\frac{k}{2}}(34)^{\frac{k}{2}})^{\frac{n-2k-4}{2k}}(12)^{\frac{k}{2}}2323(34)^{\frac{k}{2}}, & n_{2k}=4。\end{cases}$$

用与引理 2 类似的方法可证明，当 $k(\bmod 16)=0 \wedge n(\bmod 2k)=0,1,2,4$ 时，该着色 σ 是 $P(n,k)$的 4-

等全着色。

图 5 给出了 $n=33,34$ 时 $P(n,16)$ 的 4-等全着色。



(a) $\sigma(P(33,16))$ (b) $\sigma(P(34,16))$

图 5 $n=33,34$ 时 $\sigma(P(n,16))$
Fig.5 $\sigma(P(n,16))$ for $n=33,34$

由引理 2、3 得,当 $k(\mathrm{mod}\ 16)=4,8,12$ 时,$\chi''_=(P(n,k))\leqslant 4$。又因为 $\chi''_=(P(n,k))\geqslant\chi''(P(n,k))\geqslant\Delta(P(n,k))+1=4$,所以有如下定理:

定理 1 当 $k(\mathrm{mod}\ 16)=4,8,12$ 时,$\chi''_=(P(n,k))=4$。

同理,由引理 4,可证明,当 $k(\mathrm{mod}\ 16)=0\wedge n(\mathrm{mod}\ 2k)=0,1,2,4$ 时,$\chi''_=(P(n,k))=4$。进一步,有如下猜想:

猜想 1 $k(\mathrm{mod}\ 16)=0$ 时,$\chi''_=(P(n,k))=4$。

2 结论

证明了 $k(\mathrm{mod}\ 16)=4,8,12$ 以及 $k(\mathrm{mod}\ 16)=0\wedge n(\mathrm{mod}\ 2k)=0,1,2,4$ 时 $P(n,k)$ 的等全色数是 4,并进一步猜想 $k(\mathrm{mod}\ 16)=0$ 时 $P(n,k)$ 的等全色数也是 4,改进和扩展了 $P(n,k)$ 的等全色数的现有结果。

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