

# 运算图的电阻距离和基尔霍夫指数

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**摘要:**设 $G$ 为无向连通图, $S_T(G)$ 、 $Z_T(G)$ 和 $H_T(G)$ 是 $G$ 的运算图。利用电网络原理和组合方法,得到 $S_T(G)$ 、 $Z_T(G)$ 和 $H_T(G)$ 的基尔霍夫指数以及运算图的基尔霍夫指数与 $G$ 的基尔霍夫指数、度积与度和基尔霍夫指数、边数、顶点数之间的关系。

**关键词:**电阻距离;基尔霍夫指数;度积基尔霍夫指数;度和基尔霍夫指数

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## Resistance distance and Kirchhoff index in graph operation

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**Abstract:** Let  $G$  be an undirected connected graph,  $S_T(G)$ ,  $Z_T(G)$ ,  $H_T(G)$  are the operation graphs of  $G$ . By utilizing the principles of electrical networks and combinatorial methods, the Kirchhoff indices of  $S_T(G)$ ,  $Z_T(G)$ ,  $H_T(G)$  are obtained, as well as the relationships between the Kirchhoff indices of graph operations and the Kirchhoff index of  $G$ , multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index, the number of edges, and the number of vertices.

**Key words:** resistance distance; Kirchhoff index; multiplicative degree-Kirchhoff index; additive degree-Kirchhoff index

## 0 引言

设 $G=(V(G),E(G))$ 是一个无向连通图, $|V(G)|$ 为 $G$ 的阶数或顶点数, $|E(G)|$ 表示 $G$ 的边数, $d_i$ 表示 $G$ 中顶点 $i$ 的度。在连通图 $G$ 中,如果删除点 $v_k$ 使 $G$ 断开,则称 $v_k$ 为 $G$ 的割点<sup>[1]</sup>。

基尔霍夫指数是一类重要的拓扑指数。基尔霍夫指数定义为 $G$ 中所有无序顶点对之间的电阻距离之和<sup>[2]</sup>,即 $Kf(G)=\sum_{i<j}\Omega_G(v_i,v_j)$ 。Chen等<sup>[2]</sup>和Gutman等<sup>[3]</sup>定义度积基尔霍夫指数 $Kf^*(G)=\sum_{i<j}d_i d_j \Omega_G(v_i,v_j)$ ,并定义度和基尔霍夫指数 $Kf^+(G)=\sum_{i<j}(d_i+d_j)\Omega_G(v_i,v_j)$ 。

文献[4-7]分别研究细分顶点冠和细分边冠的电阻距离、细分图的冠点和冠边、广义细分点和细分边冠、普通冠和邻域冠等的电阻距离和基尔霍夫指数。超立方体网络由于自身结构和属性的独特性,成为网络科学和信息科学的热点研究对象之一。许多变形网络受到很大的关注,如折叠超立方体结构、超立方体结构的线图、细分图、全图等。殷剑宏等<sup>[8]</sup>研究超立方体网络的拉普拉斯谱,给出超立方体网络的基尔霍夫指数的解析表达式。进一步地,得到超立方体网络的线图、细分图、以及全图的基尔霍夫指数。随着化学图论的深入研究,运算图也在快速地发展。对图进行各种操作和处理后的图叫运算图,即对原图的顶点、边或结构进行某种操作生成的新图,或者计算图中顶点的度、边的权重等属性提取图中的关键特征和模式,运算图在优化算法性能和分析复杂系统分析领域也发挥着重要作用。

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除了二元运算外,一元运算(如剖分、三角剖分、顶点-面运算等)下的电阻距离和相关的拓扑指数也受到学者们的关注。Chen<sup>[9]</sup>确定图的剖分下的电阻距离;Gao等<sup>[10]</sup>得到正则图的线图、细分图、总图的基尔霍夫指数;Yang等<sup>[11]</sup>计算剖分图 $S(G)$ 、三角剖分图 $T(G)$ 的基尔霍夫指数;特别地,Wang等<sup>[12]</sup>确定正则图的三角剖分图 $R(G)$ 的基尔霍夫指数和下界;于越等<sup>[13]</sup>和Liu等<sup>[14]</sup>定义新的运算图 $H(G)$ 、 $R_T(G)$ ,并得到正则图的对应运算图的基尔霍夫指数。本文考虑3种一元运算图,并给出这些运算图的推广。

### 1 预备知识

$\Omega, \Omega^T, \Omega^S, \Omega^Z, \Omega^H$ 分别表示 $G, T(G), S_T(G), Z_T(G), H_T(G)$ 的电阻距离。

**定义 1** 在 $G$ 的边 $v_i v_j$ 添加一个新的顶点 $v_{ij}$ 得到运算图 $T(G)$ , $T(G)$ 是 $G$ 的三角剖分,如图1、2所示。

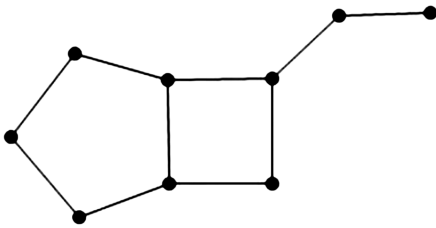


图1 图G  
Fig.1 Graph G

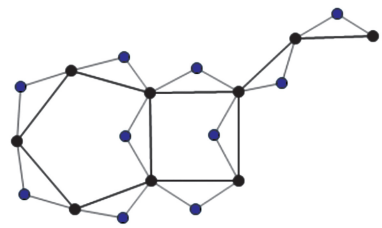


图2 运算图 $T(G)$   
Fig.2 Operation graph  $T(G)$

**定义 2** 在 $G$ 的每个顶点上插入2条长为1的新路,再结合 $T(G)$ 得到运算图 $S_T(G)$ ,如图3所示。

**定义 3** 在 $G$ 的每个顶点上插入一条长为2的新路,再结合 $T(G)$ 得到运算图 $Z_T(G)$ ,如图4所示。

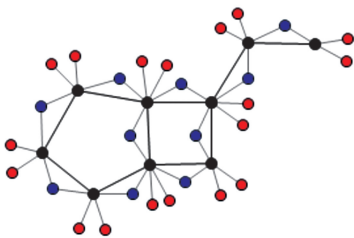


图3 运算图 $S_T(G)$   
Fig.3 Operation graph  $S_T(G)$

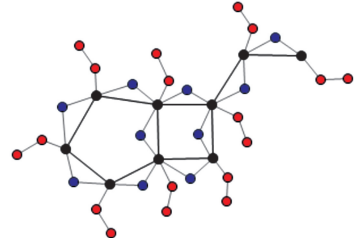


图4 运算图 $Z_T(G)$   
Fig.4 Operation graph  $Z_T(G)$

**定义 4** 设 $\Delta$ 为 $G$ 的最大度,在 $G$ 的每个顶点 $v_i$ 上加上 $\Delta - d_i$ 条长为1的路,再结合 $T(G)$ 得到运算图 $H_T(G)$ ,如图5所示。

**引理 1**<sup>[15]</sup> 设 $G$ 是连通图, $v_k$ 是 $G$ 的割点,如果 $v_i$ 和 $v_j$ 是属于 $G - v_k$ 中不同分支的顶点,则有

$$\Omega(v_i, v_j) = \Omega(v_i, v_k) + \Omega(v_k, v_j)。$$

**引理 2** (福斯特公式)<sup>[16]</sup> 设 $G$ 是一个具有 $n$ 个顶点的连通图,则 $G$ 中所有相邻顶点对之间的电阻距离之和等于 $n-1$ ,即

$$\sum_{v_k v_l \in E(G)} \Omega(v_k, v_l) = n - 1。$$

**引理 3**<sup>[12]</sup> 设 $G$ 是一个具有 $n$ 个顶点, $m$ 条边的连通图,则 $T(G)$ 的基尔霍夫指数为

$$Kf(T(G)) = \frac{2}{3} Kf(G) + \frac{1}{3} Kf^+(G) + \frac{1}{6} Kf^*(G) + \frac{3m^2 - n^2 + 2mn - 2m + n}{6}。$$

**引理 4**<sup>[9,12]</sup> 设 $G$ 是连通图, $T(G)$ 中各顶点对的电阻距离如下:

(1) 若 $v_i, v_j \in V(G)$ ,有 $\Omega^T(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j)$ ;

(2) 若 $v_i \in V(G), v_{pq} \in V'$ ,其中 $p, q = 1, 2, \dots, N, V' = \{v_{pq} | v_p v_q \in E(G)\}$ ,有

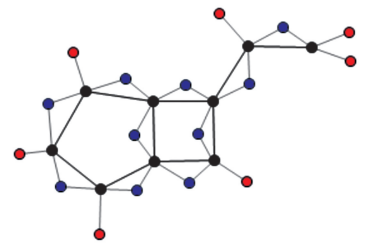


图5 运算图 $H_T(G)$   
Fig.5 Operation graph  $H_T(G)$

$$\Omega^T(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6};$$

(3) 若  $v_{pq}, v_{st} \in V'$ , 其中  $s, t = 1, 2, \dots, N$ , 有

$$\Omega^T(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_p, v_s) + \Omega(v_p, v_t) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6}.$$

## 2 运算图的电阻距离和基尔霍夫指数

### 2.1 $S_T(G)$ 的电阻距离和基尔霍夫指数

设  $V(G) = \{v_1, v_2, \dots, v_n\}$ ,  $v_k v_l \in E(G)$ ,  $v_{kl}$  表示  $S_T(G)$  中与  $v_k v_l$  相关联的顶点,  $v_i^1$  和  $v_i^2$  表示  $S_T(G)$  中与  $v_i$  相关联的 2 条边(关于  $v_i$  新添加到  $G$  的边)的顶点。令  $V' = \{v_{kl} | v_k v_l \in E(G)\}$ ,  $V'' = \{v_i^1, v_i^2 | v_i \in V(G)\}$ , 其中  $k, l = 1, 2, \dots, N$ , 则

$$V(T(G)) = V(G) \cup V',$$

$$V(S_T(G)) = V(G) \cup V' \cup V'' = V(T(G)) \cup V''.$$

由图 2、3 结构可知, 当  $v_i, v_j \in V(T(G))$  时,  $\Omega^T(v_i, v_j) = \Omega^S(v_i, v_j)$ 。

**引理 5** 设  $G$  是连通图,  $S_T(G)$  中各顶点对的电阻距离为

(1) 若  $v_i, v_j \in V(G)$ , 有  $\Omega^S(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j)$ ;

(2) 若  $v_i \in V(G)$ ,  $v_{pq} \in V'$ , 有  $\Omega^S(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}$ ;

(3) 若  $v_{pq}, v_{st} \in V'$ , 有

$$\Omega^S(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_p, v_s) + \Omega(v_p, v_t) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6};$$

(4) 若  $v_i \in V(G)$ ,  $v_j^a \in V'' (a \in \{1, 2\})$ , 有

$$\Omega^S(v_i, v_j^1) = \Omega^S(v_i, v_j^2) = 1 + \frac{2}{3} \Omega(v_i, v_j);$$

(5) 若  $v_{pq} \in V'$ ,  $v_i^a \in V'' (a \in \{1, 2\})$ , 有

$$\begin{aligned} \Omega^S(v_{pq}, v_i^1) &= \Omega^S(v_{pq}, v_i^2) = \Omega^S(v_{pq}, v_i) + \Omega^S(v_i, v_i^1) \\ &= \frac{3}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}; \end{aligned}$$

(6) 若  $v_i^a, v_j^b \in V'' (a \in \{1, 2\}, b \in \{1, 2\})$ , 则当  $i = j$  时,  $\Omega^S(v_i^1, v_i^2) = 2$ ; 当  $i \neq j$  时,  $\Omega^S(v_i^a, v_j^b) = 2 + \frac{2}{3} \Omega(v_i, v_j)$ 。

**证明** (1) 显然, 因为  $v_i, v_j \in V(T(G)) = V(G) \cup V'$ ,  $\Omega^T(v_i, v_j) = \Omega^S(v_i, v_j)$ , 所以由引理 4 知引理 5 (1) 结论成立。引理(2)、(3)类似可证。

(4) 因为

$$\Omega^S(v_i, v_i^1) = \Omega^S(v_i, v_i^2) = 1, \quad v_i \in V(G),$$

所以根据截点性质, 可得

$$\Omega^S(v_i, v_j^1) = \Omega^S(v_i, v_j^2) = \Omega^S(v_i, v_j) + \Omega^S(v_j, v_j^1) = \Omega^S(v_i, v_j) + 1.$$

由(1)知  $\Omega^S(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j)$ , 因此

$$\Omega^S(v_i, v_j^1) = \Omega^S(v_i, v_j^2) = 1 + \frac{2}{3} \Omega(v_i, v_j).$$

结论(5)、(6)类似可证。

引理5证毕。

**定理1**  $S_T(G)$ 的基尔霍夫指数为

$$\text{Kf}(S_T(G)) = 6\text{Kf}(G) + \text{Kf}^+(G) + \frac{1}{6}\text{Kf}^*(G) + \frac{m^2}{2} + \frac{11n^2}{2} + \frac{10mn}{3} - \frac{m}{3} - \frac{3n}{2}。$$

**证明** 因为  $V(S_T(G)) = V(T(G)) \cup V''$ ,所以

$$\begin{aligned} \text{Kf}(S_T(G)) &= \sum_{\{u,v\} \subset V(S_T(G))} \Omega^S(u,v) \\ &= \sum_{\{u,v\} \subset V(T(G))} \Omega^S(u,v) + \sum_{\{u,v\} \subset V''} \Omega^S(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^S(u,v) \\ &= \sum_{\{u,v\} \subset V(T(G))} \Omega^T(u,v) + \sum_{\{u,v\} \subset V''} \Omega^S(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^S(u,v) \\ &= \text{Kf}(T(G)) + \sum_{\{u,v\} \subset V''} \Omega^S(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^S(u,v)。 \end{aligned} \tag{1}$$

首先,由引理3,有

$$\text{Kf}(T(G)) = \frac{2}{3}\text{Kf}(G) + \frac{1}{3}\text{Kf}^+(G) + \frac{1}{6}\text{Kf}^*(G) + \frac{3m^2 - n^2 + 2mn - 2m + n}{6}。 \tag{2}$$

其次,当  $i \neq j$  时,  $\Omega^S(v_i^1, v_j^1) = \Omega^S(v_i^1, v_j^2) = \Omega^S(v_i^2, v_j^1) = \Omega^S(v_i^2, v_j^2)$ ,由引理5(6),得

$$\begin{aligned} \sum_{\{u,v\} \subset V''} \Omega^S(u,v) &= \sum_{i=1}^n \Omega^S(v_i^1, v_i^2) + 4 \sum_{i < j} \Omega^S(v_i^1, v_j^1) \\ &= \sum_{i=1}^n 2 + 4 \sum_{i < j} \left[ 2 + \frac{2}{3} \Omega(v_i, v_j) \right] \\ &= 2n + 4n(n-1) + \frac{8}{3}\text{Kf}(G)。 \end{aligned} \tag{3}$$

最后,考虑式(1),令

$$\sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^S(u,v) = \sum_{u \in V(G)} \sum_{v \in V''} \Omega^S(u,v) + \sum_{u \in V''} \sum_{v \in V''} \Omega^S(u,v)。$$

一方面,根据引理5(4),有

$$\begin{aligned} \sum_{u \in V(G)} \sum_{v \in V''} \Omega^S(u,v) &= \sum_{i=1}^n \sum_{j=1}^n [\Omega^S(v_i, v_j^1) + \Omega^S(v_i, v_j^2)] \\ &= 2 \sum_{i=1}^n \sum_{j=1}^n \left[ 1 + \frac{2}{3} \Omega(v_i, v_j) \right] \\ &= 2n^2 + \frac{8}{3}\text{Kf}(G); \end{aligned}$$

另一方面,根据引理5(5),有

$$\begin{aligned} \sum_{u \in V''} \sum_{v \in V''} \Omega^S(u,v) &= \sum_{v_p v_q \in E(G)} \sum_{i=1}^n [\Omega^S(v_{pq}, v_i^1) + \Omega^S(v_{pq}, v_i^2)] \\ &= 2 \sum_{v_p v_q \in E(G)} \sum_{i=1}^n \left[ \frac{3}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6} \right] \\ &= 3mn + 2 \sum_{v_p v_q \in E(G)} \sum_{i=1}^n \left[ \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} \right] - n \sum_{v_p v_q \in E(G)} \frac{\Omega(v_p, v_q)}{3} \\ &= 3mn + \frac{2}{3} \sum_{v_p v_q \in E(G)} \sum_{i=1}^n [\Omega(v_i, v_p) + \Omega(v_i, v_q)] - \frac{n}{3} \sum_{v_p v_q \in E(G)} \Omega(v_p, v_q), \end{aligned}$$

其中

$$\begin{aligned} \sum_{v_p v_q \in E(G)} \sum_{i=1}^n [\Omega(v_i, v_p) + \Omega(v_i, v_q)] &= \sum_{v_p v_q \in E(G)} [\Omega(v_p) + \Omega(v_q)] = \sum_{k=1}^n d_k \Omega(v_k) \\ &= \sum_{\{v_i, v_j\} \subset V(G)} (d_i + d_j) \Omega(v_i, v_j) = \text{Kf}^+(G)。 \end{aligned}$$

又由福斯特公式知  $\sum_{v_k v_l \in E(G)} \Omega(v_k, v_l) = n-1$ ,有

$$\sum_{u \in V'} \sum_{v \in V''} \Omega^S(u, v) = 3mn + \frac{2}{3} \text{Kf}^+(G) - \frac{n(n-1)}{3}.$$

故

$$\sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^S(u, v) = \frac{8}{3} \text{Kf}(G) + \frac{2}{3} \text{Kf}^+(G) + \frac{5n^2}{3} + 3mn + \frac{n}{3}. \tag{4}$$

综上,将式(2)、(3)、(4)代入式(1),得

$$\text{Kf}(S_T(G)) = 6\text{Kf}(G) + \text{Kf}^+(G) + \frac{1}{6} \text{Kf}^*(G) + \frac{m^2}{2} + \frac{11n^2}{2} + \frac{10mn}{3} - \frac{m}{3} - \frac{3n}{2}.$$

定理1证毕。

特别地,如果  $G$  是  $r$ -正则的,则  $m = \frac{nr}{2}$ ,  $\text{Kf}^+(G) = 2r\text{Kf}(G)$ ,  $\text{Kf}^*(G) = r^2\text{Kf}(G)$ 。 $S_T(G)$  的基尔霍夫指数可以用一种更简单的方式来表示。

**推论 1** 若  $G$  是  $r$ -正则的,则

$$\text{Kf}(S_T(G)) = \frac{(r+6)^2}{6} \text{Kf}(G) + \frac{n^2 r^2}{8} + \frac{5n^2 r}{3} + \frac{11n^2}{2} - \frac{nr}{6} - \frac{3n}{2}.$$

### 2.2 $Z_r(G)$ 的电阻距离与基尔霍夫指数

设  $V(G) = \{v_1, v_2, \dots, v_n\}$ ,  $v_k v_l \in E(G)$ ,  $v_{kl}$  表示  $Z_r(G)$  中与  $v_k v_l$  相关联的顶点,  $v_i^2$  表示  $Z_r(G)$  中与  $v_i$  相关联的边(关于  $v_i$  新添加到  $G$  的长为 2 的路)的端顶点,  $v_i^1$  表示同时与  $v_i$  和  $v_i^2$  相邻的顶点,令  $V' = \{v_{kl} \mid v_k v_l \in E(G)\}$ ,  $V'' = \{v_i^1, v_i^2 \mid v_i \in V(G)\}$ , 则

$$V(T(G)) = V(G) \cup V',$$

$$V(Z_r(G)) = V(G) \cup V' \cup V'' = V(T(G)) \cup V''.$$

由图 2、4 的结构可知,当  $v_i, v_j \in V(T(G))$  时,  $\Omega^T(v_i, v_j) = \Omega^Z(v_i, v_j)$ 。根据  $T(G)$  中的电阻距离与截点性质,  $Z_r(G)$  中的电阻距离由以下结果给出。

**引理 6** 设  $G$  是连通图,  $Z_r(G)$  中各顶点对的电阻距离为

(1) 若  $v_i, v_j \in V(G)$ , 有

$$\Omega^Z(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j);$$

(2) 若  $v_i \in V(G)$ ,  $v_{pq} \in V'$ , 有

$$\Omega^Z(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6};$$

(3) 若  $v_{pq}, v_{st} \in V'$ , 有

$$\Omega^Z(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_p, v_s) + \Omega(v_p, v_t) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6};$$

(4) 若  $v_i \in V(G)$ ,  $v_j^a \in V''(a \in \{1, 2\})$ , 有

$$\Omega^Z(v_i, v_j^1) = \Omega(v_i, v_j) + \Omega(v_j, v_j^1) = 1 + \frac{2}{3} \Omega(v_i, v_j)$$

$$\Omega^Z(v_i, v_j^2) = 2 + \frac{2}{3} \Omega(v_i, v_j);$$

(5) 若  $v_{pq} \in V'$ ,  $v_i^a \in V''(a \in \{1, 2\})$ , 有

$$\begin{aligned} \Omega^Z(v_{pq}, v_i^1) &= \Omega^Z(v_{pq}, v_i) + \Omega^Z(v_i, v_i^1) \\ &= \frac{3}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6} \end{aligned}$$

$$\Omega^Z(v_{pq}, v_i^2) = \frac{5}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6};$$

(6) 若  $v_i^a, v_j^b \in V''(a \in \{1, 2\}, b \in \{1, 2\})$ , 有

当  $i=j$ ,  $\Omega^Z(v_i^1, v_i^2) = 1$ ;

当  $i \neq j$ ,  $\Omega^Z(v_i^1, v_j^1) = 2 + \frac{2}{3}\Omega(v_i, v_j)$

$$\Omega^Z(v_i^1, v_j^2) = \Omega^Z(v_i^2, v_j^1) = 3 + \frac{2}{3}\Omega(v_i, v_j)$$

$$\Omega^Z(v_i^2, v_j^2) = 4 + \frac{2}{3}\Omega(v_i, v_j)。$$

定理 2  $Z_T(G)$  的基尔霍夫指数为

$$Kf(Z_T(G)) = 6Kf(G) + Kf^+(G) + \frac{1}{6}Kf^*(G) + \frac{m^2}{2} + \frac{17n^2}{2} + \frac{13mn}{3} - \frac{m}{3} - \frac{9n}{2}。$$

证明 因为  $V(Z_T(G)) = V(T(G)) \cup V''$ , 所以

$$\begin{aligned} Kf(Z_T(G)) &= \sum_{|u,v| \subset V(Z_T(G))} \Omega^Z(u,v) \\ &= \sum_{|u,v| \subset V(T(G))} \Omega^Z(u,v) + \sum_{|u,v| \subset V''} \Omega^Z(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^Z(u,v) \\ &= \sum_{|u,v| \subset V(T(G))} \Omega^T(u,v) + \sum_{|u,v| \subset V''} \Omega^Z(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^Z(u,v) \\ &= Kf(T(G)) + \sum_{|u,v| \subset V''} \Omega^Z(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^Z(u,v)。 \end{aligned} \tag{5}$$

首先,由引理 6(6),得

$$\begin{aligned} \sum_{|u,v| \subset V''} \Omega^Z(u,v) &= \sum_{i=1}^n \Omega^Z(v_i^1, v_i^2) + \sum_{i < j} \Omega^Z(v_i^1, v_j^1) + 2 \sum_{i < j} \Omega^Z(v_i^1, v_j^2) + \sum_{i < j} \Omega^Z(v_i^2, v_j^2) \\ &= n + \sum_{i < j} \left[ 2 + \frac{2}{3}\Omega(v_i, v_j) \right] + 2 \sum_{i < j} \left[ 3 + \frac{2}{3}\Omega(v_i, v_j) \right] + \sum_{i < j} \left[ 4 + \frac{2}{3}\Omega(v_i, v_j) \right] \\ &= n + 6n(n-1) + \frac{8}{3}Kf(G)。 \end{aligned} \tag{6}$$

其次,考虑  $\sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^Z(u,v)$ 。设

$$\sum_{u \in V(T(G))} \sum_{v \in V''} \Omega^Z(u,v) = \sum_{u \in V(G)} \sum_{v \in V''} \Omega^Z(u,v) + \sum_{u \in V'} \sum_{v \in V''} \Omega^Z(u,v)。$$

一方面,由引理 6(4),得

$$\begin{aligned} \sum_{u \in V(G)} \sum_{v \in V''} \Omega^Z(u,v) &= \sum_{i=1}^n \sum_{j=1}^n [\Omega^Z(v_i, v_j^1) + \Omega^Z(v_i, v_j^2)] \\ &= \sum_{i=1}^n \sum_{j=1}^n \left[ 1 + \frac{2}{3}\Omega(v_i, v_j) \right] + \sum_{i=1}^n \sum_{j=1}^n \left[ 2 + \frac{2}{3}\Omega(v_i, v_j) \right] \\ &= 3n^2 + \frac{8}{3}Kf(G); \end{aligned}$$

另一方面,由引理 6(5),得

$$\begin{aligned} \sum_{u \in V'} \sum_{v \in V''} \Omega^Z(u,v) &= \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n [\Omega^Z(v_{pq}, v_i^1) + \Omega^Z(v_{pq}, v_i^2)] \\ &= \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n \left[ \frac{3}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6} \right] \\ &\quad + \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n \left[ \frac{5}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6} \right] \\ &= 4mn + 2 \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n \left[ \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} \right] - n \sum_{v_p, v_q \in E(G)} \frac{\Omega(v_p, v_q)}{3} \\ &= 4mn + \frac{2}{3} \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n [\Omega(v_i, v_p) + \Omega(v_i, v_q)] - \frac{n}{3} \sum_{v_p, v_q \in E(G)} \Omega(v_p, v_q) \end{aligned}$$

$$= 4mn + \frac{2}{3} \text{Kf}^+(G) - \frac{n(n-1)}{3},$$

所以

$$\sum_{u \in V(T(G))} \sum_{v \in V'} \Omega^Z(u, v) = \frac{8}{3} \text{Kf}(G) + \frac{2}{3} \text{Kf}^+(G) + \frac{8n^2}{3} + 4mn + \frac{n}{3}. \tag{7}$$

综上,将式 (2)、(6)、(7) 代入式 (5),则有

$$\text{Kf}(Z_T(G)) = 6\text{Kf}(G) + \text{Kf}^+(G) + \frac{1}{6} \text{Kf}^*(G) + \frac{m^2}{2} + \frac{17n^2}{2} + \frac{13mn}{3} - \frac{m}{3} - \frac{9n}{2}.$$

定理 2 证毕。

**推论 2** 若  $G$  是  $r$ -正则的,则有

$$\text{Kf}(Z_T(G)) = \frac{(r+6)^2}{6} \text{Kf}(G) + \frac{n^2 r^2}{8} + \frac{13n^2 r}{6} + \frac{17n^2}{2} - \frac{nr}{6} - \frac{9n}{2}.$$

### 2.3 $H_T(G)$ 的电阻距离与基尔霍夫指数

设  $V(G) = \{v_1, v_2, \dots, v_n\}$ ,  $d_i$  是  $v_i$  的度数,  $\Delta$  表示  $G$  的最大度。对于每个顶点  $v_i \in V(G)$ , 如果  $\Delta - d_i > 0$ ,  $v_i^1, v_i^2, \dots, v_i^{\Delta-d_i}$  表示  $H_T(G)$  中与  $v_i$  相连的悬挂边的顶点, 对顶点  $v_{kl}$  的表示方法如  $S_T(G)$ 。

令  $V' = \{v_{kl} | v_k v_l \in E(G)\}$ ,  $V^* = \{v_1^1, v_1^2, \dots, v_1^{\Delta-d_1}, v_2^1, v_2^2, \dots, v_2^{\Delta-d_2}, \dots, v_n^1, v_n^2, \dots, v_n^{\Delta-d_n}\}$ ,

则

$$V(T(G)) = V(G) \cup V',$$

$$V(H_T(G)) = V(G) \cup V' \cup V^* = V(T(G)) \cup V^*.$$

由图 2、5 的结构可知, 当  $v_i, v_j \in V(T(G))$  时,  $\Omega^T(v_i, v_j) = \Omega^H(v_i, v_j)$ 。根据  $T(G)$  中的电阻距离与截点性质,  $H_T(G)$  中的电阻距离由以下结果给出。

**引理 7** 设  $G$  是一个连通图,  $H_T(G)$  中各顶点对的电阻距离如下:

(1) 若  $v_i, v_j \in V(G)$ , 有  $\Omega^H(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j)$ ;

(2) 若  $v_i \in V(G)$ ,  $v_{pq} \in V'$ , 有  $\Omega^H(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}$ ;

(3) 若  $v_{pq}, v_{st} \in V'$ , 有

$$\Omega^H(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_p, v_s) + \Omega(v_p, v_t) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6};$$

(4) 若  $v_i \in V(G)$ ,  $v_j^f \in V^*$  ( $f \in \{1, 2, \dots, \Delta - d_j\}$ ), 有

$$\Omega^H(v_i, v_j^1) = \Omega^H(v_i, v_j^2) = \dots = \Omega^H(v_i, v_j^{\Delta-d_j}) = 1 + \frac{2}{3} \Omega(v_i, v_j);$$

(5) 若  $v_{pq} \in V'$ ,  $v_i^g \in V^*$  ( $g \in \{1, 2, \dots, \Delta - d_i\}$ ), 有

$$\Omega^H(v_{pq}, v_i^1) = \Omega^H(v_{pq}, v_i^2) = \dots = \Omega^H(v_{pq}, v_i^{\Delta-d_i}) = \frac{3}{2} + \frac{\Omega(v_i, v_p)}{3} + \frac{\Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6};$$

(6) 若  $v_i^f, v_j^g \in V^*$  ( $f \in \{1, 2, \dots, \Delta - d_i\}$ ,  $g \in \{1, 2, \dots, \Delta - d_j\}$ ), 则当  $i=j$  时,  $\Omega^H(v_i^f, v_i^g) = 2$ ; 当  $i \neq j$  时,

$$\Omega^H(v_i^f, v_j^g) = 2 + \frac{2}{3} \Omega(v_i, v_j).$$

**定理 3**  $H_T(G)$  的基尔霍夫指数为

$$\begin{aligned} \text{Kf}(H_T(G)) = & \left( \left( \Delta + \frac{2}{3} \right)^2 + \frac{2}{9} \right) \text{Kf}(G) - \frac{2}{3} \left( \Delta + \frac{m}{n} + \frac{1}{2} \right) \text{Kf}^+(G) - \frac{5}{6} \text{Kf}^*(G) \\ & + (n\Delta)^2 - \frac{5n\Delta}{6} (3m-n+1) + \frac{3m^2}{2} - \frac{n^2}{6} - \frac{4mn}{3} + \frac{4m}{3} + \frac{n}{6}. \end{aligned}$$

**证明** 因为  $V(H_T(G)) = V(T(G)) \cup V^*$ , 所以

$$\text{Kf}(H_T(G)) = \sum_{\{u, v\} \subset V(H_T(G))} \Omega^H(u, v)$$

$$\begin{aligned}
 &= \sum_{\{u,v\} \subset V(T(G))} \Omega^H(u,v) + \sum_{\{u,v\} \subset V^*} \Omega^H(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V^*} \Omega^H(u,v) \\
 &= \sum_{\{u,v\} \subset V(T(G))} \Omega^T(u,v) + \sum_{\{u,v\} \subset V^*} \Omega^H(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V^*} \Omega^H(u,v) \\
 &= \text{Kf}(T(G)) + \sum_{\{u,v\} \subset V^*} \Omega^H(u,v) + \sum_{u \in V(T(G))} \sum_{v \in V^*} \Omega^H(u,v) \quad (8)
 \end{aligned}$$

首先,根据引理7(6),有

$$\begin{aligned}
 \sum_{\{u,v\} \subset V^*} \Omega^H(u,v) &= \sum_{i=1}^n \sum_{1 \leq k < l \leq \Delta-d_i} \Omega^H(v_i^k, v_i^l) + \sum_{1 \leq i < j \leq n} \sum_{k=1}^{\Delta-d_i} \sum_{l=1}^{\Delta-d_j} \Omega^H(v_i^k, v_j^l) \\
 &= \sum_{i=1}^n \sum_{1 \leq k < l \leq \Delta-d_i} 2 + \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) [2 + \Omega(v_i, v_j)] \\
 &= \sum_{i=1}^n 2 \times \binom{\Delta-d_i}{2} + \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) [2 + \Omega(v_i, v_j)] \quad (9)
 \end{aligned}$$

式(9)的第1项可化为

$$\begin{aligned}
 \sum_{i=1}^n 2 \times \binom{\Delta-d_i}{2} &= \sum_{i=1}^n (\Delta-d_i)(\Delta-d_i-1) = \sum_{i=1}^n [\Delta^2 - (2d_i+1)\Delta + d_i(d_i+1)] \\
 &= n\Delta^2 - (4m+n)\Delta + \sum_{i=1}^n d_i^2 + 2m \quad \circ
 \end{aligned}$$

式(9)的第2项可化为

$$\begin{aligned}
 &\sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) [2 + \Omega(v_i, v_j)] \\
 &= 2 \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) + \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) \Omega(v_i, v_j) \\
 &= \sum_{i=1}^n \sum_{j=1}^n (\Delta-d_i)(\Delta-d_j) - \sum_{i=1}^n (\Delta-d_i)^2 + \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) \Omega(v_i, v_j) \\
 &= \left[ \sum_{i=1}^n (\Delta-d_i) \right] \left[ \sum_{j=1}^n (\Delta-d_j) \right] - \sum_{i=1}^n (\Delta^2 - 2d_i\Delta + d_i^2) \\
 &\quad + \sum_{1 \leq i < j \leq n} (\Delta-d_i)(\Delta-d_j) \Omega(v_i, v_j) \\
 &= (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^n d_i^2 + \sum_{1 \leq i < j \leq n} [\Delta^2 - (d_i+d_j)\Delta - d_i d_j] \Omega(v_i, v_j) \\
 &= (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^n d_i^2 + \Delta^2 \text{Kf}(G) - \Delta \text{Kf}^+(G) - \text{Kf}^*(G) \quad \circ
 \end{aligned}$$

因此

$$\begin{aligned}
 &\sum_{\{u,v\} \subset V^*} \Omega^H(u,v) \\
 &= n\Delta^2 - (4m+n)\Delta + \sum_{i=1}^n d_i^2 + 2m + (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^n d_i^2 \\
 &\quad + \Delta^2 \text{Kf}(G) - \Delta \text{Kf}^+(G) - \text{Kf}^*(G) \\
 &= \Delta^2 \text{Kf}(G) - \Delta \text{Kf}^+(G) - \text{Kf}^*(G) + n^2 \Delta^2 - n\Delta(4m+1) + 2m(2m+1) \quad \circ \quad (10)
 \end{aligned}$$

考虑式(8)的第3项  $\sum_{u \in V(T(G))} \sum_{v \in V^*} \Omega^H(u,v)$ , 令

$$\sum_{u \in V(T(G))} \sum_{v \in V^*} \Omega^H(u,v) = \sum_{u \in V(G)} \sum_{v \in V^*} \Omega^H(u,v) + \sum_{u \in V^*} \sum_{v \in V^*} \Omega^H(u,v) \quad \circ$$

一方面,根据引理7(4),有

$$\begin{aligned}
 \sum_{u \in V(G)} \sum_{v \in V^*} \Omega^H(u,v) &= \sum_{i=1}^n \sum_{j=1}^n (\Delta-d_i) \left[ 1 + \frac{2}{3} \Omega(v_i, v_j) \right] \\
 &= \sum_{j=1}^n \left[ \sum_{i=1}^n (\Delta-d_i) \right] + \frac{2}{3} \Delta \sum_{i=1}^n \sum_{j=1}^n \Omega(v_i, v_j) - \frac{2}{3} \sum_{j=1}^n \sum_{i=1}^n d_i \Omega(v_i, v_j)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n \left( n\Delta - \sum_{i=1}^n d_i \right) + \frac{4}{3} \Delta \text{Kf}(G) - \frac{2}{3} \sum_{j=1}^n \sum_{i=1}^n d_i \Omega(v_i, v_j) \\
&= \sum_{j=1}^n (n\Delta - 2m) + \frac{4}{3} \Delta \text{Kf}(G) - \frac{1}{3} \sum_{j=1}^n \sum_{i=1}^n (d_i + d_j) \Omega(v_i, v_j) \\
&= n^2 \Delta - 2mn + \frac{4}{3} \Delta \text{Kf}(G) - \frac{2}{3} \text{Kf}^+(G); \tag{11}
\end{aligned}$$

另一方面,由引理7(5),有

$$\begin{aligned}
&\sum_{u \in V'} \sum_{v \in V^*} \Omega^H(u, v) \\
&= \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n [\Omega^H(v_{pq}, v_i^1) + \Omega^H(v_{pq}, v_i^2) + \cdots + \Omega^H(v_{pq}, v_i^{\Delta-d_i})] \\
&= \sum_{v_p, v_q \in E(G)} \sum_{i=1}^n \sum_{f=1}^{\Delta-d_i} \Omega^H(v_{pq}, v_i^f) \\
&= \frac{3}{2} m \left( n\Delta - \sum_{i=1}^n d_i \right) + \frac{1}{3n} \left( n\Delta - \sum_{i=1}^n d_i \right) \text{Kf}^+(G) - \frac{n-1}{6} \left( n\Delta - \sum_{i=1}^n d_i \right) \\
&= \left( \frac{3}{2} mn - \frac{n^2}{6} + \frac{n}{6} \right) \Delta + \frac{n-9m-1}{6} \sum_{i=1}^n d_i + \frac{1}{3} \Delta \text{Kf}^+(G) - \frac{1}{3n} \sum_{i=1}^n d_i \text{Kf}^+(G) \\
&= \frac{n(1+9m-n)}{6} \Delta + \frac{n-9m-1}{6} \sum_{i=1}^n d_i + \left( \frac{\Delta}{3} - \frac{1}{3n} \sum_{i=1}^n d_i \right) \text{Kf}^+(G). \tag{12}
\end{aligned}$$

综上,将式(2)、(10)、(11)、(12)代入式(8),则

$$\begin{aligned}
\text{Kf}(H_T(G)) &= \left( \left( \Delta + \frac{2}{3} \right)^2 + \frac{2}{9} \right) \text{Kf}(G) - \frac{2}{3} \left( \Delta + \frac{m}{n} + \frac{1}{2} \right) \text{Kf}^+(G) - \frac{5}{6} \text{Kf}^*(G) \\
&\quad + (n\Delta)^2 - \frac{5n\Delta}{6} (3m-n+1) + \frac{3m^2}{2} - \frac{n^2}{6} - \frac{4mn}{3} + \frac{4m}{3} + \frac{n}{6}.
\end{aligned}$$

定理3证毕。

**推论3** 若 $G$ 是 $r$ -正则的,则

$$\text{Kf}(H_T(G)) = \frac{(-11r^2 + 4r + 4)}{6} \text{Kf}(G) + \frac{n^2 r^2}{8} - \frac{n^2 r}{24} - \frac{nr}{6} - \frac{n^2}{6} + \frac{n}{6}.$$

### 3 结语

本文确定3种一元运算图的电阻距离和基尔霍夫指数。结果表明,图的电阻距离和基尔霍夫指数可用原始图的电阻距离和图不变量表示。本文不仅建立变换后的图与原图的电阻距离和基尔霍夫指标之间的关系,而且简化运算图的电阻距离和基尔霍夫指数的计算。特别地,新图的基尔霍夫指数表达式更简单,可用原图 $G$ 的基尔霍夫指数、顶点数和正则度表示。但是,对于图 $G$ 的顶点进行其他运算图,比如将图 $G$ 的每个顶点变换为一个圈或者一个长为无穷的路后,变换后的图的各项电阻距离、基尔霍夫指数、度和以及度积基尔霍夫指数的计算,尚待研究。

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