山东大学学报(理学版)2026年1月 第61卷 第1期
Journal of Shandong University(*Natural Science*), Vol.61, No.1, 2026 http://lxbwk.njournal.sdu.edu.cn

文章编号:1671-9352(2026)01-0116-11 DOI:10.6040/j.issn.1671-9352.0.2024.108

2型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 算子

吕笑凡,袁修久*,李京泰,赵学军

(空军工程大学基础部, 陕西 西安 710051)

摘要:为了描述多属性决策问题中决策者们多层的犹豫性,本文结合犹豫模糊集、q 阶三角犹豫模糊集和 q 阶犹豫模糊不确定语言集,提出2型犹豫 q 阶三角不确定语言集,定义了基于 Schweizer-Sklar 范数的模糊元的运算,证明运算的性质。为了有效处理评估属性之间相互关联的现实决策问题,将 Muirhead 平均算子推广至2型犹豫 q 阶三角不确定语言集,提出2型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子和加权 Muirhead 平均算子,并给出这些算子的计算公式和性质。最后,给出基于2型犹豫 q 阶三角不确定语言 Muirhead 平均算子的多属性决策方法,并进行算例分析。

关键词:犹豫模糊集;2型犹豫 q 阶三角不确定语言集;Muirhead 平均算子;多属性决策

中图分类号:O159 **文献标志码**:A

引用格式:吕笑凡,袁修久,李京泰,等. 2型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 算子[J]. 山东大学学报(理学版),2026,61(1):116-126.

Type-2 hesitant q-rung triangular uncertain linguistic Schweizer-Sklar-Muirhead operators

LÜ Xiaofan, YUAN Xiujiu*, LI Jingtai, ZHAO Xuejun

(Fundamentals Department, Air Force Engineering University, Xi'an 710051, Shaanxi, China)

Abstract: In order to describe the multi-layer hesitation of decision-makers in multi-attribute decision-making problems, the type-2 hesitant q-rung triangular uncertain linguistic set is defined by combining hesitant fuzzy set, q-rung triangular hesitant fuzzy set and q-rung hesitant fuzzy uncertain linguistic set. Based on the Schweizer-Sklar norm, the operation laws and properties of the type-2 hesitant q-rung triangular uncertain linguistic fuzzy elements are discussed. In addition, in order to better handle the practical problem with interrelated evaluation attributes, the Muirhead mean operator is extended to the type-2 hesitant q-rung triangular uncertain linguistic set, and the type-2 hesitant q-rung triangular uncertain linguistic Schweizer-Sklar-Muirhead mean operator and its weighted form are proposed. Moreover, the calculation formulae of the operators are given and their properties are discussed. Finally, a multi-attribute decision-making problem model based on the type-2 q-rung hesitant triangular fuzzy uncertain linguistic weighted Muirhead mean operator is established and further analyzed by a numerical example.

Key words: hesitant fuzzy set; type-2 hesitant q-rung triangular uncertain linguistic set; Muirhead mean operator; multi-attribute decision making

0 引言

多属性决策指通过适当途径从一系列方案中对方案排序或选出一个具有最高满意度的最优方案[1],广泛应用于工程、经济、管理、军事等诸多领域[2-4]。决策者们以什么形式给出评估值以及如何处理这些评估值是处理多属性决策问题的关键。Zadeh[5]提出的模糊集理论使元素与集合之间的关系有了模糊性和不确

收稿日期:2024-04-03; **网络出版时间**:2025-07-09

基金项目:国家自然科学基金资助项目(12171386)

第一作者:吕笑凡(2000—),男,硕士研究生,研究方向为仿真理论及技术. E-mail:lxfxs2022@163.com

***通信作者**:袁修久(1966—),男,教授,博士,研究方向为仿真理论及技术. E-mail:xiujiu_yuan@163.com

定性,模糊集能描述并处理决策者们的评估信息,广泛应用于多属性决策问题[6-8]。但决策者们进行评估时经常会犹豫不决,相互又不能达成一致意见。为了处理这种情况,Torra[9]提出犹豫模糊集,随后很多学者提出犹豫模糊集的拓展形式。Zhu 等[10]结合犹豫模糊集和直觉模糊集提出对偶犹豫模糊集。徐玥等[11]结合犹豫模糊集和 q 阶模糊集,提出 q 阶犹豫模糊集,放宽了隶属度与非隶属度的约束条件,允许决策者在更大的范围内来表示评估信息。任耀军等[12]结合三角模糊数和犹豫模糊集,研究 q 阶三角犹豫模糊集。

以上模糊集及其拓展形式,只从定量的角度描述元素对集合的隶属度和非隶属度。在实际问题中,决策者们更倾向于对候选方案满足属性的情况给出如"很好"、"一般"等语言术语形式的定性评价。为此,Wang 等[13]结合直觉模糊集和语言术语集提出直觉语言集,Liu 等[14]提出直觉不确定语言集。

对于信息集成算子[15-17],Huang 等[18]定义 q 阶犹豫模糊不确定语言集,并基于 Schweizer-Sklar 范数定义 q 阶犹豫模糊不确定语言 Schweizer-Sklar Bonferroni 平均算子、Maclaurin 对称平均算子以及Muirhead平均算子。Muirhead 平均算子[19]反映多个属性之间的相互关系,且其参数取特定值时退化为其他常用算子,具有很好的灵活性。

在实际决策过程中,除了决策小组各成员给出的隶属度不同外,决策小组内某些成员在进行评估时,成员自己也会出现犹豫的情况,此时,成员所给的评估值也用某种犹豫模糊元表示,因此出现多层犹豫的现象,而现有的犹豫模糊集及其推广形式无法描述处理这种情况。在决策过程中,决策者往往很难给出具体的数值,而三角模糊数与语言术语集能使评估信息更符合人们的思维习惯。为此,本文结合 2 型犹豫模糊集、三角模糊数和语言术语集,提出 2 型犹豫 q 阶三角不确定语言集概念,在此基础上基于 Schweizer-Sklar t 模和 t 余模定义了 2 型犹豫 q 阶三角不确定语言元(type-2 hesitant q-rung triangular uncertain linguistic element, T2HqTULE)的运算,并将 Muirhead 平均算子推广至 T2HqTULE,给出基于 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子的多属性决策方法。

1　预备知识

本章介绍相关基础理论知识,主要包括三角模糊数、q 阶三角犹豫模糊集、q 阶犹豫模糊不确定语言集和 Schweizer-Sklar 范数。

1.1　三角模糊数

定义 1[20]　设 $0\leqslant \underline{a}\leqslant a\leqslant \bar{a}\leqslant 1$,若模糊数 $\tilde{a}$ 的隶属度函数为

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{x-\underline{a}}{a-\underline{a}}, & x\in[\underline{a},a],\\ \dfrac{\bar{a}-x}{\bar{a}-a}, & x\in[a,\bar{a}],\\ 0, & \text{其他},\end{cases}\tag{1}$$

则称 $\tilde{a}$ 为三角模糊数,记为 $\tilde{a}=(\underline{a},a,\bar{a})$。

定义 2[21]　设三角模糊数 $\tilde{a}_1=(\underline{a}_1,a_1,\bar{a}_1)$, $\tilde{a}_2=(\underline{a}_2,a_2,\bar{a}_2)$,则三角模糊数的可能度定义为

$$P(\tilde{a}_1\geqslant\tilde{a}_2)=\lambda\ \max\left\{1-\max\left[\frac{a_2-\underline{a}_1}{a_1-\underline{a}_1+a_2-\underline{a}_2},\ 0\right],\ 0\right\}+(1-\lambda)\max\left\{1-\max\left[\frac{\bar{a}_2-a_1}{\bar{a}_1-a_1+\bar{a}_2-a_2},\ 0\right],\ 0\right\}。\tag{2}$$

1.2　q 阶三角犹豫模糊集

定义 3[12]　设 X 为论域,则定义在 X 上的 q 阶三角犹豫模糊集为

$$A=\{\langle x,\Gamma_A,\Psi_A\rangle_q \mid x\in X\},\tag{3}$$

式中:Γ_A、Ψ_A 是三角模糊数的集合,分别表示元素 x 属于集合 A 的可能的隶属度和可能的非隶属度的集合,且对于任意 $x\in X$,任意$(\underline{\mu},\mu,\bar{\mu})\in\Gamma_A$,任意$(\underline{v},v,\bar{v})\in\Psi_A$,有 $\bar{\mu}^q+\bar{v}^q\leqslant 1$。

称 $h=\langle\Gamma_h,\Psi_h\rangle_q$ 为 q 阶三角犹豫模糊元(q-rung triangular hesitant fuzzy element, q-THFE),q 阶三角犹豫模糊元全体的集合记为 q-THFE(X)。

1.3　q 阶犹豫模糊不确定语言集

定义 4[18]　设 X 为论域,则定义在 X 上的 q 阶犹豫模糊不确定语言集为

$$A=\{x,\langle[s_{\theta(x)},s_{\tau(x)}],\Gamma_A(x),\Psi_A(x)\rangle \mid x\in X\},\tag{4}$$

式中:$[s_{\theta(x)},s_{\tau(x)}]$为不确定语言变量,$\Gamma_A(x)$、$\Psi_A(x)$为区间$[0,1]$的非空有限子集,分别表示元素 x 属于不确定语言变量$[s_{\theta(x)},s_{\tau(x)}]$的可能的隶属度和可能的非隶属度的集合。

称 $h=\langle[s_{\theta(x)},s_{\tau(x)}],\Gamma_A(x),\Psi_A(x)\rangle$为 q 阶犹豫模糊不确定语言元(q-rung orthopair hesitant fuzzy uncertain linguistic element, q-ROHFULE)。q 阶犹豫模糊不确定语言元全体的集合记为 q-ROHFULE(X)。

1.4 Schweizer-Sklar 范数

定义 5[22] 设 r 为任意非正实参数,则 Schweizer-Sklar t 模和 t 余模基于二元运算 $T(\cdot,\cdot,r):[0,1]^2\to[0,1]$分别定义为

$$T_{\mathrm{SS},r}(x,y,r)=\begin{cases}(x^r+y^r-1)^{\frac{1}{r}}, & r\in(-\infty,0),\\ xy, & r=0;\end{cases}\tag{5}$$

$$T^*_{\mathrm{SS},r}(x,y,r)=\begin{cases}1-((1-x)^r+(1-y)^r-1)^r, & r\in(-\infty,0),\\ x+y-xy, & r=0。\end{cases}\tag{6}$$

2 2型犹豫 q 阶三角不确定语言集

结合犹豫模糊集、q 阶三角犹豫模糊集和 q 阶犹豫模糊不确定语言集,提出了2型犹豫 q 阶三角不确定语言集的概念。

定义 6 设 X 为论域,则定义在 X 上的2型犹豫 q 阶三角不确定语言集为

$$A=\{x,\langle([s_{\theta_1(x)},s_{\tau_1(x)}],\Gamma_1(x),\Psi_1(x)),([s_{\theta_2(x)},s_{\tau_2(x)}],\Gamma_2(x),\Psi_2(x)),\cdots,([s_{\theta_k(x)},s_{\tau_k(x)}],\Gamma_k(x),\Psi_k(x))\rangle_q \mid x\in X\},\tag{7}$$

式中:$[s_{\theta_m(x)},s_{\tau_m(x)}]$为不确定语言变量,$\Gamma_m(x)$、$\Psi_m(x)$是三角模糊数的集合,且对于任意 $x\in X$,任意 $m\in\{1,2,\cdots,k\}$,任意$(\underline{\mu}_m,\mu_m,\bar{\mu}_m)\in\Gamma_m$和任意$(\underline{v}_m,v_m,\bar{v}_m)\in\Psi_m$,有 $\bar{\mu}_m^q+\bar{v}_m^q\leqslant 1$。

称 $h=\langle([s_{\theta_1},s_{\tau_1}],\Gamma_1,\Psi_1),([s_{\theta_2},s_{\tau_2}],\Gamma_2,\Psi_2),\cdots,([s_{\theta_k},s_{\tau_k}],\Gamma_k,\Psi_k)\rangle_q$ 为2型犹豫 q 阶三角不确定语言元。记 $f_m=([s_{\theta_m},s_{\tau_m}],\Gamma_m,\Psi_m)$, $m=1,2,\cdots,k$,有 $h=\langle f_1,f_2,\cdots,f_k\rangle_q$,称 h 为元素 x 对集合 A 的主犹豫模糊元,称 f_m 为元素 x 对集合 A 的次犹豫模糊元。全体 T2HqTULE 的集合记为 T2HqTULE(X)。

为了比较2个 T2HqTULE,给出 T2HqTULE 得分函数和精确函数的定义。

定义 7 设 $h=\langle([s_{\theta_1},s_{\tau_1}],\Gamma_1,\Psi_1),([s_{\theta_2},s_{\tau_2}],\Gamma_2,\Psi_2),\cdots,([s_{\theta_k},s_{\tau_k}],\Gamma_k,\Psi_k)\rangle_q\in$ T2HqTULE(X),h 的得分函数 $S(h)$和精确函数 $H(h)$分别定义为

$$S(h)=\frac{1}{4k}\sum_{m=1}^{k}(\theta_m(x)+\tau_m(x))\left((1,1,1)+\frac{1}{|\Gamma_m|}\sum_{(\underline{\mu}_m,\mu_m,\bar{\mu}_m)\in\Gamma_m}(\underline{\mu}_m^q,\mu_m^q,\bar{\mu}_m^q)-\frac{1}{|\Psi_m|}\sum_{(\underline{v}_m,v_m,\bar{v}_m)\in\Psi_m}(\bar{v}_m^q,v_m^q,\underline{v}_m^q)\right);\tag{8}$$

$$H(h)=\frac{1}{2k}\sum_{m=1}^{k}(\theta_m(x)-\tau_m(x))\left(\frac{1}{|\Gamma_m|}\sum_{(\underline{\mu}_m,\mu_m,\bar{\mu}_m)\in\Gamma_m}(\underline{\mu}_m^q,\mu_m^q,\bar{\mu}_m^q)+\frac{1}{|\Psi_m|}\sum_{(\underline{v}_m,v_m,\bar{v}_m)\in\Psi_m}(\underline{v}_m^q,v_m^q,\bar{v}_m^q)\right)。\tag{9}$$

根据 T2HqTULE 的得分函数 $S(h)$、精确函数 $H(h)$和式(2),对 T2HqTULE 进行排序。

定义 8 设 $h_i=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X), $i=1,2$, $S(h_1)$和 $S(h_2)$分别为 h_1 和 h_2 的得分函数,$H(h_1)$和 $H(h_2)$分别为 h_1 和 h_2 的精确函数,有

(1) 如果 $S(h_1)>S(h_2)$,则 $h_1\succ h_2$;

(2) 如果 $S(h_1)=S(h_2)$,且 $H(h_1)<H(h_2)$,则 $h_1\prec h_2$;

(3) 如果 $S(h_1)=S(h_2)$,且 $H(h_1)=H(h_2)$,则 $h_1\approx h_2$。

为了讨论信息集成算子的性质,给出2型犹豫 q 阶三角不确定语言元的如下偏序关系。

定义 9 设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_k^i},s_{\tau_k^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X),且$|\Gamma_m^i|=|\Psi_m^i|$, $i=1,2$,有

$$h_1\geqslant_s h_2\Leftrightarrow f_m^1\geqslant_s f_m^2,\quad \forall m\in\{1,2,\cdots,k\},\quad s_{\theta_m^1}\geqslant s_{\theta_m^2},\quad s_{\tau_m^1}\geqslant s_{\tau_m^2},$$

$$\underline{\mu}_m^1\geqslant\underline{\mu}_m^2,\quad \mu_m^1\geqslant\mu_m^2,\quad \bar{\mu}_m^1\geqslant\bar{\mu}_m^2,\quad \underline{v}_m^1\leqslant v_m^2,\quad v_m^1\leqslant v_m^2,\quad \bar{v}_m^1\leqslant\bar{v}_m^2,$$

式中：$(\underline{\mu}_m^i,\mu_m^i,\bar{\mu}_m^i)\in\Gamma_m^i$，$(\underline{v}_m^i,v_i,\bar{v}_m^i)\in\Psi_m^i$，$i=1,2$，$m=1,2,\cdots,k$。

定义 10　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q$，$h=\langle([s_{\theta_1},s_{\tau_1}],\Gamma_1,\Psi_1),([s_{\theta_2},s_{\tau_2}],\Gamma_2,\Psi_2),\cdots,([s_{\theta_k},s_{\tau_k}],\Gamma_k,\Psi_k)\rangle_q\in$ T2HqTULE(X)，$i=1,2$，基于 Schweizer-Sklar t 模和 t 余模($r<0$)的运算定义为

$$\begin{aligned}h_1\oplus h_2&=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{f^1\oplus f^2\}\rangle_q\\&=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{\bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1)\in\Gamma^1,(\underline{v}^1,v^1,\bar{v}^1)\in\Psi^1,\\(\underline{\mu}^2,\mu^2,\bar{\mu}^2)\in\Gamma^2,(\underline{v}^2,v^2,\bar{v}^2)\in\Psi^2}}[s_{\theta^1+\theta^2},s_{\tau^1+\tau^2}],\{((1-((1-(\underline{\mu}^1)^q)^r+(1-(\underline{\mu}^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-((1-(\mu^1)^q)^r+(1-(\mu^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}},(1-((1-(\bar{\mu}^1)^q)^r+(1-(\bar{\mu}^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}})\},\\&\{(((\underline{v}^1)^{qr}+(\underline{v}^2)^{qr}-1)^{\frac{1}{qr}},((v^1)^{qr}+(v^2)^{qr}-1)^{\frac{1}{qr}},((\bar{v}^1)^{qr}+(\bar{v}^2)^{qr}-1)^{\frac{1}{qr}})\}\}\rangle_q;\end{aligned}\tag{10}$$

$$\begin{aligned}h_1\otimes h_2&=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{f^1\otimes f^2\}\rangle_q\\&=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{\bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1)\in\Gamma^1,(\underline{v}^1,v^1,\bar{v}^1)\in\Psi^1,\\(\underline{\mu}^2,\mu^2,\bar{\mu}^2)\in\Gamma^2,(\underline{v}^2,v^2,\bar{v}^2)\in\Psi^2}}[s_{\theta^1\times\theta^2},s_{\tau^1\times\tau^2}],\{(((\underline{\mu}^1)^{qr}+(\underline{\mu}^2)^{qr}-1)^{\frac{1}{qr}},((\mu^1)^{qr}+(\mu^2)^{qr}-1)^{\frac{1}{qr}},\\&((\bar{\mu}^1)^{qr}+(\bar{\mu}^2)^{qr}-1)^{\frac{1}{qr}})\},\{((1-((1-(\underline{v}^1)^q)^r+(1-(\underline{v}^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-((1-(v^1)^q)^r+(1-(v^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}},(1-((1-(\bar{v}^1)^q)^r+(1-(\bar{v}^2)^q)^r-1)^{\frac{1}{r}})^{\frac{1}{q}})\}\}\rangle_q;\end{aligned}\tag{11}$$

$$\begin{aligned}\lambda h&=\langle\{\lambda f_m,\ m=1,2,\cdots,k\}\rangle_q\\&=\langle\bigcup_{(\underline{\mu}_m,\mu_m,\bar{\mu}_m)\in\Gamma_m,(\underline{v}_m,v_m,\bar{v}_m)\in\Psi_m,}\{[s_{\lambda\theta_m},s_{\lambda\tau_m}],\{((1-(\lambda(1-\underline{\mu}_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-(\lambda(1-\mu_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},(1-(\lambda(1-\bar{\mu}_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}})\},\\&\{((\lambda\underline{v}_m^{qr}-(\lambda-1))^{\frac{1}{qr}},(\lambda v_m^{qr}-(\lambda-1))^{\frac{1}{qr}},(\lambda\bar{v}_m^{qr}-(\lambda-1))^{\frac{1}{qr}})\}\}\rangle_q;\end{aligned}\tag{12}$$

$$\begin{aligned}h^\lambda&=\langle\{f_m^\lambda,\ m=1,2,\cdots,k\}\rangle_q\\&=\langle\bigcup_{(\underline{\mu}_m,\mu_m,\bar{\mu}_m)\in\Gamma_m,(\underline{v}_m,v_m,\bar{v}_m)\in\Psi_m,}\{[s_{\theta_m^\lambda},s_{\tau_m^\lambda}],\{((\lambda\underline{\mu}_m^{qr}-(\lambda-1))^{\frac{1}{qr}},(\lambda\mu_m^{qr}-(\lambda-1))^{\frac{1}{qr}},\\&(\lambda\bar{\mu}_m^{qr}-(\lambda-1))^{\frac{1}{qr}})\},\{((1-(\lambda(1-\underline{v}_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-(\lambda(1-v_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},(1-(\lambda(1-\bar{v}_m^q)^r-(\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}})\}\}\rangle_q。\end{aligned}\tag{13}$$

定理 1　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X)，$i=1,2,3$，有

(1) $h_1\oplus h_2=h_2\oplus h_1$；

(2) $h_1\otimes h_2=h_2\otimes h_1$；

(3) $\lambda(h_1\oplus h_2)=\lambda h_1\oplus\lambda h_2$；

(4) $(h_1\otimes h_2)^\lambda=h_1^\lambda\otimes h_2^\lambda$；

(5) $(h_1\oplus h_2)\oplus h_3=h_1\oplus(h_2\oplus h_3)$；

(6) $(h_1\otimes h_2)\otimes h_3=h_1\otimes(h_2\otimes h_3)$。

证明　定理 1 中(1)和(2)由定义 10 易得,(3)和(4)的证明类似,这里给出(3)的证明。

$$\begin{aligned}\lambda(h_1\oplus h_2)&=\langle\lambda\bigcup_{(f^1\in h_1,f^2\in h_2}\{f^1\oplus f^2\}\rangle_q=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{\lambda(f^1\oplus f^2)\}\rangle_q\\&=\langle\bigcup_{f^1\in h_1,f^2\in h_2}\{\bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1)\in\Gamma^1,(\underline{v}^1,v^1,\bar{v}^1)\in\Psi^1,\\(\underline{\mu}^2,\mu^2,\bar{\mu}^2)\in\Gamma^2,(\underline{v}^2,v^2,\bar{v}^2)\in\Psi^2}}[s_{\lambda(\theta^1+\theta^2)},s_{\lambda(\tau^1+\tau^2)}],\\&\{((1-(\lambda(1-(\underline{\mu}^1)^q)^r+\lambda(1-(\underline{\mu}^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-(\lambda(1-(\mu^1)^q)^r+\lambda(1-(\mu^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}},\\&(1-(\lambda(1-(\bar{\mu}^1)^q)^r+\lambda(1-(\bar{\mu}^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}})\},\\&\{((\lambda(\underline{v}^1)^{qr}+\lambda(\underline{v}^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}},(\lambda(v^1)^{qr}+\lambda(v^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}},\\&(\lambda(\bar{v}^1)^{qr}+\lambda(\bar{v}^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}})\}\}\rangle_q,\end{aligned}$$

$$
\begin{aligned}
\lambda h_1 \oplus \lambda h_2 &= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2} \{ \lambda f^1 \oplus \lambda f^2 \} \rangle_q \\
&= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2} \{ \bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1) \in \Gamma^1,\ (\underline{v}^1,v^1,\bar{v}^1) \in \Psi^1, \\ (\underline{\mu}^2,\mu^2,\bar{\mu}^2) \in \Gamma^2,\ (\underline{v}^2,v^2,\bar{v}^2) \in \Psi^2}} [s_{\lambda(\theta^1+\theta^2)}, s_{\lambda(\tau^1+\tau^2)}], \\
&\quad \{ ((1-(\lambda(1-(\underline{\mu}^1)^q)^r+\lambda(1-(\underline{\mu}^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-(\lambda(1-(\mu^1)^q)^r+\lambda(1-(\mu^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-(\lambda(1-(\bar{\mu}^1)^q)^r+\lambda(1-(\bar{\mu}^2)^q)^r-(2\lambda-1))^{\frac{1}{r}})^{\frac{1}{q}}) \}, \\
&\quad \{ ((\lambda(\underline{v}^1)^{qr}+\lambda(\underline{v}^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}}, (\lambda(v^1)^{qr}+\lambda(v^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}}, \\
&\quad (\lambda(\bar{v}^1)^{qr}+\lambda(\bar{v}^2)^{qr}-(2\lambda-1))^{\frac{1}{qr}}) \} \} \rangle_q \\
&= \lambda(h_1 \oplus h_2)。
\end{aligned}
$$

定理 1 中(5)和(6)的证明相似,这里给出(5)的证明。

$$
\begin{aligned}
(h_1 \oplus h_2) \oplus h_3 &= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2,\, f^3 \in h_3} \{ (f^1 \oplus f^2) \oplus f^3 \} \rangle_q \\
&= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2,\, f^3 \in h_3} \{ \bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1) \in \Gamma^1,\ (\underline{v}^1,v^1,\bar{v}^1) \in \Psi^1, \\ (\underline{\mu}^2,\mu^2,\bar{\mu}^2) \in \Gamma^2,\ (\underline{v}^2,v^2,\bar{v}^2) \in \Psi^2 \\ (\underline{\mu}^3,\mu^3,\bar{\mu}^3) \in \Gamma^3,\ (\underline{v}^3,v^3,\bar{v}^3) \in \Psi^3}} [s_{\theta_1+\theta_2+\theta_3}, s_{\tau_1+\tau_2+\tau_3}], \\
&\quad \{ ((1-((1-(\underline{\mu}^1)^q)^r+(1-(\underline{\mu}^2)^q)^r+(1-(\underline{\mu}^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-((1-(\mu^1)^q)^r+(1-\mu^2)^{qr}+(1-(\mu^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-((1-(\bar{\mu}^1)^q)^r+(1-(\bar{\mu}^2)^q)^r+(1-(\bar{\mu}^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}) \}, \\
&\quad \{ (((\underline{v}^1)^{qr}+(\underline{v}^2)^{qr}+(\underline{v}^3)^{qr}-2)^{\frac{1}{qr}}, ((v^1)^{qr}+(v^2)^{qr}+(v^3)^{qr}-2)^{\frac{1}{qr}}, \\
&\quad ((\bar{v}^1)^{qr}+(\bar{v}^2)^{qr}+(\bar{v}^3)^{qr}-2)^{\frac{1}{qr}}) \} \} \rangle_q,
\end{aligned}
$$

$$
\begin{aligned}
h_1 \oplus (h_2 \oplus h_3) &= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2,\, f^3 \in h_3} \{ f^1 \oplus (f^2 \oplus f^3) \} \rangle_q \\
&= \langle \bigcup_{f^1 \in h_1,\, f^2 \in h_2,\, f^3 \in h_3} \{ \bigcup_{\substack{(\underline{\mu}^1,\mu^1,\bar{\mu}^1) \in \Gamma^1,\ (\underline{v}^1,v^1,\bar{v}^1) \in \Psi^1, \\ (\underline{\mu}^2,\mu^2,\bar{\mu}^2) \in \Gamma^2,\ (\underline{v}^2,v^2,\bar{v}^2) \in \Psi^2 \\ (\underline{\mu}^3,\mu^3,\bar{\mu}^3) \in \Gamma^3,\ (\underline{v}^3,v^3,\bar{v}^3) \in \Psi^3}} [s_{\theta_1+\theta_2+\theta_3}, s_{\tau_1+\tau_2+\tau_3}], \\
&\quad \{ ((1-((1-(\underline{\mu}^1)^q)^r+(1-(\underline{\mu}^2)^q)^r+(1-(\underline{\mu}^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-((1-(\mu^1)^q)^r+(1-\mu^2)^{qr}+(1-(\mu^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}, \\
&\quad (1-((1-(\bar{\mu}^1)^q)^r+(1-(\bar{\mu}^2)^q)+(1-(\bar{\mu}^3)^q)^r-2)^{\frac{1}{r}})^{\frac{1}{q}}) \}, \\
&\quad ((v^1)^{qr}+(v^2)^{qr}+(v^3)^{qr}-2)^{\frac{1}{qr}}, ((\bar{v}^1)^{qr}+(\bar{v}^2)^{qr}+(\bar{v}^3)^{qr}-2)^{\frac{1}{qr}}) \} \} \rangle_q \\
&= (h_1 \oplus h_2) \oplus h_3。
\end{aligned}
$$

3 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子

在多属性决策过程中,信息集成算子是综合评价的重要工具,并能有效集成决策者的评估信息。经典的犹豫模糊集算子(如犹豫模糊加权几何算子和犹豫模糊加权平均算子)[23]从数据的加权平均角度计算的。然而在实际的多属性决策问题中,各属性之间往往存在相互关系。Muirhead 平均(Muirhead mean, MM)算子描述所有决策属性之间的关系,当算子的参数向量取特定值时,MM 算子退化为其它常用的平均算子,具有很强的通用性和灵活性,广泛应用于信息集成和多属性决策。本文将 MM 算子引入到 2 型犹豫 q 阶三角不确定语言集中,基于 Schweizer-Sklar t 模和 t 余模定义 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子和加权 Muirhead 平均算子。2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead算子及其加权形式能够解决决策过程中出现的多层犹豫现象且评估属性之间相互关联的现实决策问题。

定义 11 设 $h_i=\langle f_1^i, f_2^i, \cdots, f_k^i \rangle_q = \langle ([s_{\theta_1^i}, s_{\tau_1^i}], \Gamma_1^i, \Psi_1^i), ([s_{\theta_2^i}, s_{\tau_2^i}], \Gamma_2^i, \Psi_2^i), \cdots, ([s_{\theta_{k^i}^i}, s_{\tau_{k^i}^i}], \Gamma_{k^i}^i, \Psi_{k^i}^i) \rangle_q \in$ T2HqTULE(X), $i=1,2,\cdots,n$, 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子(type-2 hesitant q-rung triangular uncertain linguistic Schweizer-Sklar-Muirhead mean, T2HqTULSSMM)定义为

$$\begin{aligned}&\mathrm{T2H}q\mathrm{TULSSMM}(h_1,h_2..,\cdots,h_n)\\&=\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}(h_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\\&=\left\langle\bigcup_{\substack{f^i\in h_i\\i=1,2,\cdots,n}}\left\{\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}(f^{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right\}\right\rangle_q,\end{aligned}\tag{14}$$

式中:$\boldsymbol{p}=(p_1,p_2,\cdots,p_j)$为算子的参数向量,$\vartheta=\{\vartheta(1),\vartheta(2),\cdots,\vartheta(j)\}$是$\{1,2,\cdots n\}$的一组排列,$S_n$ 为$\{1,2,\cdots n\}$所有排列的集合。

定理 2　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X), $i,j=1,2,\cdots,n$,则 T2HqTULSSMM 算子运算后的结果仍为 2 型犹豫 q 阶三角不确定语言元,即

$$\begin{aligned}&\mathrm{T2H}q\mathrm{TULSSMM}(h_1,h_2,\cdots,h_n)\\&=\left\langle\bigcup_{\substack{f^i\in h_i\\i=1,2,\cdots,n}}\left\{\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}(f^{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right\}\right\rangle_q\\&=\left\langle\bigcup_{\substack{f^i\in h_i\\i=1,2,\cdots,n}}\left\{\bigcup_{([s_{\theta^{\vartheta(j)}},s_{\tau^{\vartheta(j)}}],\Gamma^{\vartheta(j)},\Psi^{\vartheta(j)})\in f^{\vartheta(j)}}\left[s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(\theta^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}},s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(\tau^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right],\right.\right.\\&\left.\left.\{(L_\mu(\underline{\mu}^{\vartheta(j)}),L_\mu(\mu^{\vartheta(j)}),L_\mu(\bar{\mu}^{\vartheta(j)}))\},\{(L_v(\underline{v}^{\vartheta(j)}),L_v(v^{\vartheta(j)}),L_v(\bar{v}^{\vartheta(j)}))\}\right\}\right\rangle_q,\end{aligned}\tag{15}$$

式中:

$$(\underline{\mu}^{\vartheta(j)},\mu^{\vartheta(j)},\bar{\mu}^{\vartheta(j)})\in\Gamma^{\vartheta(j)},\quad(\underline{v}^{\vartheta(j)},v^{\vartheta(j)},\bar{v}^{\vartheta(j)})\in\Psi^{\vartheta(j)},$$

$$L_\mu(x)=\left(\frac{1}{\sum_{j=1}^{n}p_j}\left(1-\left(\frac{1}{n!}\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j(x^{qr}-1))\right)^{\frac{1}{r}}\right)^r\right)^{\frac{1}{r}}\right)^r-\left(\frac{1}{\sum_{j=1}^{n}p_j}-1\right)\right)^{\frac{1}{qr}},$$

$$L_v(x)=\left(1-\left(\frac{1}{\sum_{j=1}^{n}p_j}\left(1-\left(\frac{1}{n!}\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j(1-x^q)^r-1)\right)^{\frac{1}{r}}\right)^r\right)^{\frac{1}{r}}\right)^r-\left(\frac{1}{\sum_{j=1}^{n}p_j}-1\right)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}}。$$

证明

$$\begin{aligned}\bigotimes_{j=1}^{n}(f^{\vartheta(j)})^{p_j}=&\bigcup_{([s_{\theta^{\vartheta(j)}},s_{\tau^{\vartheta(j)}}],\Gamma^{\vartheta(j)},\Psi^{\vartheta(j)})\in f^{\vartheta(j)}}\left\{\left[s_{\prod_{j=1}^{n}(\theta^{\vartheta(j)})^{p_j}},s_{\prod_{j=1}^{n}(\tau^{\vartheta(j)})^{p_j}}\right],\left\{\left(\left(1+\sum_{j=1}^{n}(p_j((\underline{\mu}^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{qr}},\right.\right.\right.\\&\left.\left.\left(1+\sum_{j=1}^{n}(p_j((\mu^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{qr}},\left(1+\sum_{j=1}^{n}(p_j((\bar{\mu}^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{qr}}\right)\right\},\\&\left\{\left(\left(1-\left(1+\sum_{j=1}^{n}(p_j((1-(\underline{v}^{\vartheta(j)})^q)^r-1))\right)^{\frac{1}{r}}\right)^{\frac{1}{q}},\left(1-\left(1+\sum_{j=1}^{n}(p_j((1-(v^{\vartheta(j)})^q)^r-1))\right)^{\frac{1}{r}}\right)^{\frac{1}{q}},\right.\right.\\&\left.\left.\left.\left(1-\left(1+\sum_{j=1}^{n}(p_j((1-(\bar{v}^{\vartheta(j)})^q)^r-1))\right)^{\frac{1}{r}}\right)^{\frac{1}{q}}\right)\right\}\right\},\end{aligned}$$

$$\begin{aligned}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}(f^{\vartheta(j)})^{p_j}\right)=&\bigcup_{([s_{\theta^{\vartheta(j)}},s_{\tau^{\vartheta(j)}}],\Gamma^{\vartheta(j)},\Psi^{\vartheta(j)})\in f^{\vartheta(j)}}\left\{\left[s_{\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\theta_{\vartheta(j)}^{p_j}},s_{\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\tau_{\vartheta(j)}^{p_j}}\right],\right.\\&\left\{\left(\left(1-\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j((\underline{\mu}^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}},\right.\right.\\&\left(1-\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j((\mu^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}},\\&\left.\left.\left(1-\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j((\bar{\mu}^{\vartheta(j)})^{qr}-1))\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}}\right)\right\},\\&\left\{\left(\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}(p_j((1-(\underline{v}^{\vartheta(j)})^q)^r-1))\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{qr}},\right.\right.\end{aligned}$$

$$\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}\left(p_j\left(\left(1-\left(v^{\vartheta(j)}\right)^q\right)^r-1\right)\right)\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{qr}},$$

$$\left(\sum_{\vartheta(j)\in S_n}\left(1-\left(1+\sum_{j=1}^{n}\left(p_j\left(\left(1-\left(v^{\vartheta(j)}\right)^q\right)^r-1\right)\right)\right)^{\frac{1}{r}}\right)^r-(n!-1)\right)^{\frac{1}{qr}}\Big\}\Big\},$$

$$\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}\left(f^{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}=\left\langle\left\{\bigcup_{([s_{\theta^{\vartheta(j)}},s_{\tau^{\vartheta(j)}}],\Gamma^{\vartheta(j)},\Psi^{\vartheta(j)})\in f^{\vartheta(j)}}\left(\left[s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(\theta^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}},s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(\tau^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right],\right.\right.\right.$$

$$\left.\left.\left.\{(L_\mu(\underline{\mu}^{\vartheta(j)}),L_\mu(\mu^{\vartheta(j)}),L_\mu(\bar{\mu}^{\vartheta(j)}))\},\{(L_v(\underline{v}^{\vartheta(j)}),L_v(v^{\vartheta(j)}),L_v(\bar{v}^{\vartheta(j)}))\}\right)\right\}\right\rangle_q。$$

定理 3(单调性)　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q$, $h_i'=\langle f_1^{i\prime},f_2^{i\prime},\cdots,f_k^{i\prime}\rangle_q=\langle([s_{\theta_1^{i\prime}},s_{\tau_1^{i\prime}}],\Gamma_1^{i\prime},\Psi_1^{i\prime}),([s_{\theta_2^{i\prime}},s_{\tau_2^{i\prime}}],\Gamma_2^{i\prime},\Psi_2^{i\prime}),\cdots,([s_{\theta_{k^{i\prime}}^i},s_{\tau_{k^{i\prime}}^i}],\Gamma_k^{i\prime},\Psi_k^{i\prime})\rangle_q\in$ T2HqTULE(X), $i=1,2,\cdots,n$,如果 $h_i\geqslant_s h_i'$,有

$$\text{T2H}q\text{TULSSMM}(h_1,h_2,\cdots,h_n)\geqslant_s\text{T2H}q\text{TULSSMM}(h_1',h_2',\cdots,h_n')。$$

定理 4(置换不变性)　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_{k^i}^i\rangle_q$, $h_{(i)}=\langle f_{(1)}^i,f_{(2)}^i,\cdots,f_{(k^i)}^i\rangle_q\in$ T2HqTULE(X), $i=1,2,\cdots,n$,其中,$(h_{(1)},h_{(2)},\cdots,h_{(n)})$是$(h_1,h_2,\cdots,h_n)$的任意置换,有

$$\text{T2H}q\text{TULSSMM}(h_1,h_2,\cdots,h_n)=\text{T2H}q\text{TULSSMM}(h_{(1)},h_{(2)},\cdots,h_{(n)})。$$

定义 12　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X), $i=1,2,\cdots,n$, 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar 加权 Muirhead 平均算子(type-2 hesitant q-rung triangular uncertain linguistic Schweizer-Sklar weighted Muirhead mean, T2HqTULSSWMM)定义为

$$\text{T2H}q\text{TULSSWMM}(h_1,h_2..,\cdots,h_n)$$
$$=\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}\left(nw_{\vartheta(j)}h_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$
$$=\left\langle\bigcup_{\substack{f^i\in h_i\\ i=1,2,\cdots,n}}\left\{\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}\left(nw_{\vartheta(j)}f^{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right\}\right\rangle_q,\tag{16}$$

式中:$\boldsymbol{w}=(w_1,w_2,\cdots,w_j)^{\mathrm{T}}$ 为权重向量,且有 $\sum_{i=1}^{n}w_i=1$, $w_i\in[0,1]$, $\boldsymbol{p}=(p_1,p_2,\cdots,p_j)$为算子的参数向量,$\vartheta=\{\vartheta(1),\vartheta(2),\cdots,\vartheta(j)\}$是$\{1,2,\cdots j\}$的一组置换,$S_n$ 为$\{1,2,\cdots n\}$所有排列的集合。

定理 5　设 $h_i=\langle f_1^i,f_2^i,\cdots,f_k^i\rangle_q=\langle([s_{\theta_1^i},s_{\tau_1^i}],\Gamma_1^i,\Psi_1^i),([s_{\theta_2^i},s_{\tau_2^i}],\Gamma_2^i,\Psi_2^i),\cdots,([s_{\theta_{k^i}^i},s_{\tau_{k^i}^i}],\Gamma_{k^i}^i,\Psi_{k^i}^i)\rangle_q\in$ T2HqTULE(X), $i,j=1,2,\cdots,n$, T2HqTULSSWMM 算子具有单调性,则 T2HqTULSSWMM 算子运算后的结果仍为 2 型犹豫 q 阶三角不确定语言元,其表达式为

$$\text{T2H}q\text{TULSSWMM}(h_1,h_2,\cdots,h_n)$$
$$=\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}\left(nw_{\vartheta(j)}h_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$
$$=\left\langle\bigcup_{\substack{f^i\in h_i\\ i=1,2,\cdots,n}}\left\{\left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\left(\bigotimes_{j=1}^{n}\left(nw_{\vartheta(j)}f^{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right\}\right\rangle_q$$
$$=\left\langle\bigcup_{\substack{f^i\in h_i\\ i=1,2,\cdots,n}}\left\{\bigcup_{([s_{\theta^{\vartheta(j)}},s_{\tau^{\vartheta(j)}}],\Gamma^{\vartheta(j)},\Psi^{\vartheta(j)})\in f^{\vartheta(j)}}\left(\left[s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(nw_{\vartheta(j)}\theta^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}},s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}(nw_{\vartheta(j)}\tau^{\vartheta(j)})^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right],\right.\right.\right.$$
$$\left.\left.\left.\{(L_\mu^*(\underline{\mu}^{\vartheta(j)}),L_\mu^*(\mu^{\vartheta(j)}),L_\mu^*(\bar{\mu}^{\vartheta(j)}))\},\{(L_v^*(\underline{v}^{\vartheta(j)}),L_v^*(v^{\vartheta(j)}),L_v^*(\bar{v}^{\vartheta(j)}))\}\right)\right\}\right\rangle_q,\tag{17}$$

式中:

$$(\underline{\mu}^{\vartheta(j)},\mu^{\vartheta(j)},\bar{\mu}^{\vartheta(j)})\in\Gamma^{\vartheta(j)},\quad(\underline{v}^{\vartheta(j)},v^{\vartheta(j)},\bar{v}^{\vartheta(j)})\in\Psi^{\vartheta(j)},$$

$$L_\mu^*(x)=\left(\frac{1}{\sum_{j=1}^{n}p_j}\left(1-\left(\frac{1}{n!}\sum_{\vartheta(j)\in S_n}\left(1-\left(\sum_{j=1}^{n}\left(p_j\left(1-\left(nw_{\vartheta(j)}\left(1-x^q\right)^r-nw_{\vartheta(j)}+1\right)^{\frac{1}{r}}\right)^r\right.\right.\right.\right.\right.\right.$$

$$-p_j+1)-(n-1)\Big)^{\frac{1}{r}}\Big)^{r}\Big)^{\frac{1}{r}}\Big)^{r}-\Big(\frac{1}{\sum_{j=1}^{n}p_j}-1\Big)\Big)^{\frac{1}{qr}},$$

$$L_v^{*}=\Big(1-\Big(\frac{1}{\sum_{j=1}^{n}p_j}\Big(1-\Big(\frac{1}{n!}\sum_{\vartheta(j)\in S_n}\Big(1-\Big(\sum_{j=1}^{n}(p_j(1-nw_{\vartheta(j)}x^{qr}-nw_{\vartheta(j)}+1)^{\frac{1}{r}})^{r}-p_j+1)$$
$$-(n-1)\Big)^{\frac{1}{r}}\Big)^{r}\Big)^{\frac{1}{r}}\Big)^{r}-\Big(\frac{1}{\sum_{j=1}^{n}p_j}-1\Big)\Big)\Big)^{\frac{1}{r}}\Big)^{\frac{1}{q}}。$$

4　基于 2 型犹豫 q 阶三角不确定语言集的多属性决策方法

基于 2 型犹豫 q 阶三角不确定语言集的多属性决策问题,设 $B=\{B_1,B_2,\cdots,B_l\}$ 是候选方案的集合,$M=\{M_1,M_2,\cdots,M_n\}$ 是属性的集合。$\boldsymbol{w}$ 反映不同属性在决策过程中的重要程度。现有一组专家(u 位)将对这些候选方案评估,对候选方案排序或选出最符合期望的方案。第 k 位专家对候选方案 B_i 满足属性 M_j 的程度给予评估值 f_k^{ij},所有专家对候选方案 B_i 满足属性 M_j 给出的评估值用 2 型犹豫 q 阶三角不确定语言元 $h_{ij}=\{f_k^{ij},k=1,2,\cdots,u\}$ 表示,决策矩阵 $\boldsymbol{H}=(h_{ij})_{l\times n}$。

基于 2 型犹豫 q 阶三角不确定语言多属性决策方法的具体步骤:

步骤 1　利用 T2HqTULSSWMM 算子计算每个候选方案的最终评估值 $h_i=\text{T2H}q\text{TULSSWMM}(h_{i1},h_{i2},\cdots,h_{in})$。

步骤 2　计算各方案的得分函数 $S(h_i)$ 和精确函数 $H(h_i)$。

步骤 3　考虑决策者们的风险态度,确定 λ,根据每个候选方案的得分函数和式(2),得到可能度矩 $\boldsymbol{C}=(c_{ab})_{l\times l}$,由 $\boldsymbol{C}$ 得到第 i 个候选方案的总体优势度 $c_a=\sum_{b=1}^{l}c_{ab}$, $b=1,2,\cdots,l$。

步骤 4　根据方案的总体优势度进行排序,选出最理想的方案。

该方法运用 T2HqTULSSWMM 算子将专家对各个方案的各个属性的评估值进行了有效融合,计算出了各方案的最终评估值,仍为 2 型犹豫 q 阶三角不确定语言元。利用得分函数和三角模糊数的可能度计算公式计算出各方案的总体优势度,对方案进行排序或选出最理想的方案。

5　算例分析

5.1　应用实例

民用飞机备件保障能力评估是指针对民用飞机备件的订购、储备和供应等能力的实际状况与其期望能力之间的相匹配程度开展的评价工作,其目的是为确保在民用飞机运营过程中备件供应充足、及时,并且能够满足飞机维修和维护的需求。这种评估通常包括对备件库存水平、供应链管理、交付效率、备件可用性、交换率、成本效益等多方面的考量。为方便开展后续研究,评价指标应当具有覆盖性、独立性、实用性、系统性等特性,选取任务指标、应急保障指标、费效指标、服务指标构建民用飞机备件保障能力评估指标体系。

假设某民航公司为提高公司整体的备件保障能力,拟采用本文所提方法对下属 5 个子公司$\{B_1,B_2,B_3,B_4,B_5\}$开展备件保障能力评估,选择能力最优的子公司并将其备件保障模式进行推广。选择任务指标 M_1、应急保障指标 M_2、费效指标 M_3、服务指标 M_4 作为民用飞机备件保障能力评估的属性集。邀请专家对备件保障能力进行评估选。

给出语言术语集 $\bar{S}=\{S_0,S_1,S_2,S_3,S_4,S_5,S_6,S_7\}$, S_0、S_1、S_2、S_3、S_4、S_5、S_6、S_7 分别表示专家非常差、很差、差、一般、好、很好、非常好的评价。

专家们对 5 个子公司的备件保障能力进行评估后形成的决策表如表 1 所示。考虑到不同的属性指标对

评估的影响,取 $\boldsymbol{w}=(0.5,0.1,0.2,0.2)$。根据所有专家给出的评估值,当 $q\geqslant3$ 时满足 2 型犹豫 q 阶三角不确定语言集的定义要求,$q=3$, $\boldsymbol{p}=(1,1,1,1)$, $r=-1$。

表 1 2 型犹豫 q 阶三角不确定语言决策表
Table 1 Type-2 hesitant q-rung triangular uncertain linguistic decision table

子公司	M_1	M_2	M_3	M_4
B_1	$\langle([s_4,s_5],\{(0.7,0.8,0.9),(0.6,0.7,0.8)\},\{(0.1,0.2,0.3)\}),([s_3,s_5],\{(0.5,0.6,0.7)\},\{(0.2,0.3,0.4)\})\rangle_3$	$\langle([s_3,s_5],\{(0.7,0.8,0.9),(0.6,0.8,0.9)\},\{(0.2,0.3,0.4)\}),([s_4,s_5],\{(0.7,0.8,0.9)\},\{(0.1,0.2,0.3)\})\rangle_3$	$\langle([s_5,s_6],\{(0.7,0.8,0.9)\},\{(0.1,0.2,0.3)\}),([s_5,s_5],\{(0.6,0.8,0.9)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_2,s_3],\{(0.7,0.8,0.9)\},\{(0.2,0.3,0.4),(0.1,0.2,0.3)\}),([s_3,s_4],\{(0.5,0.6,0.7)\},\{(0.2,0.4,0.5)\})\rangle_3$
B_2	$\langle([s_4,s_5],\{(0.5,0.6,0.7),(0.5,0.6,0.8)\},\{(0.2,0.4,0.5)\}),([s_5,s_5],\{(0.6,0.7,0.8)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_3,s_5],\{(0.5,0.6,0.7)\},\{(0.2,0.4,0.5)\}),([s_2,s_3],\{(0.6,0.8,0.9)\},\{(0.1,0.2,0.3)\})\rangle_3$	$\langle([s_3,s_5],\{(0.7,0.8,0.9)\},\{(0.1,0.2,0.3),(0.1,0.2,0.4)\}),[s_4,s_5],\{(0.6,0.8,0.9)\},\{(0.1,0.2,0.3)\}\rangle_3$	$\langle([s_4,s_5],\{(0.6,0.7,0.8),(0.6,0.7,0.9)\},\{(0.2,0.3,0.4)\}),([s_5,s_5],\{(0.6,0.8,0.9)\},\{(0.2,0.3,0.4)\})\rangle_3$
B_3	$\langle([s_4,s_5],\{(0.6,0.7,0.8)\},\{(0.2,0.4,0.5)\}),([s_5,s_5],\{(0.6,0.8,0.9)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_3,s_4],\{(0.6,0.7,0.8)\},\{(0.2,0.3,0.4),(0.3,0.4,0.5)\}),([s_5,s_5],\{(0.7,0.8,0.9)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_4,s_4],\{(0.7,0.8,0.9),(0.6,0.8,0.9)\},\{(0.1,0.2,0.3)\}),[s_3,s_4],\{(0.7,0.8,0.9)\},\{(0.1,0.2,0.3)\}\rangle_3$	$\langle([s_4,s_5],\{(0.7,0.8,0.9),(0.6,0.7,0.9)\},\{(0.1,0.2,0.3)\}),([s_4,s_5],\{(0.7,0.8,0.9)\},\{(0.2,0.3,0.4)\})\rangle_3$
B_4	$\langle[s_5,s_6],\{(0.5,0.7,0.8)\},\{(0.3,0.4,0.5),(0.2,0.3,0.4)\}),([s_5,s_5],\{(0.6,0.8,0.9)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_2,s_2],\{(0.6,0.7,0.8),(0.6,0.7,0.9)\},\{(0.2,0.3,0.4)\}),([s_2,s_2],\{(0.5,0.6,0.7)\},\{(0.2,0.4,0.5)\})\rangle_3$	$\langle([s_2,s_3],\{(0.6,0.7,0.8),(0.5,0.6,0.7)\},\{(0.1,0.3,0.4)\}),([s_2,s_4],\{(0.5,0.7,0.8)\},\{(0.2,0.3,0.4)\})\rangle_3$	$\langle([s_3,s_5],\{(0.5,0.7,0.8)\},\{(0.2,0.3,0.4)\}),([s_4,s_5],\{(0.6,0.7,0.8)\},\{(0.1,0.2,0.3)\})\rangle_3$
B_5	$\langle([s_3,s_5],\{(0.5,0.6,0.7),(0.6,0.7,0.8)\},\{(0.3,0.4,0.5)\}),([s_4,s_5],\{(0.5,0.7,0.8)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_2,s_2],\{(0.6,0.7,0.8),(0.7,0.8,0.9)\},\{(0.2,0.3,0.4)\}).([s_2,s_3],\{(0.7,0.8,0.9)\},\{(0.1,0.2,0.3)\})\rangle_3$	$\langle([s_4,s_6],\{(0.6,0.8,0.9)\},\{(0.1,0.2,0.3)\}),([s_5,s_6],\{(0.6,0.8,0.9)\},\{(0.1,0.3,0.4)\})\rangle_3$	$\langle([s_4,s_5],\{(0.5,0.6,0.7)\},\{(0.2,0.4,0.5),(0.2,0.3,0.4)\}),([s_5,s_5],\{(0.5,0.7,0.8)\},\{(0.1,0.2,0.3)\})\rangle_3$

由表 1 可知,利用 T2HqTULSSWMM 算子对每个子公司的备件保障能力的评估信息集成,得到各子公司的备件保障能力的最终评估值,再利用定义 7 得分函数,5 个子公司的备件保障能力的得分函数分别为(2.0,2.4,2.8)、(1.9,2.3,2.8)、(2.2,2.6,3.0)、(1.5,1.8,2.2)和(1.8,2.2,2.6)。

设决策者们对风险规避持稳健态度,令 $\lambda=0.5$。利用式(2)计算可能度矩阵为

$$\boldsymbol{P}=\begin{bmatrix}0.50 & 0.57 & 0.30 & 1 & 0.7\\ 0.43 & 0.50 & 0.24 & 1 & 0.63\\ 0.70 & 0.76 & 0.50 & 1 & 0.90\\ 0 & 0 & 0 & 0.50 & 0\\ 0.29 & 0.37 & 0.10 & 1 & 0.50\end{bmatrix}。$$

最终得到每个子公司的备件保障能力的总体优势度分别为 3.08、2.80、3.86、0.50 和 2.26。5 个子公司的备件保障能力经过专家评估后的排序为 $B_3>B_1>B_2>B_5>B_4$,选择 B_3,并将 B_3 备件保障模式推广。

5.2 参数分析

当 T2HqTULSSWMM 算子中的 $\boldsymbol{p}$、$\boldsymbol{w}$ 发生变化,r、q、λ 不同时,得到子公司的备件保障能力的得分函数和总体优势度。由表 2 得知,当 $\boldsymbol{p}$ 分别取(1,0,0,0)、(1,1,0,0)、(1,1,1,0)和(1,1,1,1)时,5 个子公司的得分函数和排序出现变化,体现了决策属性相关联程度对决策结果的影响。在本文的案例中,民用飞机备件保障能力的 M_1、M_2、M_3 和 M_4 是相互联系、相互影响的,所以选取 $\boldsymbol{p}=(1,1,1,1)$是合理的。虽然最终排序结果不同,但总得到 B_3 最优,B_4 最劣,说明 T2HqTULSSWMM 算子具有一定的稳定性。

表 2　$\boldsymbol{p}$ 不同时子公司得分函数和排序表
Table 2　Score function and sorting table of subsidiary with different $\boldsymbol{p}$

参数 $\boldsymbol{p}$	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	排序情况
(1,0,0,0)	(2.7,3.2,3.7)	(2.7,3.2,3.9)	(2.9,3.4,3.9)	(2.4,3.0,3.6)	(2.5,3.1,3.7)	$B_3>B_2>B_1>B_5>B_4$
(1,1,0,0)	(2.3,2.8,3.3)	(2.4,2.9,3.4)	(2.5,3.0,3.5)	(2.0,2.4,2.9)	(2.3,2.8,3.3)	$B_3>B_2>B_5>B_1>B_4$
(1,1,1,0)	(2.1,2.4,2.9)	(2.1,2.5,3.0)	(2.2,2.6,3.0)	(1.6,1.9,2.3)	(2.0,2.5,2.9)	$B_3>B_2>B_5>B_1>B_4$
(1,1,1,1)	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$

由表 3 可知,当决策者在多属性决策过程中对属性权重修改后,各个子公司的备件保障能力的总体优势度出现较大变化,子公司的备件保障能力的最终排序也会随之改变。因此,对属性权重的赋值应该认真考虑、谨慎对待。

表 3　$\boldsymbol{w}$ 不同时子公司总体优势度和排序表
Table 3　Overall preference and sorting table of subsidiary with different $\boldsymbol{w}$

参数 $\boldsymbol{w}$	c_1	c_2	c_3	c_4	c_5	排序情况
(0.2,0.4,0.2,0.2)	2.96	2.99	4.03	0.61	1.90	$B_3>B_2>B_1>B_5>B_4$
(0.1,0.1,0.7,0.1)	3.12	2.79	3.88	0.69	2.01	$B_3>B_1>B_2>B_5>B_4$
(0.5,0.1,0.2,0.2)	3.08	2.80	3.86	0.50	2.26	$B_3>B_1>B_2>B_5>B_4$

由表 4—5 可知,B_3 最优,B_4 最劣,子公司的备件保障能力排序没有发生变化,说明 r 和 q 对最终结果的影响不大。

表 4　r 不同时子公司得分函数和排序表
Table 4　Score function and sorting table of subsidiary with different r

参数 r	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	排序情况
−1	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
−3	(2.0,2.4,2.8)	(1.8,2.3,2.7)	(2.2,2.6,3.1)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
−5	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.1)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
−7	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.1)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$

表 5　q 不同时子公司得分函数和排序表
Table 5　Score function and sorting table of subsidiary with different q

参数 q	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	排序情况
3	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
5	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
7	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$
9	(2.0,2.4,2.8)	(1.9,2.3,2.8)	(2.2,2.6,3.0)	(1.5,1.8,2.2)	(1.8,2.2,2.6)	$B_3>B_1>B_2>B_5>B_4$

由表 6 可知,在构造可能度矩阵时,改变风险因子 λ,即决策者们的风险态度出现变化时,总体优势度随之发生了变化,但不改变子公司的备件保障能力的排序。

表 6　λ 不同时子公司总体优势度和排序表
Table 6　Overall preference and sorting table of subsidiary with different λ

参数 λ	c_1	c_2	c_3	c_4	c_5	排序情况
0.1	3.06	2.85	3.84	0.50	2.25	$B_3>B_1>B_2>B_5>B_4$
0.3	3.07	2.83	3.85	0.50	2.26	$B_3>B_1>B_2>B_5>B_4$
0.5	3.08	2.80	3.86	0.50	2.26	$B_3>B_1>B_2>B_5>B_4$
0.9	3.10	2.75	3.89	0.50	2.26	$B_3>B_1>B_2>B_5>B_4$

6　结语

本文提出 2 型犹豫 q 阶三角不确定语言集的概念,定义基于 Schweizer-Sklar 范数的模糊元的运算并证明运算的性质。将 Muirhead 平均算子推广至 2 型犹豫 q 阶三角不确定语言集,提出 2 型犹豫 q 阶三角不确定语言 Schweizer-Sklar-Muirhead 平均算子及其加权形式,给出基于 2 型犹豫 q 阶三角不确定语言的 Schweizer-Sklar-Muirhead 平均算子多属性决策方法。该方法允许决策者给出更符合人们思维习惯的评估信息,应用于出

现多层犹豫现象和评估属性之间相互关联的现实决策问题。2 型犹豫 q 阶三角不确定语言集可以退化为多种犹豫模糊集的拓展形式,本文研究的信息集成算子和多属性决策方法退化为犹豫模糊集推广形式的信息集成算子和多属性决策算法。本文的结论可以进一步推广到 n 型犹豫模糊集。

参考文献:

[1] 徐泽水,赵华. 犹豫模糊集理论及应用[M]. 北京:科学出版社,2018.
XU Zeshui, ZHAO Hua. Hesitant fuzzy sets theory and applications[M]. Beijing: Science Press, 2018.

[2] WEI Guiwu. Hesitant fuzzy prioritized operators and their application to multiple attribute decision making[J]. Knowledge-Based Systems, 2012, 31:176-182.

[3] LIU Peide, WANG Peng. Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making[J]. International Journal of Intelligent Systems, 2018, 33(2):259-280.

[4] ALWAER H, CLEMENTS-CROOME D J. Key performance indicators (KPIs) and priority setting in using the multi-attribute approach for assessing sustainable intelligent buildings[J]. Building and Environment, 2010, 45(4):799-807.

[5] ZADEH L A. Fuzzy sets[J]. Information and Control, 1965, 8(3):338-353.

[6] BELLMAN R E, ZADEH L A. Decision-making in a fuzzy environment[J]. Management Science, 1970, 17(4):141-164.

[7] TRIANTAPHYLLOU E, LIN C T. Development and evaluation of five fuzzy multiattribute decision-making methods[J]. International Journal of Approximate Reasoning, 1996, 14(4):281-310.

[8] LIU Weishu, LIAO Huchang. A bibliometric analysis of fuzzy decision research during 1970—2015[J]. International Journal of Fuzzy Systems, 2017, 19:1-14.

[9] TORRA V. Hesitant fuzzy sets[J]. International Journal of Intelligent Systems, 2010, 25(6):529-539.

[10] ZHU Bin, XU Zeshui, XIA Meimei. Dual hesitant fuzzy sets[J]. Journal of Applied Mathematics, 2012, 5:1-13.

[11] 徐玥,刘练珍. q 阶犹豫模糊集及其在决策中的应用[J]. 模式识别与人工智能,2018,31(9):816-836.
XU Yue, LIU Lianzhen. q-rung hesitant fuzzy fets and its application to multi-criteria decision-making[J]. Pattern Recognition and Artificial Intelligence, 2018, 31(9):816-836.

[12] 任耀军,袁修久,黄林. q 阶三角犹豫模糊 BM 算子及其多属性决策应用[J]. 系统工程与电子技术,2022,44(1):181-191.
REN Yaojun, YUAN Xiujiu, HUANG Lin. q-rung hesitant triangular fuzzy BM operator and its application in multipe criteria decision making[J]. Systems Engineering and Electronics, 2022, 44(1):181-191.

[13] WANG Jianqiang, WANG Dandan, ZHANG Hongyu, et al. Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators[J]. Soft Computing, 2015, 19:389-405.

[14] LIU Peide, JIN Fang. Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making[J]. Information Sciences, 2012, 205:58-71.

[15] CHIAO Kuoping. Multi-criteria decision making with interval type 2 fuzzy Bonferroni mean[J]. Expert Systems with Applications, 2021, 176:114789.

[16] GARG H, NAZ S, ZIAA F, et al. A ranking method based on Muirhead mean operator for group decision making with complex interval-valued q-rung orthopair fuzzy numbers[J]. Soft Computing, 2021, 25(22):14001-14027.

[17] RANI P, MISHRA A R. Fermatean fuzzy Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection[J]. Expert Systems with Applications, 2021, 182:115267.

[18] HUANG Lin, YUAN Xiujiu, REN Yaojun. The q-rung orthopair hesitant fuzzy uncertain linguistic aggregation operators and their application in multi-attribute decision making[J]. IEEE Access, 2020, 8:187084-187113.

[19] MUIRHEAD R F. Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters[J]. Proceedings of the Edinburgh Mathematical Society, 1902, 21:144-162.

[20] VAN LAARHOVEN P J M, PEDRYCZ W. A fuzzy extension of Saaty's priority theory[J]. Fuzzy Sets and Systems, 1983, 11(1/3):229-241.

[21] XU Zeishui. Fuzzy harmonic mean operators[J]. International Journal of Intelligent Systems, 2009, 24(2):152-172.

[22] SCHWEIZER B. Associative functions and abstract semigroups[J]. Publicationes Mathematicae (Debrecen), 1963, 10:69-81.

[23] XIA Meimei, XU Zeishui. Hesitant fuzzy information aggregation in decision making[J]. International Journal of Approximate Reasoning, 2011, 52(3):395-407.

(编辑:陈丽萍)