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关于两类特征和的递推性质

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摘要:利用解析的方法、广义 Gauss 和的性质,在模素数 p 的条件下,得到两类特征和 $A_k(p)$ 和 $T_k(p)$ 的递推公式,本文结果用于解决对角同余方程解的个数问题。

关键词:特征和;广义 Gauss 和;二次剩余;递推公式

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Two kinds of character sums and their recurrence properties

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Abstract: Methods of analytic number theory, together with the properties of generalized Gauss sums modulo p are used to derive some third order recurrence formulas involving character sums $A_k(p)$ and $T_k(p)$ under certain conditions for an odd prime p. These results can be applied to solving problems regarding the number of solutions to the general diagonal equation.

Key words: character sums; generalized Gauss sums; quadratic residues; recurrence formulas

1　引言及主要结果

设任意整数 $q\geqslant 3$,χ 是任意模 q 的 Dirichlet 特征,定义广义高斯和 $G(m,k,\chi;q)$ 为

$$G(m,k,\chi;q)=\sum_{a=1}^{q}\chi(a)\mathrm{e}\left(\frac{ma^k}{q}\right),$$

式中 $\mathrm{e}(y)=\mathrm{e}^{2\pi \mathrm{i}y}$。若 $m=k=1$,则 $G(m,k,\chi;q)=\tau(\chi)$ 为经典高斯和。关于高斯和的性质以及相关理论参见文献[1]。

高斯和 $G(m,k,\chi;q)$ 是解析数论中经典的研究内容,许多学者都对其进行了广泛而深入的研究,并得到一系列重要的结论[2-8]。特别地,Weil[2] 给出了关于高斯和上界的重要结论:

$$G(m,k,\chi;p)\leqslant(k+1)\sqrt{p}。$$

张文鹏等[5] 给出三次广义高斯和的四次幂均值,即

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\left(\frac{a^3}{p}\right)\right|^4=5p^3-18p^2+20p+1+\frac{U^5}{p}+5pU-5U^3-4U^2+4U,$$

其中 $U=\sum_{a=1}^{p}\mathrm{e}\left(\frac{a^3}{p}\right)$ 是一个实数。

特征和的三次线性递推公式为[6],对于奇素数 p,任意的正整数 s 和 h,

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$$\sum_{\substack{a_1=1\\ a_1^h+a_2^h+\cdots+a_s^h\equiv 0 \pmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_s(a_s),$$

其中$\chi_i(i=1,2,\cdots,s)$是任意模 p 的 Dirichlet 特征。作为应用,利用以上特征和的计算结果,得到对角方程解的个数,即

$$x_1^h+x_2^h+\cdots+x_s^h\equiv 0 \pmod p,$$

其中所有的 $x_i(i=1,2,\cdots,s)$都是模 p 的二次剩余(或者二次非剩余)。本文将考虑更为一般的对角方程解的个数,即

$$x_1^h+x_2^h+\cdots+x_s^h\equiv z \pmod p,\quad z\in\mathbb{F}_p^*:\mathbb{F}_p\setminus\{0\}, \tag{1}$$

以及

$$x_1^h+x_2^h+\cdots+yx_s^h\equiv 0 \pmod p,\quad y\in\mathbb{F}_p^*:\mathbb{F}_p\setminus\{0\}, \tag{2}$$

其中 $x_i(i=1,2,\cdots,s)$均是模 p 的二次剩余(或者二次非剩余)。关于数论中对角方程解的个数问题的研究结果可参考文献[9-16]。

本文中,在模奇素数 p 的条件下,研究以下 2 种特征和的计算问题,对任意的正整数 s 和 h,

$$A_s(h,z,\chi_1,\cdots,\chi_s;p)=\sum_{\substack{a_1=1\\ a_1^h+a_2^h+\cdots+a_s^h\equiv z \pmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_s(a_s), \tag{3}$$

以及

$$T_s(h,\chi_1,\cdots,\chi_s;p)=\sum_{\substack{a_1=1\\ a_1^h+a_2^h+\cdots+ya_s^h\equiv 0 \pmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_s(a_s), \tag{4}$$

其中 $z\in\mathbb{F}_p^*:\mathbb{F}_p\setminus\{0\}$, $y\in\mathbb{F}_p^*$ 以及$\chi_i(i=1,2,\cdots,k)$是模 p 的 Dirichlet 特征。

利用模 p 简化剩余系的性质,特征和式(3)可变为 Jacobi 和,即

$$\begin{aligned}A_s(h,z,\chi_1,\cdots,\chi_s;p)&=\sum_{\substack{a_1=1\\ a_1^h+a_2^h+\cdots+a_s^h\equiv z \pmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_s(a_s)\\&=\sum_{\substack{a_1=1\\ a_1+a_2+\cdots+a_s\equiv z \pmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\chi_1^{\bar{h}}(a_1)\chi_2^{\bar{h}}(a_2)\cdots\chi_s^{\bar{h}}(a_s)。\end{aligned}$$

本文中,利用广义高斯和的性质,得到特征和式(3)、(4)的一些计算结果,主要给出 $A_s(3,z,\chi_2,\cdots,\chi_2;p)$和 $T_s(3,\chi_2,\cdots,\chi_2;p)$精确的计算公式,其中$\chi_2=\left(\frac{*}{p}\right)$表示模 p 的 Legendre 符号。为了方便起见,记 $A_s(3,z,\chi_2,\cdots,\chi_2;p)=A_s(p)$以及 $T_s(3,\chi_2,\cdots,\chi_2;p)=T_s(p)$,本文给出 $A_s(p)$和 $T_s(p)$的三阶递推公式。

定理 1 设 p 为奇素数,$(3,p-1)=1$,对于任意的整数 s 和 h,则

$$A_s(p)=\begin{cases}(-1)^{\frac{h(p-1)}{2}+1}p^{h-1}, & 若\ s=2h,\\ (-1)^{\frac{h(p-1)}{2}+p-1}p^h\left(\dfrac{z}{p}\right), & 若\ s=2h+1。\end{cases}$$

定理 2 设 p 为奇素数,$p\equiv 1 \pmod 6$,χ 是模 p 的任意三阶特征,则

$$A_1(p)=\left(\frac{z}{p}\right)\cdot(1+\psi(z)+\bar{\psi}(z));$$

$$\begin{aligned}A_3(p)=&7p\left(\frac{-z}{p}\right)+\frac{C}{p}\left(\frac{-z}{p}\right)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+6p\left(\frac{-z}{p}\right)(\psi(z)+\bar{\psi}(z))\\&+\frac{3C}{p}\left(\frac{-z}{p}\right)(\bar{\psi}(z)\tau^3(\chi_2\psi)+\psi(z)\tau^3(\overline{\chi_2\psi}));\end{aligned}$$

$$\begin{aligned}A_5(p)=&51p^2\left(\frac{z}{p}\right)+\frac{1}{p}\left(\frac{-z}{p}\right)(\bar{\psi}(z)\tau^6(\chi_2\psi)+\psi(z)\tau^6(\overline{\chi_2\psi}))+45p^2\left(\frac{z}{p}\right)(\psi(z)+\bar{\psi}(z))\\&+15C\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+5C\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\psi(z)+\tau^3(\overline{\chi_2\psi})\bar{\psi}(z))\end{aligned}$$

$$+30C\left(\frac{z}{p}\right)\left(\tau^3(\chi_2\psi)\bar{\psi}(z)+\tau^3(\overline{\chi_2\psi})\psi(z)\right)。$$

对任意的整数 $n\geqslant 3$，有递推公式

$$A_{2n+1}(p)=9\cdot\left(\frac{-1}{p}\right)p\cdot A_{2n-1}(p)+\left[6C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))-12p^2\right]\cdot A_{2n-3}(p)$$
$$+\left[6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))\right]\cdot A_{2n-5}(p)。$$

定理 3　设 p 为奇素数，$p\equiv 1\pmod 6$，若 z 是模 p 的三次剩余，2 是模 p 的三次剩余，则

$$A_1(p)=3\left(\frac{z}{p}\right),\quad A_3(p)=\left(\frac{-z}{p}\right)\cdot 27(p-4b^2),$$
$$A_5(p)=\left(\frac{z}{p}\right)243\cdot(p^2-6pb^2+3b^4),$$

对任意的整数 $n\geqslant 3$，有递推公式

$$A_{2n+1}(p)=(-1)^{(p-1)/2}9p\cdot A_{2n-1}(p)-162pb^2\cdot A_{2n-3}(p)$$
$$+(-1)^{(p-1)/2}729pb^4\cdot A_{2n-5}(p)。$$

由于特征和 $A_s(p)$ 以及特征和 $T_s(p)$ 关系密切，因此可以得到以下定理。

定理 4　设 p 为奇素数，$(3,p-1)=1$，对于任意的整数 s 以及 h，那么

$$T_s(p)=\begin{cases}\left(\frac{-1}{p}\right)\cdot(p-1)\cdot A_{s-1}, & \text{若 } s=2h,\\ 0, & \text{若 } s=2h+1。\end{cases}$$

注 1　首先，可以得到 $A_{2n+1}(p)\equiv 0\pmod{3^n p}$ 以及 $T_{2n}(p)\equiv 0\pmod{3^n p}$，其中 $n\geqslant 3$；其次，由于含有 $\tau(\psi)(\tau^2(\chi_2\psi)+2\tau(\chi_2)\tau(\overline{\chi_2\psi}))$，因此很难得到 $A_{2n}(p)$ 的精确表达式，这仍然是一个公开的问题；最后，利用本文结果，可以解决高次对角方程解的个数问题，利用解析的方法，将同余方程解的个数转化为特征和的计算问题。根据特征和 $A_s(p)$ 以及特征和 $T_s(p)$ 的计算结果，可以给出同余方程

$$x_1^h+x_2^h+\cdots+x_k^h\equiv z\pmod p$$

解的个数 $N(h,k,z,p)$，以及同余方程

$$x_1^h+x_2^h+\cdots+yx_k^h\equiv 0\pmod p$$

解的个数 $T(h,k,y,p)$，其中，$z\in\mathbb{F}_p^*:\mathbb{F}_p\backslash\{0\}$，$x_i(i=1,2,\cdots,k)$ 均是模 p 的二次剩余（或者二次非剩余）。

例 1　设 p 为奇素数，$(3,p-1)=1$，则

$$N(3,3,1,p)=\frac{1}{8}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a^3+b^3+c^3\equiv 1\ (\mathrm{mod}\ p)}}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)\left(1+\left(\frac{c}{p}\right)\right)$$
$$=\frac{1}{8}\left[(p-1)+(p-2)^2+3+3(-1)^{\frac{p-1}{2}}+p(-1)^{\frac{p-1}{2}}\right]。$$

例 2　设 p 为满足 $p\equiv 1\pmod{24}$ 的奇素数，那么

$$T(3,2,1,p)=\frac{1}{4}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a^3+b^3\equiv 0\ (\mathrm{mod}\ p)}}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)=\frac{3}{2}(p-1)。$$

2　若干引理

为了证明主要定理的，首先给出以下引理。

引理 1　设 p 为奇素数，$p\equiv 1\bmod 3$，$C=\tau(\chi_2)$，对任意满足 $(m,p)=1$ 的整数 m，记 $U(m,p)=\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)$，有恒等式

$$U^7(m,p)=9C^2U^5(m,p)+\left[6C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))-12p^2\right]\cdot U^3(m,p)$$
$$+\left[6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))\right]\cdot U(m,p),$$

其中$\left(\frac{*}{p}\right)=\chi_2$ 表示模 p 的 Legendre 符号。

证明 注意到 $p\equiv 1 \pmod 3$，假如 h 是立方数，那么 $x^3\equiv h \pmod p$ 有 3 个零点。假如 h 是非立方数，那么 $x^3\equiv h\pmod p$ 没有零点，因此

$$\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)=\sum_{\substack{a=1\\ h\text{是立方数}}}^{p-1}3\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{mh}{p}\right)。\tag{5}$$

由于 ψ 是三阶的 Dirichlet 特征，那么

$$1+\psi(a)+\psi^2(a)=\begin{cases}0, & \text{若 } a \text{ 是立方数},\\ 3, & \text{若 } a \text{ 是非立方数}。\end{cases}\tag{6}$$

将式(6)代入式(5)，注意到 $\psi^2=\bar{\psi}$，得

$$\begin{aligned}U(m,p)&=\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)=\sum_{a=1}^{p-1}(1+\psi(a)+\psi^2(a))\chi_2(a)\mathrm{e}\left(\frac{ma}{p}\right)\\&=\chi_2(m)\tau(\chi_2)+\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)+\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi})。\end{aligned}\tag{7}$$

由于 $\psi^2=\bar{\psi}$, $\chi_2^2=\chi_0$, $C^2=\tau^2(\chi_2)=\chi_2(-1)p$，并且 $\tau(\chi_2\psi)\tau(\overline{\chi_2\psi})=\chi_2(-1)p=C^2$。由式(7)，得

$$\begin{aligned}U^2(m,p)&=(\chi_2(m)\tau(\chi_2)+\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)+\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi}))^2\\&=\tau^2(\chi_2)+2\tau(\chi_2)(\bar{\psi}(m)\tau(\chi_2\psi)+\psi(m)\tau(\overline{\chi_2\psi}))+\psi(m)\tau^2(\chi_2\psi)+2\chi_2(-1)p+\bar{\psi}(m)\tau^2(\overline{\chi_2\psi})\\&=3C^2+2C(\bar{\psi}(m)\tau(\chi_2\psi)+\psi(m)\tau(\overline{\chi_2\psi}))+\psi(m)\tau^2(\chi_2\psi)+\bar{\psi}(m)\tau^2(\overline{\chi_2\psi})\\&=3C^2+\bar{\psi}(m)(2C\tau(\chi_2\psi)+\tau^2(\overline{\chi_2\psi}))+\psi(m)(2C\tau(\overline{\chi_2\psi})+\tau^2(\chi_2\psi)),\end{aligned}\tag{8}$$

有恒等式 $(x+y)^3=x^3+3x^2y+3xy^2+y^3$。由式(8)，得

$$\begin{aligned}&U^6(m,p)-9C^2U^4(m,p)+27C^4U^2(m,p)-27C^6\\&=(U^2(m,p)-3C^2)^3\\&=[\bar{\psi}(m)(2C\tau(\chi_2\psi)+\tau^2(\overline{\chi_2\psi}))+\psi(m)(2C\tau(\overline{\chi_2\psi})+\tau^2(\chi_2\psi))]^3\\&=(2C\tau(\chi_2\psi)+\tau^2(\overline{\chi_2\psi}))^3+(2C\tau(\overline{\chi_2\psi})+\tau^2(\chi_2\psi))^3\\&\quad+3(2C\tau(\chi_2\psi)+\tau^2(\overline{\chi_2\psi}))(2C\tau(\overline{\chi_2\psi})+\tau^2(\chi_2\psi))(U^2(m,p)-3C^2)\\&=14C^3[\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})]+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})+24C^2p^2\\&\quad+3[5p^2+2C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))]\cdot(U^2(m,p)-3C^2),\end{aligned}$$

由此可得

$$\begin{aligned}U^6(m,p)&=9C^2U^4(m,p)+[6C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))-12p^2]\cdot U^2(m,p)\\&+6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3[\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})]。\end{aligned}\tag{9}$$

式(9)两边同乘 $U(m,p)$，得

$$\begin{aligned}U^7(m,p)&=9C^2U^5(m,p)+[6C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))-12p^2]\cdot U^3(m,p)\\&+[6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))]\cdot U(m,p)。\end{aligned}$$

引理 1 证毕。

引理 2 设 p 为满足 $p\equiv 1\pmod 3$ 的奇素数，有恒等式

$$\sum_{m=1}^{p-1}U(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)=p\chi_2(z)\cdot(1+\psi(z)+\bar{\psi}(z)),$$

$$\begin{aligned}&\sum_{m=1}^{p-1}U^3(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)\\&=7p^2\chi_2(-z)+C\chi_2(-z)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+6p^2\chi_2(-z)(\psi(z)+\bar{\psi}(z))\end{aligned}$$

$$+3C\chi_2(-z)(\bar{\psi}(z)\tau^3(\chi_2\psi)+\psi(z)\tau^3(\overline{\chi_2\psi})),$$

$$\sum_{m=1}^{p-1} U^5(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=51p^3\left(\frac{z}{p}\right)+\left(\frac{-z}{p}\right)(\bar{\psi}(z)\tau^6(\chi_2\psi)+\psi(z)\tau^6(\overline{\chi_2\psi}))+45p^3\left(\frac{z}{p}\right)(\psi(z)+\bar{\psi}(z))$$

$$+15Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+5Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\psi(z)+\tau^3(\overline{\chi_2\psi})\bar{\psi}(z))$$

$$+30Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\bar{\psi}(z)+\tau^3(\overline{\chi_2\psi})\psi(z))。$$

证明 由高斯和的性质以及式(7),得

$$\sum_{m=1}^{p-1} U(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\sum_{m=1}^{p-1}(\chi_2(m)\tau(\chi_2)+\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)+\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi}))\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\tau(\chi_2)\chi_2(-z)\tau(\chi_2)+\tau(\chi_2\psi)\chi_2(-z)\psi(-z)\tau(\overline{\chi_2\psi})+\tau(\overline{\chi_2\psi})\chi_2(-z)\bar{\psi}(-z)\tau(\chi_2\psi)$$

$$=p\chi_2(z)+p\chi_2(z)\psi(z)+p\chi_2(z)\bar{\psi}(z)$$

$$=p\chi_2(z)\cdot(1+\psi(z)+\bar{\psi}(z))。$$

由于 $C^2=\tau^2(\chi_2)=\chi_2(-1)p$ 以及 $\tau(\chi_2\psi)\tau(\overline{\chi_2\psi})=\chi_2(-1)p=C^2$, $\chi_2^2=\chi_0$,则

$$\sum_{m=1}^{p-1} U^3(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\sum_{m=1}^{p-1}(\chi_2(m)\tau(\chi_2)+\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)+\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi}))^3\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\sum_{m=1}^{p-1}(7C^3\chi_2(m)+\chi_2(m)\tau^3(\chi_2\psi)+\chi_2(m)\tau^3(\overline{\chi_2\psi})+6C^2\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)$$

$$+6C^2\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi})+3C\chi_2(m)\psi(m)\tau^2(\chi_2\psi)+3C\chi_2(m)\bar{\psi}(m)\tau^2(\overline{\chi_2\psi}))\mathrm{e}\left(\frac{-mz}{p}\right)$$

$$=7p^2\chi_2(-z)+C\chi_2(-z)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+6p^2\chi_2(-z)(\psi(z)+\bar{\psi}(z))$$

$$+3C\chi_2(-z)(\bar{\psi}(z)\tau^3(\chi_2\psi)+\psi(z)\tau^3(\overline{\chi_2\psi}))。$$

类似地,由式(7)得

$$\sum_{m=1}^{p-1} U^5(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\sum_{m=1}^{p-1}(\chi_2(m)\tau(\chi_2)+\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)+\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi}))^5\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\sum_{m=1}^{p-1}[C^5\chi_2(m)+\chi_2(m)\psi(m)\tau^5(\chi_2\psi)+\chi_2(m)\bar{\psi}(m)\tau^5(\overline{\chi_2\psi})+5C^4\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)$$

$$+5C^4\chi_2(m)\psi(m)\tau(\overline{\chi_2\psi})+5C\chi_2(m)\bar{\psi}(m)\tau^4(\chi_2\psi)+5C\chi_2(m)\psi(m)\tau^4(\overline{\chi_2\psi})$$

$$+5\chi_2(m)\tau^4(\chi_2\psi)\tau(\overline{\chi_2\psi})+5\chi_2(m)\tau(\chi_2\psi)\tau^4(\overline{\chi_2\psi})+10C^3\chi_2(m)\psi(m)\tau^2(\chi_2\psi)$$

$$+10C^3\chi_2(m)\bar{\psi}(m)\tau^2(\overline{\chi_2\psi})+10C^2\chi_2(m)\tau^3(\chi_2\psi)+10C^2\chi_2(m)\tau^3(\overline{\chi_2\psi})$$

$$+10\chi_2(m)\tau^3(\chi_2\psi)\bar{\psi}(m)\tau^2(\overline{\chi_2\psi})+10\psi(m)\tau^2(\chi_2\psi)\chi_2(m)\tau^3(\overline{\chi_2\psi})$$

$$+20C^3\chi_2(m)\tau(\chi_2\psi)\tau(\overline{\chi_2\psi})+20C\chi_2(m)\psi(m)\tau^3(\chi_2\psi)\tau(\overline{\chi_2\psi})$$

$$+20C\chi_2(m)\bar{\psi}(m)\tau(\chi_2\psi)\tau^3(\overline{\chi_2\psi})+30C^2\chi_2(m)\bar{\psi}(m)\tau^2(\chi_2\psi)\tau(\overline{\chi_2\psi})$$

$$
\begin{aligned}
&+30C^2\chi_2(m)\psi(m)\tau(\chi_2\psi)\tau^2(\overline{\chi_2\psi})+30C\chi_2(m)\tau^2(\chi_2\psi)\tau^2(\overline{\chi_2\psi})\,]\,\mathrm{e}\left(\frac{-zm}{p}\right)\\
=&p^3\left(\frac{z}{p}\right)+\left(\frac{-z}{p}\right)\bar{\psi}(z)\tau^6(\chi_2\psi)+\left(\frac{-z}{p}\right)\psi(z)\tau^6(\overline{\chi_2\psi})+5p^3\left(\frac{z}{p}\right)\psi(z)+5p^3\left(\frac{z}{p}\right)\bar{\psi}(z)\\
&+5Cp\tau^3(\chi_2\psi)\left(\frac{z}{p}\right)\psi(z)+5Cp\tau^3(\overline{\chi_2\psi})\left(\frac{z}{p}\right)\bar{\psi}(z)+5Cp\tau^3(\chi_2\psi)\left(\frac{z}{p}\right)\\
&+5Cp\tau^3(\overline{\chi_2\psi})\left(\frac{z}{p}\right)+10Cp\tau^3(\chi_2\psi)\left(\frac{z}{p}\right)\bar{\psi}(z)+10Cp\tau^3(\overline{\chi_2\psi})\left(\frac{z}{p}\right)\psi(z)\\
&+10Cp\tau^3(\chi_2\psi)\left(\frac{z}{p}\right)+10Cp\tau^3(\overline{\chi_2\psi})\left(\frac{z}{p}\right)+10p^3\left(\frac{z}{p}\right)\psi(z)+10p^3\left(\frac{z}{p}\right)\bar{\psi}(z)\\
&+20p^3\left(\frac{z}{p}\right)+20Cp\tau^3(\chi_2\psi)\left(\frac{z}{p}\right)\bar{\psi}(z)+20Cp\tau^3(\overline{\chi_2\psi})\left(\frac{z}{p}\right)\psi(z)\\
&+30p^3\left(\frac{z}{p}\right)\psi(z)+30p^3\left(\frac{z}{p}\right)\bar{\psi}(z)+30p^3\left(\frac{z}{p}\right)\\
=&51p^3\left(\frac{z}{p}\right)+\left(\frac{-z}{p}\right)(\bar{\psi}(z)\tau^6(\chi_2\psi)+\psi(z)\tau^6(\overline{\chi_2\psi}))+45p^3\left(\frac{z}{p}\right)(\psi(z)+\bar{\psi}(z))\\
&+15Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+5Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\psi(z)+\tau^3(\overline{\chi_2\psi})\bar{\psi}(z))\\
&+30Cp\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\bar{\psi}(z)+\tau^3(\overline{\chi_2\psi})\psi(z)),
\end{aligned}
$$

引理 2 证毕。

引理 3[6] 设 p 为满足 $p\equiv1(\mathrm{mod}\ 6)$ 的奇素数，ψ 为任意模 p 的三阶特征，若 2 是模 p 的三次剩余，则

$$\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})=\frac{\tau^3(\chi_2)}{p}\cdot(d^2-2p),$$

其中 d 由 $4p=d^2+27b^2$ 和 $d\equiv1(\mathrm{mod}\ 3)$ 唯一确定的。

注 2 注意到 $\tau(\chi_2\psi)\tau(\overline{\chi_2\psi})=\tau^2(\chi_2)=\left(\frac{-1}{p}\right)p$，由引理 3，则

$$
\begin{aligned}
\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})&=(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))^2-2\tau^3(\chi_2\psi)\tau^3(\overline{\chi_2\psi})\\
&=\tau^2(\chi_2)\cdot(d^4+2p^2-4pd^2)。
\end{aligned}
\tag{10}
$$

3 定理 1、2、3 的证明

由于 $(3,p-1)=1$，当 a 通过模 p 的简化剩余系时，a^3 也通过模 p 的简化剩余系，因此，对任意满足 $(m,p)=1$ 的整数 m，由高斯和的性质得

$$
\begin{aligned}
U(m,p)&=\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)=\sum_{a=1}^{p-1}\left(\frac{a^3}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)\\
&=\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma}{p}\right)=\left(\frac{m}{p}\right)\tau(\chi_2)。
\end{aligned}
\tag{11}
$$

由 $A_s(p)$ 的定义、三角和的性质以及式(11)得

$$
\begin{aligned}
A_s(p)&=\sum_{\substack{a_1=1\\a_1^3+a_2^3+\cdots+a_s^3\equiv z}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{\substack{a_s=1\\(\mathrm{mod}\ p)}}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_s}{p}\right)\\
&=\frac{1}{p}\sum_{m=0}^{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)\right)^s\mathrm{e}\left(\frac{-zm}{p}\right)\\
&=\frac{1}{p}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)\right)^s\mathrm{e}\left(\frac{-zm}{p}\right)
\end{aligned}
$$

$$=\frac{1}{p}\sum_{m=1}^{p-1}U^{s}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\frac{1}{p}\tau^{s}(\chi_2)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)^{s}\mathrm{e}\left(\frac{-zm}{p}\right)。\tag{12}$$

对于满足$(z,m)=1$的整数m,当$s=2h$时,

$$A_s(p)=\frac{1}{p}\tau^{2h}(\chi_2)\sum_{m=1}^{p-1}\mathrm{e}\left(\frac{-zm}{p}\right)=-\frac{1}{p}\tau^{2h}(\chi_2)$$

$$=(-1)^{\frac{h(p-1)}{2}+1}p^{h-1},\tag{13}$$

当$s=2h+1$时,

$$A_s(p)=\frac{1}{p}\tau^{2h+1}(\chi_2)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\frac{1}{p}\tau^{2h+2}(\chi_2)\cdot\left(\frac{-1}{p}\right)\cdot\left(\frac{z}{p}\right)$$

$$=(-1)^{\frac{h(p-1)}{2}+p-1}p^{h}\left(\frac{z}{p}\right)。\tag{14}$$

定理 1 证毕。

下面证明定理 2。由

$$A_s(p)=\frac{1}{p}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\mathrm{e}\left(\frac{ma^3}{p}\right)\right)^{s}\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\frac{1}{p}\sum_{m=1}^{p-1}U^{s}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right),\tag{15}$$

再由式(15)以及引理 2,得

$$A_1(p)=\frac{1}{p}\sum_{m=1}^{p-1}U(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)=\chi_2(z)\cdot(1+\psi(z)+\bar{\psi}(z)),\tag{16}$$

$$A_3(p)=\frac{1}{p}\sum_{m=1}^{p-1}U^{3}(m,p)\mathrm{e}\left(\frac{-m}{p}\right)$$

$$=7p\chi_2(-z)+\frac{C}{p}\chi_2(-z)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+6p\chi_2(-z)(\psi(z)+\bar{\psi}(z))$$

$$+\frac{3C}{p}\chi_2(-z)(\bar{\psi}(z)\tau^3(\chi_2\psi)+\psi(z)\tau^3(\overline{\chi_2\psi})),\tag{17}$$

$$A_5(p)=\frac{1}{p}\sum_{m=1}^{p-1}U^{5}(m,p)\mathrm{e}\left(\frac{-m}{p}\right)$$

$$=51p^2\left(\frac{z}{p}\right)+\frac{1}{p}\left(\frac{-z}{p}\right)(\bar{\psi}(z)\tau^6(\chi_2\psi)+\psi(z)\tau^6(\overline{\chi_2\psi}))+45p^2\left(\frac{z}{p}\right)(\psi(z)+\bar{\psi}(z))$$

$$+15C\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))+5C\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\psi(z)+\tau^3(\overline{\chi_2\psi})\bar{\psi}(z))$$

$$+30C\left(\frac{z}{p}\right)(\tau^3(\chi_2\psi)\bar{\psi}(z)+\tau^3(\overline{\chi_2\psi})\psi(z))。\tag{18}$$

若$n\geqslant1$,由式(15)和引理 1 得

$$A_{2n+1}(p)=\frac{1}{p}\sum_{m=1}^{p-1}U^{2n+1}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\frac{1}{p}\sum_{m=1}^{p-1}U^{2n-6}(m,p)\cdot U^{7}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$=\frac{9C^2}{p}\sum_{m=1}^{p-1}U^{2n-1}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$+[6C(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi}))-12p^2]\frac{1}{p}\sum_{m=1}^{p-1}U^{2n-3}(m,p)\mathrm{e}\left(\frac{-zm}{p}\right)$$

$$
\begin{aligned}
&+\left[6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3\left(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})\right)\right]\\
&\times\frac{1}{p}\sum_{m=1}^{p-1}U^{2n-5}(m,p)\,\mathrm{e}\left(\frac{-zm}{p}\right)\\
=\;&9\cdot\left(\frac{-1}{p}\right)p\cdot A_{2n-1}(p)+\left[6C\left(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})\right)-12p^2\right]\cdot A_{2n-3}(p)\\
&+\left[6C^2p^2+\tau^6(\chi_2\psi)+\tau^6(\overline{\chi_2\psi})-4C^3\left(\tau^3(\chi_2\psi)+\tau^3(\overline{\chi_2\psi})\right)\right]\cdot A_{2n-5}(p)。
\end{aligned}
\tag{19}
$$

由式(16)—(19)，则定理2证毕。

由引理3和定理2，

$$A_1(p)=3\left(\frac{z}{p}\right),\quad A_3(p)=\left(\frac{-z}{p}\right)\cdot 27(p-4b^2),$$

$$A_5(p)=\left(\frac{z}{p}\right)243\cdot(p^2-6pb^2+3b^4),$$

对任意整数 $n\geqslant 1$，有递推公式

$$
\begin{aligned}
A_{2n+1}(p)=&(-1)^{(p-1)/2}9p\cdot A_{2n-1}(p)-162pb^2\cdot A_{2n-3}(p)\\
&+(-1)^{(p-1)/2}729pb^4\cdot A_{2n-5}(p)。
\end{aligned}
$$

定理3证毕。

4 定理4的证明

利用广义高斯和以及简化剩余系的性质，我们知道 $A_s(p)$ 以及 $T_s(p)$ 有着密切的联系，以下将完成定理4的证明。

$$
\begin{aligned}
T_s(p)=&\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+ya_s^3\equiv 0\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_s}{p}\right)\\
=&\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv -ya_s^3\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_s}{p}\right)\\
=&\sum_{\substack{a_1=1\\ -a_1^3a_s^3-a_2^3a_s^3-\cdots-a_{s-1}^3a_s^3\equiv -ya_s^3\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{-a_1a_s}{p}\right)\left(\frac{-a_2a_s}{p}\right)\cdots\left(\frac{-a_{s-1}a_s}{p}\right)\left(\frac{a_s}{p}\right)\\
=&\left(\frac{-1}{p}\right)^{s-1}\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv y\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_s}{p}\right)^{s}。
\end{aligned}
\tag{20}
$$

若 $s=2h$ 时，

$$
\begin{aligned}
T_s(p)=&\left(\frac{-1}{p}\right)\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv y\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_{s-1}}{p}\right)\\
=&\left(\frac{-1}{p}\right)\cdot(p-1)\cdot\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv y\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{s-1}=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_{s-1}}{p}\right)\\
=&\left(\frac{-1}{p}\right)\cdot(p-1)\cdot A_{s-1}。
\end{aligned}
\tag{21}
$$

若 $s=2h-1$ 时，

$$
\begin{aligned}
T_s(p)=&\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv y\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_s=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_{s-1}}{p}\right)\left(\frac{a_s}{p}\right)\\
=&\sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{s-1}^3\equiv y\ (\bmod p)}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{s-1}=1}^{p-1}\left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_{s-1}}{p}\right)\sum_{a_s=1}^{p-1}\left(\frac{a_s}{p}\right)\\
=&0。
\end{aligned}
\tag{22}
$$

由式(21)—(22),定理 4 证毕。

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