

一类(3+1)维宏观群体平衡方程的矩方法和尺度变换群分析及自相似解

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摘要:探究一类带齐次碰撞核的(3+1)维宏观群体平衡(积分偏微分)方程的解析解法及自相似解。采用矩方法将(3+1)维积分偏微分方程转化成(2+1)维偏微分矩方程。利用尺度变换群方法获得了(3+1)维积分偏微分方程和(2+1)维偏微分矩方程的尺度变换群、约化的(2+1)维积分偏微分方程、约化的(1+1)维常微分方程、自相似解、精确解,分析解的动力学性态。

关键词:宏观群体平衡方程;矩方法;尺度变换群;自相似解

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Method of moments, scaling transformation group analysis and self-similar solutions of a class of (3+1)-dimensional macroscopic population balance equation

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Abstract: Self-similar solutions and approaches for analytically solving a class of (3+1)-dimensional macroscopic population balance (integro-partial differential) equations with homogeneous collision kernels are studied. (3+1)-dimensional integro-partial differential equation is successfully transformed into (2+1)-dimensional partial differential moment equations by use of the method of moments. The admitted scaling transformation groups, reduced (2+1)-dimensional integro-partial differential equations, reduced (1+1)-dimensional ordinary differential equations, self-similar solutions, explicit exact solutions of (3+1)-dimensional integro-partial differential equations and (2+1)-dimensional partial differential moment equations are presented by use of the method of scaling transformation analysis. The dynamic behaviour analysis of the obtained solutions are also given.

Key words: macroscopic population balance equation; moment method; scaling transformation group; self-similar solution

0 引言

宏观群体平衡方程^[1-4]可用于预测相空间微粒系统中粒子或颗粒的诸多过程,在制药和混合肥料等实体工业领域有广泛的应用^[5-6]。因为群体平衡方程含积分项的复杂性,精确解的探寻比较困难^[7-15],所以一般常采用数值方法进行研究^[5-6]。管道中塞流粒子碰撞和凝聚过程的群体平衡方程^[4]为

$$\frac{\partial f(x,y,t)}{\partial t} + \frac{\partial}{\partial x} [v(x,t)f(x,y,t)] = B(x,y,t) - D(x,y,t), \quad (1)$$

式中: t 为时间, y 为粒子的质量,外部坐标 x 刻画粒子的位置状态, $f(x,y,t)$ 为粒子单向流动凝聚的密度分布, $v(x,t)$ 为粒子在同一条直线上沿着管道单向运动的速度。如果忽略粒子的所有其它固有特征(如粒径、

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尺寸、体积)及其在化学环境影响下的生长,仅考虑粒子自身的质量,则有

$$\frac{dx}{dt} = v(x, t), \quad \frac{dy}{dt} = 0.$$

质量分别为 s 和 $y-s$ 的小粒子互相碰撞,凝聚成质量是 y 的大粒子的出生率和死亡率分别为:

$$B(x, y, t) = \frac{1}{2} K(x, t) \int_0^\infty f(x, y-s, t) f(x, s, t) ds, \quad (2)$$

$$D(x, y, t) = K(x, t) f(x, y, t) \int_0^\infty f(x, s, t) ds. \quad (3)$$

$K(x, t)$ 为碰撞因子,它仅取决于环境,而不依赖于粒子的形状、尺寸和质量。 $B(x, y, t) - D(x, y, t)$ 为粒子的净形成率,式(2)、(3)代入方程(1),得到

$$\begin{aligned} & \frac{\partial f(x, y, t)}{\partial t} + \frac{\partial}{\partial x} [v(x, t) f(x, y, t)] \\ & = K(x, t) \left[\frac{1}{2} \int_0^\infty f(x, y-s, t) f(x, s, t) ds - f(x, y, t) \int_0^\infty f(x, s, t) ds \right]. \end{aligned} \quad (4)$$

1 (3+1) 维宏观群体平衡方程

假设粒子以恒定速率沿轴线单向运动,碰撞核是 β 次齐次的,但不一定满足对称性,对常数 λ , 满足 $K(\lambda x, \lambda t) = \lambda^\beta K(x, t)$, 但 $K(x, t) = K(t, x)$ 不一定成立,则考虑恒定速率和齐次碰撞核

$$v(x, t) = v_i (i=1, 2), \quad K(x, t) = \kappa_1 x^p t^q, \quad K(x, t) = \kappa_2 x^\alpha + \kappa_3 t^\alpha, \quad (5)$$

其中 $v_1, v_2, \kappa_1, \kappa_2, \kappa_3, p, q, \alpha$ 都是常数。式(5)代入方程(4),得到

$$\frac{\partial f(x, y, t)}{\partial t} + v_1 \frac{\partial f(x, y, t)}{\partial x} = \kappa_1 x^p t^q \left[\frac{1}{2} \int_0^\infty f(x, y-s, t) f(x, s, t) ds - f(x, y, t) \int_0^\infty f(x, s, t) ds \right], \quad (6)$$

$$\begin{aligned} & \frac{\partial f(x, y, t)}{\partial t} + v_2 \frac{\partial f(x, y, t)}{\partial x} \\ & = (\kappa_2 x^\alpha + \kappa_3 t^\alpha) \left[\frac{1}{2} \int_0^\infty f(x, y-s, t) f(x, s, t) ds - f(x, y, t) \int_0^\infty f(x, s, t) ds \right]. \end{aligned} \quad (7)$$

方程(4)、(6)、(7)的精确解常采用数值方法^[3-6]研究,例如文献[4]利用矩方法探究方程(4)的数值解。群体平衡方程的解析求解方法有改进的李群方法^[16-18],尺度变换群方法^[8-13, 16],相似方法^[15]和矩方法^[19]等。本文利用矩方法研究方程(4)的矩方程组,采用尺度变换群方法研究矩方程及方程(6)、(7)的算子、约化方程、自相似解、精确解及解的动力学性态。

2 方程(6)、(7)的尺度变换群分析

考虑方程(6)的变换系统

$$\bar{x} = xa^{\lambda_1}, \quad \bar{y} = ya^{\lambda_2}, \quad \bar{t} = ta^{\lambda_3}, \quad \bar{f} = fa^\mu, \quad (8)$$

其中 $\lambda_i (i=1, 2, 3), \mu, a$ 是常数,式(8)对应的算子为

$$X = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 t \frac{\partial}{\partial t} + \mu f \frac{\partial}{\partial f}. \quad (9)$$

由式(8)得

$$\bar{f}(\bar{x}, \bar{y}, \bar{t}) = f(\bar{x}a^{-\lambda_1}, \bar{y}a^{-\lambda_2}, \bar{t}a^{-\lambda_3})a^\mu, \quad \bar{f}(\bar{x}, \bar{y}-\bar{s}, \bar{t}) = f(\bar{x}a^{-\lambda_1}, \bar{y}a^{-\lambda_2} - \bar{s}a^{-\lambda_2}, \bar{t}a^{-\lambda_3})a^\mu. \quad (10)$$

式(8)、(10)代入方程,得

$$\frac{\partial \bar{f}(\bar{x}, \bar{y}, \bar{t})}{\partial \bar{t}} + v_1 \frac{\partial \bar{f}(\bar{x}, \bar{y}, \bar{t})}{\partial \bar{x}} = \kappa_1 \bar{x}^p \bar{t}^q \left[\frac{1}{2} \int_0^\infty \bar{f}(\bar{x}, \bar{y}-\bar{s}, \bar{t}) \bar{f}(\bar{x}, \bar{s}, \bar{t}) d\bar{s} - \bar{f}(\bar{x}, \bar{y}, \bar{t}) \int_0^\infty \bar{f}(\bar{x}, \bar{s}, \bar{t}) d\bar{s} \right],$$

进一步地,得

$$\begin{aligned}
 & a^{\mu-\lambda_3} \frac{\partial f(x,y,t)}{\partial t} + \nu_1 a^{\mu-\lambda_1} \frac{\partial f(x,y,t)}{\partial x} \\
 & = \kappa_1 a^{2\mu+\lambda_2+p\lambda_1+q\lambda_3} x^p t^q \left[\frac{1}{2} \int_0^\infty f(x,y-s,t) f(x,s,t) ds - f(x,y,t) \int_0^\infty f(x,s,t) ds \right]. \tag{11}
 \end{aligned}$$

利用假设条件,式(8)将方程(6)的任一解变成同一方程的解,由方程(11),得到

$$\lambda_3 = \lambda_1, \quad \mu = -\lambda_2 - (p+q+1)\lambda_1. \tag{12}$$

式(12)代入式(9),得

$$X_1 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (p+q+1)y \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (p+q+1)f \frac{\partial}{\partial f}.$$

类似地,考虑方程(7),得

$$Y_1 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (\alpha+1)y \frac{\partial}{\partial y}, \quad Y_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (\alpha+1)f \frac{\partial}{\partial f}.$$

3 方程(6)、(7)的自相似解

3.1 方程(6)的自相似解

子李代数 $\text{span}\{X_1, X_2\}$ 对应的群不变量仅有一个,因此方程(6)的自相似解没有找到。假设 $J=J(x,y,t,f)$, 则方程 $X_1 J=0$ 的特征方程为

$$\frac{df}{0} = \frac{dx}{x} = \frac{dt}{t} = \frac{dy}{-(p+q+1)y}.$$

算子 X_1 的群不变量为 $J_1=f, J_2=xt^{-1}, J_3=yt^{p+q+1}$, 因此方程(6)的解为

$$f(x,y,t) = g(u,z), \quad u=yt^{p+q+1}, \quad z=xt^{-1},$$

其中 $g(u,z)$ 满足

$$\begin{aligned}
 & (p+q+1)u \frac{\partial g(u,z)}{\partial u} + (\nu_1-z) \frac{\partial g(u,z)}{\partial z} \\
 & = \kappa_1 z^p \left[\frac{1}{2} \int_0^\infty g(u-\theta,z) g(\theta,z) d\theta - g(u,z) \int_0^\infty g(\theta,z) d\theta \right]. \tag{13}
 \end{aligned}$$

记 $J=J(x,y,t,f)$, 方程 $X_2 J=0$ 的特征方程为

$$\frac{df}{-(p+q+1)f} = \frac{dx}{x} = \frac{dt}{t} = \frac{dy}{0}.$$

算子 X_2 的群不变量为 $J_1=y, J_2=xt^{-1}, J_3=ft^{p+q+1}$, 因此方程(6)的解为

$$f(x,y,t) = t^{-(p+q+1)} g(y,z), \quad z=xt^{-1},$$

其中 $g(y,z)$ 满足

$$(\nu_1-z) \frac{\partial g(y,z)}{\partial z} - (p+q+1)g(y,z) = \kappa_1 z^p \left[\frac{1}{2} \int_0^\infty g(y-s,z) g(s,z) ds - g(y,z) \int_0^\infty g(s,z) ds \right]. \tag{14}$$

3.2 方程(7)的自相似解

子李代数 $\text{span}\{Y_1, Y_2\}$ 对应的群不变量仅找到一个,因此方程(7)的自相似解没有找到。假设 $J=J(x,y,t,f)$, 则方程 $Y_1 J=0$ 的特征方程为

$$\frac{df}{0} = \frac{dx}{x} = \frac{dt}{t} = \frac{dy}{-(\alpha+1)y}.$$

算子 Y_1 的群不变量为 $J_1=f, J_2=xt^{-1}, J_3=yt^{\alpha+1}$, 因此方程(7)的解为

$$f(x,y,t) = \varphi(u,z), \quad u=yt^{\alpha+1}, \quad z=xt^{-1},$$

其中 $\varphi(u,z)$ 满足

$$\begin{aligned}
 & (\alpha+1)u \frac{\partial \varphi(u,z)}{\partial u} + (\nu_2-z) \frac{\partial \varphi(u,z)}{\partial z} \\
 & = (\kappa_2 z^\alpha + \kappa_3) \left[\frac{1}{2} \int_0^\infty \varphi(u-\theta,z) \varphi(u,\theta) d\theta - \varphi(u,z) \int_0^\infty \varphi(\theta,z) d\theta \right]. \tag{15}
 \end{aligned}$$

设 $J=J(x,y,t,f)$, 方程 $Y_2 J=0$ 的特征方程为

$$\frac{df}{-(\alpha+1)f} = \frac{dx}{x} = \frac{dt}{t} = \frac{dy}{0}.$$

算子 Y_2 的群不变量为 $J_1=y$, $J_2=xt^{-1}$, $J_3=ft^{\alpha+1}$, 因此方程(7)的解为

$$f(x,y,t) = t^{-(\alpha+1)} \varphi(y,z), \quad z=xt^{-1},$$

其中 $\varphi(y,z)$ 满足

$$\begin{aligned} & (\nu_2 - z) \frac{\partial \varphi(y,z)}{\partial z} - (\alpha+1) \varphi(y,z) \\ &= (\kappa_2 z^\alpha + \kappa_3) \left[\frac{1}{2} \int_0^\infty \varphi(y-s,z) \varphi(s,z) ds - \varphi(y,z) \int_0^\infty \varphi(s,z) ds \right]. \end{aligned} \quad (16)$$

3.3 约化方程(13)–(16)的尺度变换群分析

若能找到方程(13)–(16)的算子和群不变量, 就可以再次将这些方程约化为(1+1)维常微分方程, 以及获得对应的自相似解。采用类似的方法, 可以获得方程(13)–(16)的算子:

$$Z_1 = u \frac{\partial}{\partial u} - g \frac{\partial}{\partial g}, \quad Z_2 = y \frac{\partial}{\partial y} - g \frac{\partial}{\partial g}, \quad Z_3 = u \frac{\partial}{\partial u} - \varphi \frac{\partial}{\partial \varphi}, \quad Z_4 = y \frac{\partial}{\partial y} - \varphi \frac{\partial}{\partial \varphi}.$$

算子 $Z_i (i=1,2,3,4)$ 的群不变量都存在, 但鉴于广义积分的收敛性条件, 导致对应的自相似解没有找到。例如算子 Z_1 的群不变量为 $J_1=z$, $J_2=gu$, 自相似解为 $g(u,z) = u^{-1} \varphi(z)$, 但当将解代入方程(13)时, 广义积分的收敛性条件, 导致函数 $\varphi(z)$ 没有找到, 因此对应的自相似解没有找到。

4 矩方法及自相似解

4.1 方程(4)的矩方程组

设 $f(x,y,t)$ 满足方程(4), j 阶矩定义为

$$m_j(x,t) = \int_0^\infty y^j f(x,y,t) dy, \quad j=0,1,2,\dots, \quad (17)$$

其中 $m_0(x,t)$ 为粒子的总数量, $m_1(x,t)$ 为粒子的总质量, $m_2(x,t)$ 为粒子密度分布的方差, 高阶矩 $m_j(x,t) (j=3,4,\dots)$ 的含义参见文献[10]。方程(4)两边关于 y 从 0 到 ∞ 积分, 得到

$$\begin{aligned} & \int_0^\infty \frac{\partial f(x,y,t)}{\partial t} dy + \int_0^\infty \frac{\partial [v(x,t)f(x,y,t)]}{\partial x} dy \\ &= K(x,t) \left[\frac{1}{2} \int_0^\infty \left(\int_0^\infty f(x,y-s,t) f(x,s,t) ds \right) dy - \int_0^\infty f(x,y,t) dy \int_0^\infty f(x,s,t) ds \right]. \end{aligned} \quad (18)$$

假设 $f(x,y,t)$ 和 $v(x,t)$ 满足求导与积分运算的可交换性条件, 则由方程(18), 得到

$$\begin{aligned} & \frac{\partial}{\partial t} \int_0^\infty f(x,y,t) dy + \frac{\partial}{\partial x} \left[v(x,t) \int_0^\infty f(x,y,t) dy \right] \\ &= K(x,t) \left[\frac{1}{2} \int_0^\infty \left(\int_0^\infty f(x,y-s,t) f(x,s,t) ds \right) dy - \left(\int_0^\infty f(x,y,t) dy \right)^2 \right]. \end{aligned} \quad (19)$$

交换方程(19)中二重积分项的积分次序, 得到

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty f(x,y-s,t) f(x,s,t) ds \right) dy &= \int_0^\infty \left(\int_0^\infty f(x,y-s,t) f(x,s,t) dy \right) ds \\ &= \int_0^\infty \left(f(x,s,t) \int_0^\infty f(x,y-s,t) dy \right) ds. \end{aligned} \quad (20)$$

注意到 $s \leq y$, 若令变换 $\theta=y-s$, 则

$$\int_0^\infty f(x,y-s,t) dy = \int_{-s}^0 f(x,\theta,t) d\theta + \int_0^\infty f(x,\theta,t) d\theta = \int_0^\infty f(x,\theta,t) d\theta. \quad (21)$$

因此, 针对方程(4), 由式(17)–(21)得

$$\frac{\partial m_0(x,t)}{\partial t} + \frac{\partial [v(x,t)m_0(x,t)]}{\partial x} = -\frac{1}{2} K(x,t) m_0^2(x,t). \quad (22)$$

方程(22)的解用于描述粒子互相碰撞时, 粒子凝聚的总数量在空间的演化性态分布。

一般地, 在方程(4)两边同时乘以 $y^n (n=1,2,\dots)$, 关于 y 从 0 到 ∞ 同时积分, 记组合系数为 $C_n^i =$

$\frac{n!}{i!(n-i)!}$, 得到

$$\frac{\partial m_1(x,t)}{\partial t} + \frac{\partial [v(x,t)m_1(x,t)]}{\partial x} = 0, \tag{23}$$

$$\frac{\partial m_2(x,t)}{\partial t} + \frac{\partial [v(x,t)m_2(x,t)]}{\partial x} = K(x,t)m_1^2(x,t), \tag{24}$$

$$\frac{\partial m_3(x,t)}{\partial t} + \frac{\partial [v(x,t)m_3(x,t)]}{\partial x} = 3K(x,t)m_1(x,t)m_2(x,t), \tag{25}$$

...

$$\frac{\partial m_n(x,t)}{\partial t} + \frac{\partial [v(x,t)m_n(x,t)]}{\partial x} = K(x,t) \left[\frac{1}{2} \sum_{i=0}^n C_n^i m_i(x,t)m_{n-i}(x,t) - m_0(x,t)m_n(x,t) \right]. \tag{26}$$

采用递归方法求解方程(23)–(26), 特别地, 若考虑粒子作等速单向运动, 碰撞系数是常数, 选取 $v(x,t) = \nu_0$, $K(x,t) = \kappa_0$, 其中 ν_0, κ_0 均是常数, $\varphi_j (j=1, 2, \dots)$ 是一元函数, 则

$$m_1(x,t) = \varphi_1(x - \nu_0 t),$$

$$m_2(x,t) = \kappa_0 t \varphi_1^2(x - \nu_0 t) + \varphi_2(x - \nu_0 t),$$

$$m_3(x,t) = \frac{3}{2} \kappa_0^2 t^2 \varphi_1^3(x - \nu_0 t) + 3\kappa_0 t \varphi_1(x - \nu_0 t) \varphi_2(x - \nu_0 t) + \varphi_3(x - \nu_0 t),$$

...

4.2 矩方程的尺度变换群分析

采用式(5)和方程(22), 得到

$$\frac{\partial m_0(x,t)}{\partial t} + \nu_1 \frac{\partial m_0(x,t)}{\partial x} = -\frac{1}{2} \kappa_1 x^p t^q m_0^2(x,t), \tag{27}$$

$$\frac{\partial m_0(x,t)}{\partial t} + \nu_2 \frac{\partial m_0(x,t)}{\partial x} = -\frac{1}{2} (\kappa_2 x^\alpha + \kappa_3 t^\alpha) m_0^2(x,t). \tag{28}$$

考虑方程(27)的变换系统, 有

$$\bar{x} = b^{\beta_1} x, \quad \bar{t} = b^{\beta_2} t, \quad \bar{m}_0 = b^\beta m_0, \tag{29}$$

其中 $b, \beta_1, \beta_2, \beta$ 是常数, 式(29)对应的算子为

$$\bar{Y} = \beta_1 x \frac{\partial}{\partial x} + \beta_2 t \frac{\partial}{\partial t} + \beta m_0 \frac{\partial}{\partial m_0}. \tag{30}$$

将式(29)代入方程

$$\frac{\partial \bar{m}_0(\bar{x}, \bar{t})}{\partial \bar{t}} + \nu_1 \frac{\partial \bar{m}_0(\bar{x}, \bar{t})}{\partial \bar{x}} = -\frac{1}{2} \kappa_1 \bar{x}^p \bar{t}^q \bar{m}_0^2(\bar{x}, \bar{t}),$$

得到

$$a^{\beta-\beta_2} \frac{\partial m_0(x,t)}{\partial t} + a^{\beta-\beta_1} \nu_1 \frac{\partial m_0(x,t)}{\partial x} = -\frac{1}{2} \kappa_1 a^{2\beta+\beta_1+q\beta_2} x^p t^q m_0^2(x,t). \tag{31}$$

由假设条件, 式(29)将方程(27)的任一解变成同一方程的解, 则由方程(31), 得

$$\beta_2 = \beta_1, \quad \beta = -(p+q+1)\beta_1. \tag{32}$$

将式(32)代入式(30), 得

$$Y = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (p+q+1) m_0 \frac{\partial}{\partial m_0}.$$

类似地, 针对方程(28), 得

$$Z = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - (\alpha+1) m_0 \frac{\partial}{\partial m_0}.$$

4.3 矩方程(27)的自相似解

设 $V = V(x, t, m_0)$, 则方程 $YV = 0$ 的特征方程为

$$\frac{dx}{x} = \frac{dt}{t} = \frac{dm_0}{-(p+q+1)m_0}.$$

算子 Y 的群不变量为 $J_1 = m_0 t^{p+q+1}$, $J_2 = xt^{-1}$, 因此, 方程(27)的解为

$$m_0(x, t) = t^{-(p+q+1)} \varphi(z), \quad z = xt^{-1}, \quad (33)$$

其中 $\varphi(z)$ 满足

$$(\nu_1 - z)\varphi' = (p+q+1)\varphi - \frac{1}{2}\kappa_1 z^p \varphi^2. \quad (34)$$

方程(34)的解为

$$\varphi(z) = (z - \nu_1)^{-(p+q+1)} \left(c - \frac{\kappa_1}{2} \int \frac{z^p}{(z - \nu_1)^{p+q+2}} dz \right)^{-1}, \quad (35)$$

其中 c 是常数。由式(33)和(35), 得到方程(27)的解为

$$m_0(x, t) = (x - \nu_1 t)^{-(p+q+1)} \left(c - \frac{\kappa_1}{2} \int \frac{z^p}{(z - \nu_1)^{p+q+2}} dz \right)^{-1}, \quad z = xt^{-1}. \quad (36)$$

取 $q = -(p+1)$, $p = 0, \pm 1, \dots, \pm 5$, 由式(36), 得到方程(27)的解的表达式如下:

$$m_0(x, t) = \frac{2}{c - \kappa_1 \ln |z - \nu_1|}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 t^{-1},$$

$$m_0(x, t) = \frac{2\nu_1}{\kappa_1 (c\nu_1 + \ln |z| - \ln |z - \nu_1|)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^{-1},$$

$$m_0(x, t) = \frac{2\nu_1^2 z}{\kappa_1 [c\nu_1^2 z + z(\ln |z| - \ln |z - \nu_1|) - \nu_1]}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^{-2} t,$$

$$m_0(x, t) = \frac{4\nu_1^3 z^2}{\kappa_1 [2c\nu_1^3 z^2 - 2\nu_1 z + 2z^2(\ln |z| - \ln |z - \nu_1|) - \nu_1^2]}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^{-3} t^2,$$

$$m_0(x, t) = \frac{12\nu_1^4 z^3}{\kappa_1 [6c\nu_1^4 z^3 - 6\nu_1 z^2 - 3\nu_1^2 z + 6z^3(\ln |z| - \ln |z - \nu_1|) - 2\nu_1^3]}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^{-4} t^3,$$

$$m_0(x, t) = \frac{24\nu_1^5 z^4}{\kappa_1 [12c\nu_1^5 z^4 - 12\nu_1 z^3 - 6\nu_1^2 z^2 - 4\nu_1^3 z + 12z^4(\ln |z| - \ln |z - \nu_1|) - 3\nu_1^4]}, \quad z = xt^{-1},$$

$$K(x, t) = \kappa_1 x^{-5} t^4,$$

$$m_0(x, t) = \frac{2}{\kappa_1 (c - \nu_1 \ln |z - \nu_1| - z)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x t^{-2},$$

$$m_0(x, t) = \frac{4}{\kappa_1 (2c - 2\nu_1^2 \ln |z - \nu_1| - z^2 - 2\nu_1 z)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^2 t^{-3},$$

$$m_0(x, t) = \frac{12}{\kappa_1 (6c - 6\nu_1^3 \ln |z - \nu_1| - 2z^3 - 3\nu_1 z^2 - 6\nu_1^2 z)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^3 t^{-4},$$

$$m_0(x, t) = \frac{24}{\kappa_1 (12c - 12\nu_1^4 \ln |z - \nu_1| - 3z^4 - 4\nu_1 z^3 - 6\nu_1^2 z^2 - 12\nu_1^3 z)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^4 t^{-5},$$

$$m_0(x, t) = \frac{120}{\kappa_1 (60c - 60\nu_1^5 \ln |z - \nu_1| - 12z^5 - 15\nu_1 z^4 - 20\nu_1^2 z^3 - 30\nu_1^3 z^2 - 60\nu_1^4 z)}, \quad z = xt^{-1},$$

$$K(x, t) = \kappa_1 x^5 t^{-6},$$

$$m_0(x, t) = \left(\frac{\kappa_1}{2} \int \frac{z^p}{\nu_1 - z} dz \right)^{-1}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1 x^p t^{-(p+1)},$$

$$m_0(x, t) = \frac{2(\nu_1 - \sigma)}{\kappa_1 (x - \sigma t) + 2c(\nu_1 - \sigma)}, \quad z = xt^{-1}, \quad K(x, t) = \kappa_1,$$

其中 $\sigma, \nu_1, \kappa_1, c$ 为常数。

假设 $p=q=0$, σ 是常数, 设方程(27)的解为

$$m_0(x, t) = \varphi(\xi), \quad \xi = x - \sigma t,$$

其中 $\varphi(\xi)$ 满足

$$(\nu_1 - \sigma)\varphi' = -\frac{1}{2}\kappa_1\varphi^2. \tag{37}$$

求解方程(37), 得到方程(27)的解为

$$m_0(x, t) = \frac{2(\nu_1 - \sigma)}{\kappa_1(x - \sigma t) + 2c(\nu_1 - \sigma)}. \tag{38}$$

观察解(38)发现, 当 $x \rightarrow \infty$ 时, $m_0(x, t) \rightarrow 0$, 记 $m_0(\infty, t) = 0$; 当 $t \rightarrow \infty$ 时, $m_0(x, t) \rightarrow 0$, 记 $m_0(x, \infty) = 0$, 表明粒子的碰撞和凝聚随着时间的演化, 解(38)是渐近稳定的, 对应的边值和初值条件分别为:

$$m_0(0, t) = \frac{2(\sigma - \nu_1)}{\kappa_1\sigma t + 2c(\sigma - \nu_1)}, \quad m_0(\infty, t) = 0;$$

$$m_0(x, 0) = \frac{2(\nu_1 - \sigma)}{\kappa_1 x + 2c(\nu_1 - \sigma)}, \quad m_0(x, \infty) = 0.$$

类似地, 得到解(38)在 $[0, L] \times [0, T]$ 上的边值和初值条件, 其中 L, T 都是正常数。取 $\kappa_1 = 0.5, \nu_1 = 0.1, \sigma = -0.001, c = 4$, 解(38)在 $[0, 8] \times [0, 8]$ 上, 图 1(a) 为粒子凝聚的总数量动力学性态演化投影。边值条件 $m_0(0, t)$ 在 $[0, 8 \times 10^4]$ 上的动力学性态演化见图 1(b)。初值条件 $m_0(x, 0)$ 在 $[0, 8]$ 上的动力学性态演化见图 1(c)。方程(27)和(28)的解的边值和初值条件及动力学演化性态可类似分析, 略。

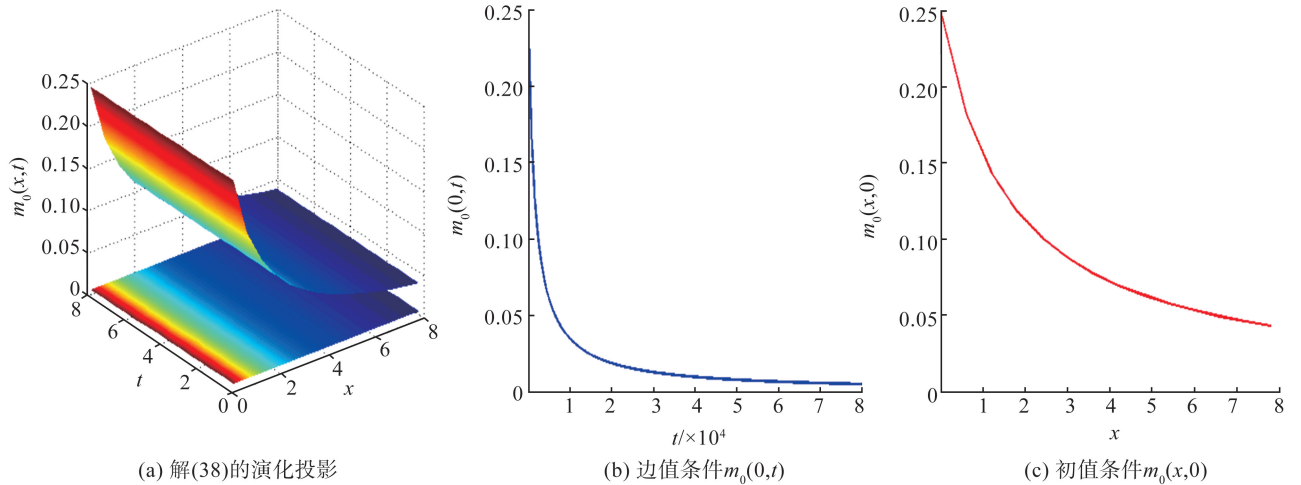


图 1 解(38)及边值和初值条件的动力学演化性态分布

Fig.1 Distributions of dynamic evolution of solution (38) and boundary and initial conditions

4.4 矩方程(28)的自相似解

设 $V = V(x, t, m_0)$, 则方程 $ZV = 0$ 的特征方程为

$$\frac{dx}{x} = \frac{dt}{t} = \frac{dm_0}{-(\alpha+1)m_0}.$$

算子 Z 的群不变量为 $J_1 = m_0 t^{\alpha+1}, J_2 = xt^{-1}$, 因此方程(28)的解为

$$m_0(x, t) = t^{-(\alpha+1)}\varphi(z), \quad z = xt^{-1}, \tag{39}$$

其中 $\varphi(z)$ 满足

$$(\nu_2 - z)\varphi' = (\alpha+1)\varphi - \frac{1}{2}(\kappa_2 z^\alpha + \kappa_3)\varphi^2. \tag{40}$$

方程(40)的解为

$$\varphi(z) = (z - \nu_2)^{-(\alpha+1)} \left(c - \frac{1}{2} \int \frac{\kappa_2 z^\alpha + \kappa_3}{(z - \nu_2)^{\alpha+2}} dz \right)^{-1}, \tag{41}$$

其中 c 是常数。采用式(40),得到方程(28)的解为

$$m_0(x,t) = (x-\nu_2 t)^{-(\alpha+1)} \left(c - \frac{1}{2} \int \frac{\kappa_2 z^\alpha + \kappa_3}{(z-\nu_2)^{\alpha+2}} dz \right)^{-1}, \quad z = xt^{-1}. \quad (42)$$

取 $\alpha = 0, \pm 1, \dots, \pm 6, \pm \frac{1}{2}, \pm \frac{1}{3}, \frac{3}{2}, n$, 其中 $n = 2, 3, 4, \dots$, 利用式(42),得到方程(28)的解如下:

$$m_0(x,t) = \frac{2\nu_2(\nu_2-z)}{(x-\nu_2 t) [2c\nu_2(\nu_2-z) - (\kappa_2 + \kappa_3)z]}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 + \kappa_3,$$

$$m_0(x,t) = \frac{2\nu_2}{2c\nu_2 + \kappa_2 \ln|z| - (\kappa_2 + \kappa_3\nu_2) \ln|z-\nu_2|}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{-1} + \kappa_3 t^{-1},$$

$$m_0(x,t) = \frac{2(x-\nu_2 t)z}{\kappa_2 + 2cz - \kappa_3 z^2}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{-2} + \kappa_3 t^{-2},$$

$$m_0(x,t) = \frac{4(x-\nu_2 t)^2 z^2}{(2\nu_2-z)\kappa_3 z^3 + (2z-\nu_2)\kappa_2 + 4cz^2}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{-3} + \kappa_3 t^{-3},$$

$$m_0(x,t) = \frac{6(x-\nu_2 t)^3 z^3}{(\nu_2^2 - 3\nu_2 z + 3z^2)\kappa_2 + 6cz^3 - (3\nu_2^2 - 3\nu_2 z + z^2)\kappa_3 z^4}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{-4} + \kappa_3 t^{-4},$$

$$m_0(x,t) = \frac{8(x-\nu_2 t)^4 z^4}{(2\nu_2^2 - 2\nu_2 z + z^2)(2\nu_2-z)\kappa_3 z^5 + 8cz^4 - A}, \quad A = (\nu_2^2 - 2\nu_2 z + 2z^2)(\nu_2 - 2z)\kappa_2, \quad z = xt^{-1},$$

$$K(x,t) = \kappa_2 x^{-5} + \kappa_3 t^{-5},$$

$$m_0(x,t) = \frac{10(x-\nu_2 t)^5 z^5}{(\nu_2^4 - 5\nu_2^3 z + 10\nu_2^2 z^2 - 10\nu_2 z^3 + 5z^4)\kappa_2 + 10cz^5 - B}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{-6} + \kappa_3 t^{-6},$$

$$B = (5\nu_2^4 - 10\nu_2^3 z + 10\nu_2^2 z^2 - 5\nu_2 z^3 + z^4)\kappa_3 z^6,$$

$$m_0(x,t) = \frac{4\nu_2(\nu_2-z)^2}{(x-\nu_2 t)^2 (4c\nu_2(\nu_2-z)^2 + \kappa_2 z^2 + \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x + \kappa_3 t,$$

$$m_0(x,t) = \frac{6\nu_2(\nu_2-z)^3}{(x-\nu_2 t)^3 (6c\nu_2(\nu_2-z)^3 - \kappa_2 z^3 - \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^2 + \kappa_3 t^2,$$

$$m_0(x,t) = \frac{8\nu_2(\nu_2-z)^4}{(x-\nu_2 t)^4 (8c\nu_2(\nu_2-z)^4 + \kappa_2 z^4 + \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^3 + \kappa_3 t^3,$$

$$m_0(x,t) = \frac{10\nu_2(\nu_2-z)^5}{(x-\nu_2 t)^5 (10c\nu_2(\nu_2-z)^5 - \kappa_2 z^5 - \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^4 + \kappa_3 t^4,$$

$$m_0(x,t) = \frac{12\nu_2(\nu_2-z)^6}{(x-\nu_2 t)^6 (12c\nu_2(\nu_2-z)^6 + \kappa_2 z^6 + \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^5 + \kappa_3 t^5,$$

$$m_0(x,t) = \frac{14\nu_2(\nu_2-z)^7}{(x-\nu_2 t)^7 (14c\nu_2(\nu_2-z)^7 - \kappa_2 z^7 - \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^6 + \kappa_3 t^6,$$

$$m_0(x,t) = \frac{2n\nu_2(\nu_2-z)^n}{(x-\nu_2 t)^n (2nc\nu_2(\nu_2-z)^n + (-1)^n \kappa_2 z^n + (-1)^n \kappa_3 \nu_2)}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 x^{n-1} + \kappa_3 t^{n-1},$$

$$m_0(x,t) = \frac{3\nu_2 \sqrt{(z-\nu_2)^3}}{\sqrt{(x-\nu_2 t)^3} [\kappa_2 \sqrt{z^3} + \kappa_3 \nu_2 + (\kappa_2 + 3c\nu_2) \sqrt{(z-\nu_2)^3}]}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 \sqrt{x} + \kappa_3 \sqrt{t},$$

$$m_0(x,t) = \frac{\nu_2 \sqrt{z-\nu_2}}{\sqrt{x-\nu_2 t} [\sqrt{z} \kappa_2 + \kappa_3 \nu_2 + (\kappa_2 + c\nu_2) \sqrt{z-\nu_2}]}, \quad z = xt^{-1}, \quad K(x,t) = \frac{\kappa_2}{\sqrt{x}} + \frac{\kappa_3}{\sqrt{t}},$$

$$m_0(x,t) = \frac{8\nu_2 \sqrt[3]{(z-\nu_2)^4}}{\sqrt[3]{(x-\nu_2 t)^4} [3(\kappa_2 \sqrt[3]{z^4} + \kappa_3 \nu_2) + 8c\nu_2 \sqrt[3]{(z-\nu_2)^4}]}, \quad z = xt^{-1}, \quad K(x,t) = \kappa_2 \sqrt[3]{x} + \kappa_3 \sqrt[3]{t},$$

$$m_0(x,t) = \frac{4\nu_2 \sqrt[3]{z(z-\nu_2)^2}}{\sqrt[3]{(x-\nu_2 t)^2} [3(\kappa_3 \nu_2 \sqrt[3]{z} + \kappa_2 z) + 4c\nu_2 \sqrt[3]{z(z-\nu_2)^2}]}, \quad z = xt^{-1}, \quad K(x,t) = \frac{\kappa_2}{\sqrt[3]{x}} + \frac{\kappa_3}{\sqrt[3]{t}},$$

$$m_0(x,t) = \frac{5\nu_2 \sqrt{(z-\nu_2)^5}}{\sqrt{(x-\nu_2 t)^5} [\kappa_2 \sqrt{z^5} + \kappa_3 \nu_2 + (\kappa_2 + 5c\nu_2) \sqrt{(z-\nu_2)^5}]}, \quad z = xt^{-1},$$

$$K(x,t) = \kappa_2 \sqrt{x^3} + \kappa_3 \sqrt{t^3},$$

其中 $\nu_2, \kappa_2, \kappa_3, c$ 为常数。

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