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形式三角矩阵环上相对于对偶对的 Gorenstein 平坦模和维数

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摘要:设 $T=\begin{pmatrix}A & 0\\ U & B\end{pmatrix}$ 是形式三角矩阵环,其中 A 和 B 是环,U 是 (B,A)-双模。$\mathcal{GF}_{(\mathcal{F}(R),\mathcal{Y})}$ 表示相对于完备对偶对 $(\mathcal{X},\mathcal{Y})$ 的 Gorenstein 平坦左 R-模类。假设 $(\mathcal{C}_1,\mathcal{C}_2)$ 和 $(\mathcal{D}_1,\mathcal{D}_2)$ 分别为 A 环和 B 环上的完备对偶对,$(\mathcal{U}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})$ 是由 $(\mathcal{C}_1,\mathcal{C}_2)$ 和 $(\mathcal{D}_1,\mathcal{D}_2)$ 诱导的 T 环上的完备对偶对。证明若 U_A 有有限平坦维数,$\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ 关于扩张封闭,则左 T-模 $\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}\in \mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ 当且仅当 $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$,$M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$,$\varphi^M:U\otimes_A M_1\to M_2$ 是单同态。此外还给出左 T-模的相对于完备对偶对 $(\mathcal{U}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})$ 的 Gorenstein 平坦维数的估计。

关键词:形式三角矩阵环;相对于完备对偶对的 Gorenstein 平坦模;相对于完备对偶对的 Gorenstein 平坦维数
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Gorenstein flat modules and dimensions with respect to duality pairs over formal triangular matrix rings

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Abstract: Let $T=\begin{pmatrix}A & 0\\ U & B\end{pmatrix}$ be a formal triangular matrix ring, where A and B are rings and U is a (B,A)-bimodule. $\mathcal{GF}_{(\mathcal{F}(R),\mathcal{Y})}$ denotes the class of all Gorenstein flat left R-modules with respect to a complete duality pair $(\mathcal{X},\mathcal{Y})$. Assume that $(\mathcal{C}_1,\mathcal{C}_2)$ and $(\mathcal{D}_1,\mathcal{D}_2)$ are complete duality pairs over the ring A and the ring B respectively, and $(\mathcal{U}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})$ is a complete duality pair over the ring T induced by $(\mathcal{C}_1,\mathcal{C}_2)$ and $(\mathcal{D}_1,\mathcal{D}_2)$. It is proven that, if U_A has finite flat dimension and $\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ is closed under extensions, then a left T-module $\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ if and only if $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$, $M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$, and $\varphi^M: U\otimes_A M_1\to M_2$ is a monomorphism. Furthermore, an estimate of Gorenstein flat dimension with respect to the complete duality pair $(\mathcal{U}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})$ of a left T-module is given.

Key words: formal triangular matrix ring; Gorenstein flat module with respect to a complete duality pair; Gorenstein flat dimension with respect to a complete duality pair

0 引言

本文提到的环均指有单位元的非零结合环,模均指酉模。设 A 和 B 是环,U 是 (B,A)-双模,$T=$

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$\begin{pmatrix}A & 0\\ U & B\end{pmatrix}=\left\{\begin{pmatrix}a & 0\\ u & b\end{pmatrix}: a\in A,\ b\in B,\ u\in U\right\}$,其中 T 关于矩阵加法和矩阵乘法构成一个环,称形式三角矩阵环。Enochs 等[1]在任意环上引入了 Gorenstein 投射、内射和平坦模,并且研究 Gorenstein 同调维数。Encohs等[2]刻画形式三角矩阵环上的 Gorenstein 投射模和内射模。Mao[3]刻画形式三角矩阵环上的 Gorenstein 平坦模,证明对于左 T-模 $M=\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$,如果 T 是右凝聚环,${}_BU$ 有有限平坦维数,U_A 有有限平坦维数或内射维数,那么 M 是 Gorenstein 平坦左 T-模当且仅当 M_1 是 Gorenstein 平坦左 A-模,$M_2/\mathrm{Im}(\varphi^M)$ 是 Gorenstein 平坦左 B-模,$\varphi^M: U\otimes_A M_1\to M_2$ 是单同态,此外还给出了左 T-模的 Gorenstein 平坦维数的估计。Mao[4]利用 A 环和 B 环上的完备对偶对构造 T 环上的完备对偶对。Gillespie[5]引入 Gorenstein $(\mathscr{L},\mathscr{A})$-投射,内射和平坦模,即相对于完备对偶对$(\mathscr{L},\mathscr{A})$的 Gorenstein 投射,内射和平坦模,并且证明了如果$(\mathscr{L},\mathscr{A})$是完全对偶对,那么 $\mathscr{L}$ 包含平坦模类。Becerril[6]在文献[5]的基础上进一步研究$(\mathscr{F},\mathscr{A})$-Gorenstein 平坦模的性质,并且给出模的相对于二元组$(\mathscr{F},\mathscr{A})$的有限 Gorenstein 平坦维数和弱 Gorenstein 整体维数的一些刻画,其中 $\mathscr{F}$ 是平坦左 R-模类,$\mathscr{A}$ 是右 R-模类。

受此启发,本文进一步研究文献[3]中的主要结果。设 T 是形式三角矩阵环,$(\mathscr{C}_1,\mathscr{C}_2)$ 和 $(\mathscr{D}_1,\mathscr{D}_2)$ 分别为 A 环和 B 环上的完备对偶对,$(\mathscr{U}^{\mathscr{C}_1}_{\mathscr{D}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$ 是由 $(\mathscr{C}_1,\mathscr{C}_2)$ 和 $(\mathscr{D}_1,\mathscr{D}_2)$ 诱导的 T 环上的完备对偶对。$\mathscr{GF}_{(\mathscr{F}(R),\mathscr{Y})}$ 表示相对于完备对偶对$(\mathscr{X},\mathscr{Y})$的 Gorenstein 平坦左 R-模类。本文刻画 T 环上相对于完备对偶对$(\mathscr{U}^{\mathscr{C}_1}_{\mathscr{D}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$的 Gorenstein 平坦模,即如果 T 是形式三角矩阵环,U_A 有有限平坦维数,$\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 关于扩张封闭,那么左 T-模 $\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 当且仅当 $M_1\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$, $M_2/\mathrm{Im}(\varphi^M)\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$, $\varphi^M: U\otimes_A M_1\to M_2$ 是单同态。此外还给出了左 T-模的相对于完备对偶对$(\mathscr{U}^{\mathscr{C}_1}_{\mathscr{D}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$的 Gorenstein 平坦维数的估计。

1　预备知识

在本文中,R-模表示左 R-模,R^{op}-模表示右 R-模,$\mathrm{Mod}(R)$ 和 $\mathrm{Mod}(R^{\mathrm{op}})$ 分别表示左 R-模范畴和右 R-模范畴,$\mathscr{P}(R)$ 和 $\mathscr{F}(R)$ 分别表示投射 R-模类和平坦 R-模类,1_M 表示恒等态射 $M\to M$, $M^+=\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ 表示 R-模 M 的示性模。

$T=\begin{pmatrix}A & 0\\ U & B\end{pmatrix}$ 表示形式三角矩阵环,其中 A 和 B 是环,U 是(B,A)-双模。由文献[7]的定理 1.5 知,T-模范畴 $\mathrm{Mod}(T)$ 等价于范畴 Ω,Ω 中的对象是三元组 $M=\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$,其中 $M_1\in\mathrm{Mod}(A)$, $M_2\in\mathrm{Mod}(B)$, $\varphi^M: U\otimes_A M_1\to M_2$ 是 B-模同态;设 $M=\begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$ 和 $N=\begin{pmatrix}N_1\\ N_2\end{pmatrix}_{\varphi^N}$ 是 Ω 中的对象,Ω 中的态射 $M\to N$ 是态射对 $\begin{pmatrix}f_1\\ f_2\end{pmatrix}$,其中 $f_1\in\mathrm{Hom}_A(M_1,N_1)$, $f_2\in\mathrm{Hom}_B(M_2,N_2)$,并且使得图

$$\begin{array}{ccc} U\otimes_A M_1 & \xrightarrow{1_U\otimes_A f_1} & U\otimes_A N_1\\ {\scriptstyle\varphi^M}\downarrow & & \downarrow{\scriptstyle\varphi^N}\\ M_2 & \xrightarrow{f_2} & N_2 \end{array}$$

可交换。同样地,T^{op}-模范畴 $\mathrm{Mod}(T^{\mathrm{op}})$ 等价于范畴 Γ,Γ 中的对象是三元组 $W=(W_1,W_2)_{\varphi_W}$,其中 $W_1\in\mathrm{Mod}(A^{\mathrm{op}})$, $W_2\in\mathrm{Mod}(B^{\mathrm{op}})$, $\varphi_W: W_2\otimes_B U\to W_1$ 是 A-模同态;设 $W=(W_1,W_2)_{\varphi_W}$ 和 $X=(X_1,X_2)_{\varphi_X}$ 是 Γ 中的对象,Γ 中的态射 $W\to X$ 是态射对(g_1,g_2),其中 $g_1\in\mathrm{Hom}_A(W_1,X_1)$, $g_2\in\mathrm{Hom}_B(W_2,X_2)$,并且使得图

$$\begin{array}{ccc} W_2\otimes_B U & \xrightarrow{g_2\otimes_B 1_U} & X_2\otimes_B U \\ \downarrow{\scriptstyle\varphi_W} & & \downarrow{\scriptstyle\varphi_X} \\ W_1 & \xrightarrow{g_1} & X_1 \end{array}$$

可交换。在本文中，将 T-模范畴 $\mathrm{Mod}(T)$ 和范畴 Ω 不加区别，T^{op}-模范畴 $\mathrm{Mod}(T^{\mathrm{op}})$ 和范畴 Γ 不加区别。在不产生混淆的情况下，有时会省略下标 φ^M 和 φ_W。

注意到 T-模序列 $0\to\begin{pmatrix}M_1'\\M_2'\end{pmatrix}_{\varphi^{M'}}\xrightarrow{\binom{f_1}{f_2}}\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^{M}}\xrightarrow{\binom{g_1}{g_2}}\begin{pmatrix}M_1''\\M_2''\end{pmatrix}_{\varphi^{M''}}\to 0$ 正合当且仅当独立的 A-模序列 $0\to M_1'\xrightarrow{f_1}M_1\xrightarrow{g_1}M_1''\to 0$ 和 B-模序列 $0\to M_2'\xrightarrow{f_2}M_2\xrightarrow{g_2}M_2''\to 0$ 均正合。

命题 1[8]　设 $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是 T-模，$W=(W_1,W_2)_{\varphi_W}$ 是 T^{op}-模，则有 Abel 群同构：

$$W\otimes_T M\cong(W_1\otimes_A M_1\oplus W_2\otimes_B M_2)/H,$$

其中 H 由所有形如 $(\varphi_W(w_2\otimes u))\otimes x_1-w_2\otimes\varphi^M(u\otimes x_1)$ 的元素生成，并且 $x_1\in M_1$，$w_2\in W_2$，$u\in U$。

定义 1[9]　称 $(\mathscr{X},\mathscr{Y})$ 是 R 环上的对偶对，如果 $\mathscr{X}$ 是 R-模类，$\mathscr{Y}$ 是 R^{op}-模类，并且满足下列条件：

(1) $M\in\mathscr{X}$ 当且仅当 $M^+\in\mathscr{Y}$；

(2) $\mathscr{Y}$ 关于直和项和有限直和封闭。

称对偶对 $\{\mathscr{X},\mathscr{Y}\}$ 是对称对偶对，如果 $(\mathscr{X},\mathscr{Y})$ 和 $(\mathscr{Y},\mathscr{X})$ 是对偶对。称对偶对 $(\mathscr{X},\mathscr{Y})$ 是完全对偶对，如果 $\mathscr{X}$ 包含 R-模 ${}_RR$，并且 $\mathscr{X}$ 关于直和和扩张封闭。称对偶对 $(\mathscr{X},\mathscr{Y})$ 是完备对偶对，如果 $\{\mathscr{X},\mathscr{Y}\}$ 是对称对偶对，并且 $(\mathscr{X},\mathscr{Y})$ 是完全对偶对。

命题 2[5]　若 $(\mathscr{X},\mathscr{Y})$ 是完全对偶对，则 $\mathscr{X}$ 包含投射 R-模类，$\mathscr{Y}$ 包含内射 R^{op}-模类。事实上，$\mathscr{X}$ 关于正向极限封闭，因此也包含平坦 R-模类。

设 $\mathscr{C}_1$ 是 A-模类，$\mathscr{D}_1$ 是 B-模类，定义 T-模的类 $\mathscr{U}_{\mathscr{D}_1}^{\mathscr{C}_1}=\left\{\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}:M_1\in\mathscr{C}_1,\ M_2\in\mathscr{D}_1\right\}$。类似地，定义 T^{op}-模的类 $\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2}$，此时 $\mathscr{C}_2$ 是 A^{op}-模类，$\mathscr{D}_2$ 是 B^{op}-模类。

命题 3[4]　设 $\mathscr{C}_1$ 是 A-模类，$\mathscr{C}_2$ 是 A^{op}-模类，$\mathscr{D}_1$ 是 B-模类，$\mathscr{D}_2$ 是 B^{op}-模类，则有：

(1) $(\mathscr{C}_1,\mathscr{C}_2)$ 和 $(\mathscr{D}_1,\mathscr{D}_2)$ 是对偶对当且仅当 $(\mathscr{U}_{\mathscr{D}_1}^{\mathscr{C}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$ 是对偶对；

(2) $(\mathscr{C}_1,\mathscr{C}_2)$ 和 $(\mathscr{D}_1,\mathscr{D}_2)$ 是对称对偶对当且仅当 $(\mathscr{U}_{\mathscr{D}_1}^{\mathscr{C}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$ 是对称对偶对；

(3) 如果 ${}_BU\in\mathscr{D}_1$，那么 $(\mathscr{C}_1,\mathscr{C}_2)$ 和 $(\mathscr{D}_1,\mathscr{D}_2)$ 是完全对偶对当且仅当 $(\mathscr{U}_{\mathscr{D}_1}^{\mathscr{C}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$ 是完全对偶对。

本文始终假设 $(\mathscr{C}_1,\mathscr{C}_2)$，$(\mathscr{D}_1,\mathscr{D}_2)$ 和 $(\mathscr{U}_{\mathscr{D}_1}^{\mathscr{C}_1},\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})$ 是完备对偶对。

2　形式三角矩阵环上相对于完备对偶对的 Gorenstein 平坦模和维数

定义 2[5]　设 $(\mathscr{X},\mathscr{Y})$ 是 R 环上的完备对偶对，M 是 R-模，称 M 是 Gorenstein$(\mathscr{X},\mathscr{Y})$-平坦模，如果存在平坦 R-模的正合序列

$$\mathbf{F}:\cdots\to F^{-2}\to F^{-1}\to F^0\to F^1\to\cdots$$

使得 $M\cong\mathrm{Ker}(F^0\to F^1)$，并且对任意的 $Y\in\mathscr{Y}$，$Y\otimes_R\mathbf{F}$ 是正合的。用 $\mathscr{GF}_{(\mathscr{F}(R),\mathscr{Y})}$ 表示 Gorenstein$(\mathscr{X},\mathscr{Y})$-平坦 R-模类。

由文献[10]的命题 1.14 知，$M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是平坦 T-模当且仅当 M_1 是平坦 A-模，$M_2/\mathrm{Im}(\varphi^M)$ 是平坦 B-模，φ^M 是单同态。

定理 1　设 T 是形式三角矩阵环，U_A 有有限平坦维数，$M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是 T-模，$\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 关于扩张封

闭，则以下等价：

(1) $M\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$；

(2) $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$，$M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$，$\varphi^M$ 是单同态。

在此情形下，$U\otimes_A M_1\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$ 当且仅当 $M_2\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$。

证明 (1)⇒(2)。首先证明 $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$。因为 $M\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$，所以存在平坦 T-模的正合序列

$$\mathbf{\Delta}:\cdots\to\begin{pmatrix}F_1^{-1}\\F_2^{-1}\end{pmatrix}_{\varphi^{-1}}\xrightarrow{\begin{pmatrix}\sigma_1^{-1}\\\sigma_2^{-1}\end{pmatrix}}\begin{pmatrix}F_1^{0}\\F_2^{0}\end{pmatrix}_{\varphi^{0}}\xrightarrow{\begin{pmatrix}\sigma_1^{0}\\\sigma_2^{0}\end{pmatrix}}\begin{pmatrix}F_1^{1}\\F_2^{1}\end{pmatrix}_{\varphi^{1}}\xrightarrow{\begin{pmatrix}\sigma_1^{1}\\\sigma_2^{1}\end{pmatrix}}\begin{pmatrix}F_1^{2}\\F_2^{2}\end{pmatrix}_{\varphi^{2}}\to\cdots,$$

使得 $M\cong\mathrm{Ker}\begin{pmatrix}\sigma_1^0\\\sigma_2^0\end{pmatrix}$，并且对任意的 $H\in\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2}$，$H\otimes_T\mathbf{\Delta}$ 是正合的。由文献[10]的命题 1.14 知，有平坦 A-模的正合序列

$$\mathbf{\Lambda}_1:\cdots\to F_1^{-1}\xrightarrow{\sigma_1^{-1}}F_1^0\xrightarrow{\sigma_1^0}F_1^1\xrightarrow{\sigma_1^1}F_1^2\to\cdots,$$

使得 $M_1\cong\mathrm{Ker}(\sigma_1^0)$。对任意的 $E\in\mathcal{C}_2$，显然有 $(E,0)\in\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2}$。进而由命题 1 知 $E\otimes_A\mathbf{\Lambda}_1\cong(E,0)\otimes_T\mathbf{\Delta}$。因为 $(E,0)\otimes_T\mathbf{\Delta}$ 是正合的，所以 $E\otimes_A\mathbf{\Lambda}_1$ 是正合的，因此 $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$。

其次证明 φ^M 是单同态。设 $\lambda_1:M_1\to F_1^0$ 和 $\lambda_2:M_2\to F_2^0$ 是嵌入同态。考虑 $\mathrm{Mod}(B)$ 上的交换图

$$\begin{array}{ccc}U\otimes_A M_1&\xrightarrow{1_U\otimes\lambda_1}&U\otimes_A F_1^0\\ {\scriptstyle\varphi^M}\downarrow&&\downarrow{\scriptstyle\varphi^0}\\ M_2&\xrightarrow{\lambda_2}&F_2^0\end{array}$$

。

因为 U_A 有有限平坦维数，所以由文献[2]的引理 2.3 知 $U\otimes_A\mathbf{\Lambda}_1$ 是正合的。因此 $1_U\otimes\lambda_1$ 是单同态。由文献[10]的命题 1.14 知 φ^0 是单同态。因此由上面的交换图可得 φ^M 是单同态。

最后证明 $M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$。对任意的 $i\in\mathbf{Z}$，存在 $\overline{\sigma_2^i}:F_2^i/\mathrm{Im}(\varphi^i)\to F_2^{i+1}/\mathrm{Im}(\varphi^{i+1})$，使得图

$$\begin{array}{ccccccc}&&\vdots&&\vdots&&\vdots\\&&\downarrow&&\downarrow&&\downarrow\\0\to&&U\otimes_A F_1^{-1}&\xrightarrow{\varphi^{-1}}&F_2^{-1}&\longrightarrow&F_2^{-1}/\mathrm{Im}(\varphi^{-1})\to0\\&&{\scriptstyle 1_U\otimes\sigma_1^{-1}}\downarrow&&{\scriptstyle\sigma_2^{-1}}\downarrow&&{\scriptstyle\overline{\sigma_2^{-1}}}\downarrow\\0\to&&U\otimes_A F_1^{0}&\xrightarrow{\varphi^{0}}&F_2^{0}&\longrightarrow&F_2^{0}/\mathrm{Im}(\varphi^{0})\to0\\&&{\scriptstyle 1_U\otimes\sigma_1^{0}}\downarrow&&{\scriptstyle\sigma_2^{0}}\downarrow&&{\scriptstyle\overline{\sigma_2^{0}}}\downarrow\\0\to&&U\otimes_A F_1^{1}&\xrightarrow{\varphi^{1}}&F_2^{1}&\longrightarrow&F_2^{1}/\mathrm{Im}(\varphi^{1})\to0\\&&{\scriptstyle 1_U\otimes\sigma_1^{1}}\downarrow&&{\scriptstyle\sigma_2^{1}}\downarrow&&{\scriptstyle\overline{\sigma_2^{1}}}\downarrow\\0\to&&U\otimes_A F_1^{2}&\xrightarrow{\varphi^{2}}&F_2^{2}&\longrightarrow&F_2^{2}/\mathrm{Im}(\varphi^{2})\to0\\&&\downarrow&&\downarrow&&\downarrow\\&&\vdots&&\vdots&&\vdots\end{array}$$

可交换。因为第一列和第二列是正合的，所以由文献[10]的命题 1.14 和文献[11]的定理 6.3 知，有平坦 B-模的正合序列

$$\mathbf{\Xi}:\cdots\to F_2^{-1}/\mathrm{Im}(\varphi^{-1})\xrightarrow{\overline{\sigma_2^{-1}}}F_2^0/\mathrm{Im}(\varphi^0)\xrightarrow{\overline{\sigma_2^{0}}}F_2^1/\mathrm{Im}(\varphi^1)\xrightarrow{\overline{\sigma_2^{1}}}F_2^2/\mathrm{Im}(\varphi^2)\to\cdots,$$

使得 $M_2/\mathrm{Im}(\varphi^M)\cong\mathrm{Ker}(\overline{\sigma_2^0})$。对任意的 $G\in\mathcal{D}_2$，易知 $(0,G)\in\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2}$。每一个 B-模的短正合列

$$0\to U\otimes_A F_1^i\xrightarrow{\varphi^i}F_2^i\to F_2^i/\mathrm{Im}(\varphi^i)\to0$$

都可以诱导正合序列 $G\otimes_B U\otimes_A F_1^i \xrightarrow{1_G\otimes\varphi^i} G\otimes_B F_2^i \to G\otimes_B(F_2^i/\mathrm{Im}(\varphi^i))\to 0$。则由命题 1 可得 $G\otimes_B(F_2^i/\mathrm{Im}(\varphi^i))\cong(G\otimes_B F_2^i)/\mathrm{Im}(1_G\otimes\varphi^i)\cong(0,G)\otimes_T\begin{pmatrix}F_1^i\\F_2^i\end{pmatrix}_{\varphi^i}$。进而 $G\otimes_B\mathbf{\Xi}\cong(0,G)\otimes_T\mathbf{\Delta}$ 是正合的。因此 $M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$。

(2)⇒(1)。因为 φ^M 是单同态,所以存在 $\mathrm{Mod}(T)$ 中的正合序列

$$0\to\begin{pmatrix}M_1\\U\otimes_A M_1\end{pmatrix}\to\begin{pmatrix}M_1\\M_2\end{pmatrix}\to\begin{pmatrix}0\\M_2/\mathrm{Im}(\varphi^M)\end{pmatrix}\to 0。$$

首先证明 $\begin{pmatrix}M_1\\U\otimes_A M_1\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。因为 $M_1\in\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$,所以存在平坦 A-模的正合序列

$$\mathbf{\Lambda}:\cdots\longrightarrow F^{-1}\xrightarrow{\sigma^{-1}}F^0\xrightarrow{\sigma^0}F^1\xrightarrow{\sigma^1}F^2\longrightarrow\cdots,$$

使得 $M_1\cong\mathrm{Ker}(\sigma^0)$,并且对任意的 $E\in\mathcal{C}_2$, $E\otimes_A\mathbf{\Lambda}$ 是正合的。因为 U_A 有有限平坦维数,所以由文献[2]的引理 2.3 知 $U\otimes_A\mathbf{\Lambda}$ 是正合的。因此有平坦 T-模的正合序列

$$\mathbf{\Upsilon}:\cdots\to\begin{pmatrix}F^{-1}\\U\otimes_A F^{-1}\end{pmatrix}\xrightarrow{\begin{pmatrix}\sigma^{-1}\\1_U\otimes\sigma^{-1}\end{pmatrix}}\begin{pmatrix}F^0\\U\otimes_A F^0\end{pmatrix}\xrightarrow{\begin{pmatrix}\sigma^0\\1_U\otimes\sigma^0\end{pmatrix}}\begin{pmatrix}F^1\\U\otimes_A F^1\end{pmatrix}\to\cdots$$

使得 $\begin{pmatrix}M_1\\U\otimes_A M_1\end{pmatrix}\cong\mathrm{Ker}\begin{pmatrix}\sigma^0\\1_U\otimes\sigma^0\end{pmatrix}$。对任意的 $(W_1,W_2)\in\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2}$,存在 $\mathrm{Mod}(T^{\mathrm{op}})$ 中的短正合列

$$0\to(W_1,0)\to(W_1,W_2)\to(0,W_2)\to 0。$$

因为对任意的 $i\in\mathbf{Z}$, $\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}$ 是平坦 T-模,所以有短正合列

$$0\to(W_1,0)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}\to(W_1,W_2)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}\to(0,W_2)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}\to 0。$$

于是由命题 1 知 $(0,W_2)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}\cong(W_2\otimes_B U\otimes_A F^i)/(W_2\otimes_B U\otimes_A F^i)=0$,因此 $(W_1,W_2)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}\cong(W_1,0)\otimes_T\begin{pmatrix}F^i\\U\otimes_A F^i\end{pmatrix}$。又因为 $W_1\in\mathcal{C}_1$,所以 $(W_1,W_2)\otimes_T\mathbf{\Upsilon}\cong(W_1,0)\otimes_T\mathbf{\Upsilon}\cong W_1\otimes_A\mathbf{\Lambda}$ 是正合的。故 $\begin{pmatrix}M_1\\U\otimes_A M_1\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。

其次证明 $\begin{pmatrix}0\\M_2/\mathrm{Im}(\varphi^M)\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。因为 $M_2/\mathrm{Im}(\varphi^M)\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$,所以存在平坦 B-模的正合序列

$$\mathbf{\Theta}:\cdots\longrightarrow Q^{-1}\xrightarrow{f^{-1}}Q^0\xrightarrow{f^0}Q^1\xrightarrow{f^1}Q^2\longrightarrow\cdots$$

使得 $M_2/\mathrm{Im}(\varphi^M)\cong\mathrm{Ker}(f^0)$,并且对任意的 $G\in\mathcal{D}_2$, $G\otimes_B\mathbf{\Theta}$ 是正合的。因此有平坦 T-模的正合序列

$$\begin{pmatrix}0\\\mathbf{\Theta}\end{pmatrix}:\cdots\to\begin{pmatrix}0\\Q^{-1}\end{pmatrix}\xrightarrow{\begin{pmatrix}0\\f^{-1}\end{pmatrix}}\begin{pmatrix}0\\Q^0\end{pmatrix}\xrightarrow{\begin{pmatrix}0\\f^0\end{pmatrix}}\begin{pmatrix}0\\Q^1\end{pmatrix}\xrightarrow{\begin{pmatrix}0\\f^1\end{pmatrix}}\begin{pmatrix}0\\Q^2\end{pmatrix}\to\cdots$$

使得 $\begin{pmatrix}0\\M_2/\mathrm{Im}(\varphi^M)\end{pmatrix}\cong\mathrm{Ker}\begin{pmatrix}0\\f^0\end{pmatrix}$。对任意的 $(X_1,X_2)\in\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2}$, $(X_1,X_2)\otimes_T\begin{pmatrix}0\\\mathbf{\Theta}\end{pmatrix}\cong X_2\otimes_B\mathbf{\Theta}$ 是正合的。故 $\begin{pmatrix}0\\M_2/\mathrm{Im}(\varphi^M)\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。

因为 $\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ 关于扩张封闭,所以 $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。

最后,由 φ^M 是单同态知,存在 Mod(B)上的短正合列 $0\to U\otimes_A M_1\xrightarrow{\varphi^M}M_2\to M_2/\mathrm{Im}(\varphi^M)\to 0$,进而有 Mod($T$)上的短正合列

$$0\to X=\begin{pmatrix}0\\U\otimes_A M_1\end{pmatrix}\to Y=\begin{pmatrix}0\\M_2\end{pmatrix}\to Z=\begin{pmatrix}0\\M_2/\mathrm{Im}(\varphi^M)\end{pmatrix}\to 0。$$

因为 $\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 关于扩张封闭,所以易证 $\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 是投射可解的。则当 $Z\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 时,$X\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 当且仅当 $Y\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$。因此由前面的证明可知 $U\otimes_A M_1\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$ 当且仅当 $M_2\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。

当 $A=B=U$ 时,$\mathscr{C}_1$ 和 $\mathscr{D}_1$ 是 A-模类,$\mathscr{C}_2$ 和 $\mathscr{D}_2$ 是 A^{op}-模类。由定理 1 易得推论 1。

推论 1 设 $T(A)=\begin{pmatrix}A&0\\A&A\end{pmatrix}$是形式三角矩阵环,$M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是 $T(A)$-模, $\mathscr{GF}_{(\mathscr{F}(T(A)),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 关于扩张封闭,$\mathscr{C}_1=\mathscr{D}_1$, $\mathscr{C}_2=\mathscr{D}_2$,则以下等价:

(1) $M\in\mathscr{GF}_{(\mathscr{F}(T(A)),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$;

(2) $M_1,M_2/\mathrm{Im}(\varphi^M)\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$,$\varphi^M$ 是单同态;

(3) $M_2,M_2/\mathrm{Im}(\varphi^M)\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$,$\varphi^M$ 是单同态。

下面考虑形式三角矩阵环上模的相对于完备对偶对的 Gorenstein 平坦维数。

设 X 是 R-模。记 $Gfd_{(\mathscr{F}(R),\mathscr{Y})}(X)=\inf\{n\in\mathbf{Z}$:存在 R-模的正合序列 $0\to G_n\to\cdots\to G_0\to X\to 0$,其中每个 $G_i\in\mathscr{GF}_{(\mathscr{F}(R),\mathscr{Y})}\}$,称 $Gfd_{(\mathscr{F}(R),\mathscr{Y})}(X)$ 是 X 的 Gorenstein$(\mathscr{X},\mathscr{Y})$-平坦维数[6]。如果这样的 n 不存在,记 $Gfd_{(\mathscr{F}(R),\mathscr{Y})}(X)=\infty$。记 $lGFD_{(\mathscr{F}(R),\mathscr{Y})}(R)=\sup\{Gfd_{(\mathscr{F}(R),\mathscr{Y})}(X)$:$X$ 是任意的 R-模$\}$,称 $lGFD_{(\mathscr{F}(R),\mathscr{Y})}(R)$是 R 环的左整体 Gorenstein$(\mathscr{X},\mathscr{Y})$-平坦维数。

引理 1 设 $lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B)<\infty$,${}_BU$ 是平坦 B-模,U_A 有有限平坦维数,$\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$ 关于扩张封闭。如果 $X\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$,那么 $U\otimes_A X\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。

证明 因为 $lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B)<\infty$, $\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$关于扩张封闭,所以由文献[6]的命题 4.1 知任意 B^{op}-模的平坦维数有限。进而由文献[12]的引理 1.2.3 知结论成立。

定理 2 设 T 是形式三角矩阵环,$M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是 T-模,$lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B)<\infty$, U_A 有有限平坦维数,${}_BU$ 是平坦 B-模,$\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 和 $\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$ 关于扩张封闭,则

$$\begin{aligned}&\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1),\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}\leqslant Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)\\&\leqslant\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1)+1,\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}。\end{aligned}$$

证明 首先证明

$$\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1),\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}\leqslant Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)。$$

设 $Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)=m<\infty$,则存在 Mod($T$)中的正合序列

$$0\to\begin{pmatrix}N_1^m\\N_2^m\end{pmatrix}_{\varphi^m}\xrightarrow{\begin{pmatrix}\sigma_1^m\\\sigma_2^m\end{pmatrix}}\begin{pmatrix}N_1^{m-1}\\N_2^{m-1}\end{pmatrix}_{\varphi^{m-1}}\to\cdots\to\begin{pmatrix}N_1^0\\N_2^0\end{pmatrix}_{\varphi^0}\xrightarrow{\begin{pmatrix}\sigma_1^0\\\sigma_2^0\end{pmatrix}}\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}\to 0,$$

其中每个$\begin{pmatrix}N_1^i\\N_2^i\end{pmatrix}_{\varphi^i}\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$。由定理 1 知每个 $N_1^i\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$。因此由引理 1 知每个 $U\otimes_A N_1^i\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。再由定理 1 知每个$N_2^i\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。注意到有如下两个正合序列

$$0\longrightarrow N_1^m\xrightarrow{\sigma_1^m}N_1^{m-1}\longrightarrow\cdots\longrightarrow N_1^0\xrightarrow{\sigma_1^0}M_1\longrightarrow 0,$$

$$0\longrightarrow N_2^m\xrightarrow{\sigma_2^m}N_2^{m-1}\longrightarrow\cdots\longrightarrow N_2^0\xrightarrow{\sigma_2^0}M_2\longrightarrow 0。$$

因此 $\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1),\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}\leqslant m=Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)$。

下面证明

$$Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)\leqslant\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1)+1,\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}。$$

设 $\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1)+1,\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}=n<\infty$，则存在 Mod($A$)中的正合序列

$$0\longrightarrow G_{n-1}\xrightarrow{f_{n-1}}G_{n-2}\xrightarrow{f_{n-2}}\cdots\rightarrow G_1\xrightarrow{f_1}G_0\xrightarrow{f_0}M_1\longrightarrow 0,$$

其中每个 $G_i\in\mathscr{GF}_{(\mathscr{F}(A),\mathscr{C}_2)}$。对任意的 $0\leqslant i\leqslant n-1$，令 $K_1^i=\mathrm{Ker}(f_{i-1})$，$\pi_i:G_i\rightarrow K_1^i$。显然 π_i 是满同态。存在正合序列 $P_0\xrightarrow{g_0}M_2\rightarrow 0$，其中 P_0 是投射 B-模。定义 $h_0:(U\otimes_A G_0)\oplus P_0\rightarrow M_2$，其中对任意的 $u\in U$，$c_0\in G_0$，$x_0\in P_0$，有 $h_0(u\otimes c_0,x_0)=\varphi^M(u\otimes f_0(c_0))+g_0(x_0)$。则 h_0 是满同态。因此有 Mod(T)中的短正合列

$$0\longrightarrow\begin{pmatrix}K_1^1\\K_2^1\end{pmatrix}_{\psi^1}\longrightarrow\begin{pmatrix}G_0\\(U\otimes_A G_0)\oplus P_0\end{pmatrix}_{\varphi^{G_0}}\xrightarrow{\begin{pmatrix}f_0\\h_0\end{pmatrix}}\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}\longrightarrow 0。$$

同理，存在正合序列 $P_1\xrightarrow{g_1}K_2^1\rightarrow 0$，其中 P_1 是投射 B-模。定义 $h_1:(U\otimes_A G_1)\oplus P_1\rightarrow K_2^1$，其中对任意的 $u\in U$，$c_1\in G_1$，$x_1\in P_1$，有 $h_1(u\otimes c_1,x_1)=\psi^1(u\otimes\pi_1(c_1))+g_1(x_1)$。则 h_1 是满同态。因此有 Mod(T)中的短正合列

$$0\longrightarrow\begin{pmatrix}K_1^2\\K_2^2\end{pmatrix}_{\psi^2}\longrightarrow\begin{pmatrix}G_1\\(U\otimes_A G_1)\oplus P_1\end{pmatrix}_{\varphi^{G_1}}\xrightarrow{\begin{pmatrix}\pi_1\\h_1\end{pmatrix}}\begin{pmatrix}K_1^1\\K_2^1\end{pmatrix}_{\psi^1}\longrightarrow 0。$$

继续该过程，则有 Mod(T)中的正合序列

$$0\rightarrow\begin{pmatrix}0\\K_2^n\end{pmatrix}_{\psi^n}\rightarrow\begin{pmatrix}G_{n-1}\\(U\otimes_A G_{n-1})\oplus P_{n-1}\end{pmatrix}_{\varphi^{G_{n-1}}}\rightarrow\begin{pmatrix}G_{n-2}\\(U\otimes_A G_{n-2})\oplus P_{n-2}\end{pmatrix}_{\varphi^{G_{n-2}}}\rightarrow$$

$$\cdots\rightarrow\begin{pmatrix}G_1\\(U\otimes_A G_1)\oplus P_1\end{pmatrix}_{\varphi^{G_1}}\rightarrow\begin{pmatrix}G_0\\(U\otimes_A G_0)\oplus P_0\end{pmatrix}_{\varphi^{G_0}}\rightarrow\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}\rightarrow 0。$$

由引理 1 知每个 $U\otimes_A G_i\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。易证每个$(U\otimes_A G_i)\oplus P_i\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。因为 $Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\leqslant n$，所以由文献[6]的引理 3.10 知 $K_2^n\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。易证每个 $\varphi^{G_i}:U\otimes_A G_i\rightarrow(U\otimes_A G_i)\oplus P_i$ 是单同态，每个 $((U\otimes_A G_i)\oplus P_i)/\mathrm{Im}(\varphi^{G_i})\cong P_i\in\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$。因此由定理 1 知 $\begin{pmatrix}G_i\\(U\otimes_A G_i)\oplus P_i\end{pmatrix}_{\varphi^{G_i}}\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$，由定理 1 的证明知 $\begin{pmatrix}0\\K_2^n\end{pmatrix}_{\psi^n}\in\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$，故 $Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(M)\leqslant n=\max\{Gfd_{(\mathscr{F}(A),\mathscr{C}_2)}(M_1)+1,\ Gfd_{(\mathscr{F}(B),\mathscr{D}_2)}(M_2)\}$。

定理 3　设 T 是形式三角矩阵环，${}_BU\neq 0$ 是平坦 B-模，U_A 有有限平坦维数，$\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$ 和 $\mathscr{GF}_{(\mathscr{F}(B),\mathscr{D}_2)}$ 关于扩张封闭，则

$$\max\{lGFD_{(\mathscr{F}(A),\mathscr{C}_2)}(A),\ lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B),1\}\leqslant lGFD_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(T)$$
$$\leqslant\max\{lGFD_{(\mathscr{F}(A),\mathscr{C}_2)}(A)+1,\ lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B)\}。$$

证明　首先证明

$$\max\{lGFD_{(\mathscr{F}(A),\mathscr{C}_2)}(A),\ lGFD_{(\mathscr{F}(B),\mathscr{D}_2)}(B),1\}\leqslant lGFD_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(T)。$$

设 $lGFD_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}(T)=m<\infty$。因为 ${}_BU\neq 0$，所以由定理 1 知 T-模 $\begin{pmatrix}A\\0\end{pmatrix}\notin\mathscr{GF}_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}$，进而 $1\leqslant Gfd_{(\mathscr{F}(T),\mathscr{U}_{\mathscr{C}_2,\mathscr{D}_2})}\begin{pmatrix}A\\0\end{pmatrix}\leqslant m$。设 N 是任意的 B-模，则存在 B-模的正合序列

$$0\rightarrow K_m\rightarrow P_{m-1}\rightarrow\cdots\rightarrow P_0\rightarrow N\rightarrow 0,$$

其中每个 P_i 都是投射 B-模。则有 Mod(T)上的正合序列

$$0\rightarrow\begin{pmatrix}0\\K_m\end{pmatrix}\rightarrow\begin{pmatrix}0\\P_{m-1}\end{pmatrix}\rightarrow\cdots\rightarrow\begin{pmatrix}0\\P_0\end{pmatrix}\rightarrow\begin{pmatrix}0\\N\end{pmatrix}\rightarrow 0,$$

其中每个$\begin{pmatrix}0\\P_i\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。因为 $Gfd_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}\begin{pmatrix}0\\N\end{pmatrix}\leqslant lGFD_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T)=m$，所以由文献[6]的引理3.10知$\begin{pmatrix}0\\K_m\end{pmatrix}\in\mathcal{GF}_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$。进而由定理1知 $K_m\in\mathcal{GF}_{(\mathcal{F}(B),\mathcal{D}_2)}$，则 $Gfd_{(\mathcal{F}(B),\mathcal{D}_2)}(N)\leqslant m$。因此 $lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)\leqslant m$。设 Y 是任意的 A-模，则由定理2知

$$Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(Y)\leqslant Gfd_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}\begin{pmatrix}Y\\0\end{pmatrix}\leqslant lGFD_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T)=m。$$

因此 $lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)\leqslant m$。故 $\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A),\ lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B),1\}\leqslant lGFD_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T)$。

下面证明

$$lGFD_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T)\leqslant\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)+1,\ lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)\}。$$

设 $\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)+1,\ lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)\}<\infty$，则 $lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)<\infty$。设 $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}$是任意的 T-模，则由定理2知

$$\begin{aligned}Gfd_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(M)&\leqslant\max\{Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(M_1)+1,\ Gfd_{(\mathcal{F}(B),\mathcal{D}_2)}(M_2)\}\\&\leqslant\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)+1,\ lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)\}。\end{aligned}$$

故 $lGFD_{(\mathcal{F}(T),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T)\leqslant\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)+1,\ lGFD_{(\mathcal{F}(B),\mathcal{D}_2)}(B)\}$。

当 $A=B=U$ 时，$\mathcal{C}_1$ 和 $\mathcal{D}_1$ 是 A-模类，$\mathcal{C}_2$ 和 $\mathcal{D}_2$ 是 A^{op}-模类。由定理2、3易得推论2。

推论2 设 $T(A)=\begin{pmatrix}A&0\\A&A\end{pmatrix}$是形式三角矩阵环，$\mathcal{GF}_{(\mathcal{F}(T(A)),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}$ 和 $\mathcal{GF}_{(\mathcal{F}(A),\mathcal{C}_2)}$ 关于扩张封闭，$\mathcal{C}_1=\mathcal{D}_1$，$\mathcal{C}_2=\mathcal{D}_2$。

（1）如果 $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$ 是 $T(A)$-模，$lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)<\infty$，那么

$$\begin{aligned}\max\{Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(M_1),\ Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(M_2)\}&\leqslant Gfd_{(\mathcal{F}(T(A)),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(M)\\&\leqslant\max\{Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(M_1)+1,\ Gfd_{(\mathcal{F}(A),\mathcal{C}_2)}(M_2)\}；\end{aligned}$$

（2）$\max\{lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A),1\}\leqslant lGFD_{(\mathcal{F}(T(A)),\mathcal{U}_{\mathcal{C}_2,\mathcal{D}_2})}(T(A))\leqslant lGFD_{(\mathcal{F}(A),\mathcal{C}_2)}(A)+1$。

参考文献：

[1] ENOCHS E E, JENDA O M G. Relative homological algebra[M]. Berlin: Walter De Gruyter, 2000.

[2] ENOCHS E E, IZURDIAGA M C, TORRECILLAS B. Gorenstein conditions over triangular matrix rings[J]. Journal of Pure and Applied Algebra, 2014, 218(8):1544-1554.

[3] MAO Lixin. Gorenstein flat modules and dimensions over formal triangular matrix rings[J]. Journal of Pure and Applied Algebra, 2020, 224(4):106207.

[4] MAO Lixin. Duality pairs and FP-injective modules over formal triangular matrix rings[J]. Communications in Algebra, 2020, 48(12):5296-5310.

[5] GILLESPIE J. Duality pairs and stable module categories[J]. Journal of Pure and Applied Algebra, 2019, 223(8):3425-3435.

[6] BECERRIL V. ($\mathcal{F}$, $\mathcal{A}$)-Gorenstein flat homological dimensions[J]. Journal of the Korean Mathematical Society, 2022, 59(6):1203-1227.

[7] GREEN E L. On the representation theory of rings in matrix form[J]. Pacific Journal of Mathematics, 1982, 100(1):123-138.

[8] KRYLOV P, TUGANBAEV A. Formal matrices[M]. Berlin: Springer, 2017.

[9] HOLM H, JØRGENSEN P. Cotorsion pairs induced by duality pairs[J]. Journal of Commutative Algebra, 2009, 1(4):621-633.

[10] FOSSUM R M, GRIFFITH P A, REITEN I. Trivial extensions of Abelian categories[M]. New York: Springer, 1975.

[11] ROTMAN J J. An introduction to homological algebra[M]. New York: Academic Press, 1979.

[12] 牟婷. 相对于对偶对的Gorenstein平坦维数和相对奇点范畴[D]. 兰州：西北师范大学，2022.
MOU Ting. Gorenstein flat dimensions with respect to duality pairs and relative singularity categories[D]. Lanzhou: Northwest Normal University, 2022.

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