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# 新的(2+1)维 Boussinesq 方程的孤立子解

郭学军,曹玉雷*
(南阳理工学院数理学院, 河南 南阳 473004)

**摘要**:研究(2+1)维的 Boussinesq 方程,该系统是 Boussinesq 方程的一个多维推广版本。利用 Hirota 的双线性方法构造(2+1)维 Boussinesq 方程的孤立子解,并分析孤立子解的局部特征,给出了孤立子解的动力学行为。此外,在特殊参数限制下得到了共振的孤立子解。由于共振碰撞,孤立子解呈现“V”型,不再是传统的交叉型。更高阶的共振孤立子解的动力学行为更加复杂多样,由基本的共振孤立子叠加而成。

**关键词**:(2+1)-维 Boussinesq 方程;双线性方法;孤立子解;动力学
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# Soliton solutions of the new (2+1)-dimensional Boussinesq equation

GUO Xuejun, CAO Yulei*
(School of Mathematics and Physics, Nanyang Institute of Technology, Nanyang 473004, Henan, China)

**Abstract**: The Boussinesq equation is an important physical model, it can describe the propagation of shallow water waves with small amplitude. The (2+1)-dimensional Boussinesq equation is investigated, which is a multidimensional extension of the typical Boussinesq equation. The soliton solutions of (2+1)-dimensional Boussinesq equation are constructed by using the Hirota bilinear method. The local characteristics of the soliton solutions are also analyzed, and the dynamic behaviors of these soliton solutions are given analytically. Additionally, under the limitation of special parameters, we also obtain the resonance soliton solutions, and due to the resonance collision, the soliton solution presents a "V" shape, which is no longer the traditional cross type. The dynamics of higher-order resonance soliton solutions are more complex and diverse, consisting of the superposition of fundamental resonant solitons.

**Key words**: (2+1)-dimensional Boussinesq equation; bilinear method; soliton solution; dynamics

## 0 引言

双线性方法是一种构造非线性发展方程孤子解的常用方法,又叫广田直接法。1971 年首次被日本的数学家 Hirota 提出[1],也被称作双线性导数法。反散射方法和达布变换方法求解非线性方程的精确解都依赖于方程的 Lax 对,使用这些方法很难求得非线性发展方程的精确解。但是 Hirota 双线性方法被提出以后,很快就被推广应用于各种非线性发展方程精确解的构造。相对于其它方法,双线性方法有一个最显著的优点是它不依赖于谱问题和 Lax 对[2]。也即无论方程是否具有 Lax 对,都有可能通过双线性变换,把方程化为双线性型来直接求解。然而 Hirota 双线性方法也有其局限性,不是每个非线性方程都具有双线性形式。并且即便求出方程的低阶孤立子解,也不能证明该方程存在任意 $n$ 阶孤立子解。但是这丝毫不影响 Hirota 方法在可积系统中的地位[3-7]。

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**第一作者**:郭学军(1966— ),男,教授,研究方向为孤立子与可积系统. E-mail: gxuejun66@163.com
*****通信作者**:曹玉雷(1993— ),男,讲师,博士,研究方向为孤立子与可积系统. E-mail: caoyulei@mail.ustc.edu.cn

利用 Hirota 双线性方法求解可分为以下 3 步:第 1 步,选取合适的变量替换,将一般的孤子方程转化成关于 $\tau$ 函数的双线性双形式。第 2 步,利用摄动展开方法把 $\tau$ 函数写成级数的形式,然后将此级数展开式带入双线性方程中,求得各项的表达式。并将展开式截断,就可以获得非线性发展方程的单孤子解,双孤子解,三孤子解。第 3 步,利用归纳法,得到 N 阶孤子解的解[8]。需要注意的是,不是所有的非线性发展方程都具有 Lax 对,归纳法未必可以实现。

Hirota 双线性方法求解非线性发展方程精确解简洁且高效。双线性方法在处理(1+1)维系统时[9-12],其方法简单易懂。但是对于高维系统,随着自变量的增加,其难度也在增加。因此,用双线性方法研究高维系统的精确解具有一定的挑战性,并且精确求解高维非线性偏微分方程也引起了学者们的广泛关注[13-16]。很自然的,人们往往会将低维系统的结果推广到高维系统,这就更加激励人们研究高维系统。

本文主要研究一个新的高维非线性可积系统,即(2+1)维的 Boussinesq 方程[17]

$$u_{tt}-u_{xx}-\beta(u^2)_{xx}-\gamma u_{xxxx}+\frac{\alpha^2}{4}u_{yy}+\alpha u_{yt}=0, \tag{1}$$

其中 $\alpha,\beta,\gamma$ 为任意常数。该系统在经典(1+1)维 Boussinesq 方程 $u_{tt}-u_{xx}-\beta(u^2)_{xx}-\gamma u_{xxxx}=0$ 的基础上增加耗散项 $u_{yy}$ 和 $u_{yt}$ 得到,可以描述重力波在水面上的传播。

## 1　(2+1)维 Boussinesq 方程的孤立子解

通过变量变换

$$u=6\frac{\gamma}{\beta}(\ln f)_{xx}, \tag{2}$$

其中 $f\equiv f(x,y,t)$, 得到(2+1)维 Boussinesq 方程以下形式的双线性方程:

$$(D_t^2-D_x^2-D_x^4+\frac{\alpha^2}{4}D_y^2+\alpha D_yD_t)ff=0, \tag{3}$$

即

$$f_{tt}f-f_t^2-f_{xx}f+f_x^2-\gamma(f_{xxxx}f-4f_xf_{xxx}+3f_{xx}^2)+\frac{\alpha^2}{4}(f_{yy}f-f_y)+\alpha(f_{yt}f-f_yf_t)=0。 \tag{4}$$

接下来,通过 Hirota 的双线性方法来构造(2+1)维 Boussinesq 方程的多孤立子解。

### 1.1　一阶孤立子解

首先考虑最简单的孤立子解,令

$$f=1+\mathrm{e}^{\eta_1},\quad \eta_1=w_1t+p_1x+q_1y+\eta_{01},$$

将 $f$ 和 $\eta_1$ 带入方程(4),得到以下色散关系:

$$w_1=-\frac{1}{2}\alpha q_1-\sqrt{\gamma p_1^4+p_1^2}。$$

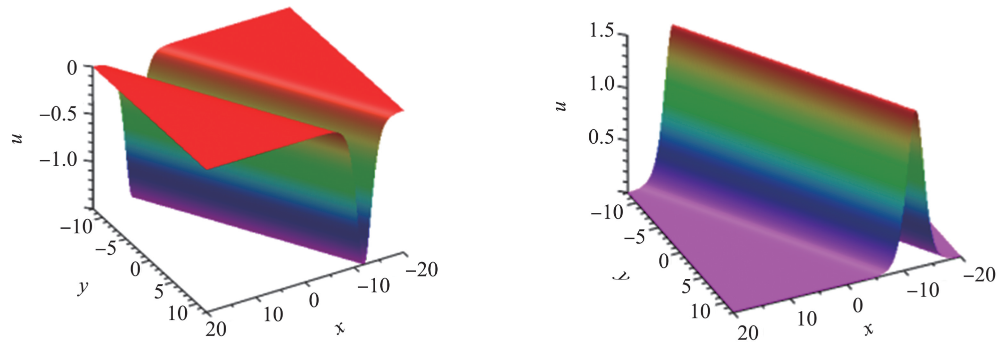
根据方程(2),即得到(2+1)维 Boussinesq 方程的一阶孤立子解

$$u=\frac{6\gamma p_1^2\mathrm{e}^{-t\sqrt{p_1^2(\gamma p_1^2+1)}+\frac{1}{2}(-\alpha t+2y)q_1+p_1x+\eta_{01}}}{(1+\mathrm{e}^{-\mathrm{i}\omega t-t\sqrt{p_1^2(\gamma p_1^2+1)}+\frac{1}{2}(-\alpha t+2y)q_1+p_1x+\eta_{01}})^2\beta}。$$

从以上单孤立子解的表达式很容易看出,一阶孤立子解的轨迹为

$$L:-t\sqrt{p_1^2(\gamma p_1^2+1)}+\frac{1}{2}(-\alpha t+2y)q_1+p_1x+\eta_{01}=0。$$

当 $\beta\gamma<0$ 时,一阶孤立子解的振幅小于其背景平面,因此,该一阶孤立子解为暗孤立子, 如图 1(a)所示, $\alpha=1$, $\beta=-1$, $\gamma=1$, $q_1=1$, $p_1=1$。当 $\beta\gamma>0$ 时,一阶孤立子解的振幅大于其背景平面,因此该孤立子解为一阶明孤立子,如图 1(b)所示,$\alpha=1$, $\beta=1$, $\gamma=1$, $q_1=1$, $p_1=1$。




(a) 暗孤立子　　(b) 明孤立子

图 1　(2+1)维 Boussinesq 方程的一阶孤立子解

Fig.1　First-order soliton solutions of (2+1)-dimensional Boussinesq equation

## 1.2　二阶孤立子解

为了构造(2+1)维 Boussinesq 方程的二阶孤立子解,假设

$$\begin{cases} f=1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2}, \\ \eta_1=p_1x+w_1t+q_1y+\eta_{01}, \\ \eta_2=p_2x+w_2t+q_1y+\eta_{02}, \end{cases}$$

将上述式子代入方程(4),得

$$w_1=-\frac{1}{2}\alpha q_1+\sqrt{p_1^2(\gamma p_1^2+1)},\quad w_2=-\frac{1}{2}\alpha q_2+\sqrt{p_2^2(\gamma p_2^2+1)},$$

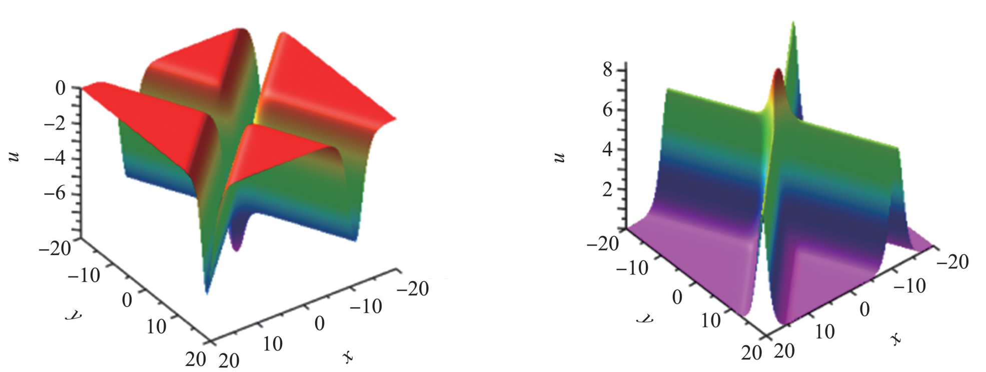
$$A_{12}=\frac{2\alpha p_1^3p_2-3\gamma p_1^2p_2^2+2\gamma p_1p_2^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_1^2(\gamma p_1^2+1)}+p_1p_2}{2\alpha p_1^3p_2+3\gamma p_1^2p_2^2+2\gamma p_1p_2^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_1^2(\gamma p_1^2+1)}+p_1p_2},$$

进一步地,取 $p_1=-1$, $p_2=1$, $q_1=1$, $q_2=\frac{1}{2}$, 根据方程(2)即得到(2+1)维 Boussinesq 方程的二阶孤立子解

$$u=\frac{6\left((4\gamma+1)e^{l_1+l_3}+(4\gamma+4)e^{l_1}+(\gamma+1)e^{l_2}+(4\gamma+1)e^{l_1+l_2}+(\gamma+1)e^{l_3}\right)\gamma(\gamma+1)}{\left(\left((4\gamma+1)e^{l_1}+(\gamma+1)(e^{l_2}+e^{l_3}+1)\right)^2\beta\right)},$$

式中, $l_1=2t\sqrt{\gamma+1}-3t+\frac{3}{2}y$, $l_2=t\sqrt{\gamma+1}-t+x+\frac{1}{2}y$, $l_3=t\sqrt{\gamma+1}-2t+x+y$。

类似于(2+1)维 Boussinesq 方程单孤立子解的讨论, $\beta\gamma<0$ 时,该二孤立子解是交叉型的暗孤立子,如图 2(a)所示, $\alpha=4$, $\beta=-1$, $\gamma=4$。当 $\beta\gamma>0$ 时,该双孤立子解是交叉型的明孤立子,如图 2(b)所示, $\alpha=4$, $\beta=1$, $\gamma=4$。




(a) 暗孤立子　　(b) 明孤立子

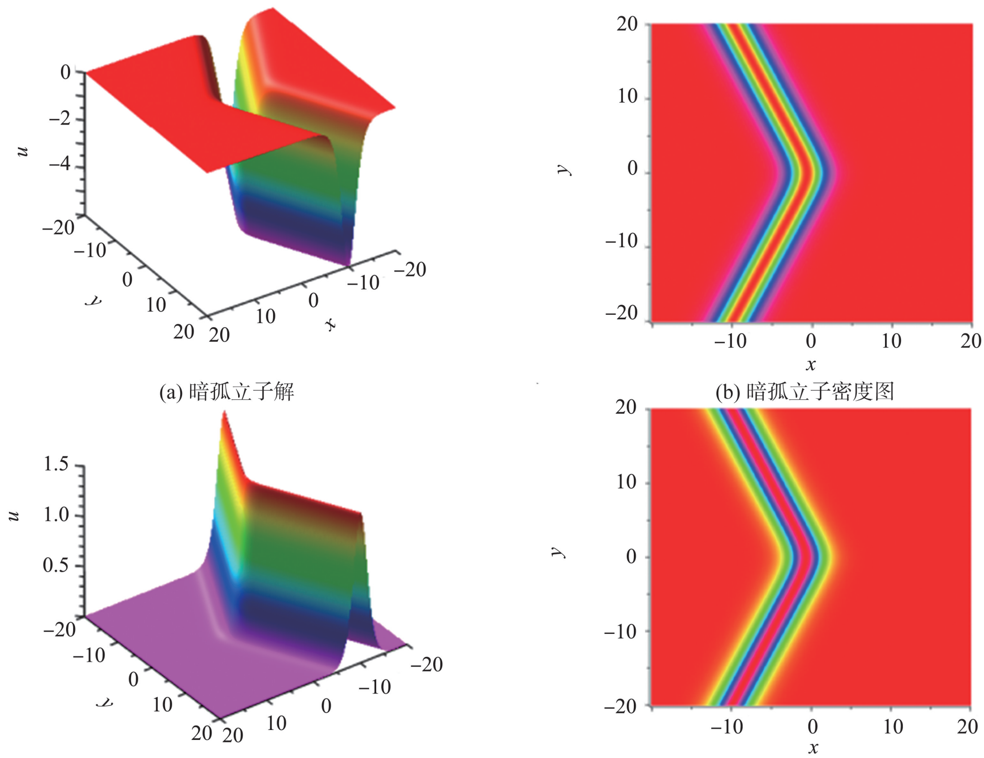
图 2　(2+1)维 Boussinesq 方程的二阶孤立子解

Fig.2　Second-order soliton solutions of (2+1)-dimensional Boussinesq equation

下面考虑该系统共振的孤立子解,线孤子的相移可以通过控制精确解中的相关参数来调节。特别地,相移为无穷大时,孤立子的波矢和频率满足 Miles 共振条件[18],即得到共振孤立子解。若取 $p_1=1$, $p_2=1$, $q_1=-\frac{1}{2}$, $q_2=\frac{1}{2}$,很显然得到 $A_{12}=0$,满足 Miles 共振条件,此时(2+1)维 Boussinesq 方程的两个孤立子发生共振碰撞,不再是交叉型。由于共振碰撞二阶孤立子解变成"V"型,根据方程(2)即得到(2+1)维 Boussinesq 方程的双孤立子解为

$$u=\frac{6\gamma\left(e^{\frac{1}{4}t\alpha+t\sqrt{\gamma+1}+x-\frac{1}{2}y}+e^{-\frac{1}{4}t\alpha+t\sqrt{\gamma+1}+x+\frac{1}{2}y}\right)}{\left(1+e^{\frac{1}{4}t\alpha+t\sqrt{\gamma+1}+x-\frac{1}{2}y}+e^{-\frac{1}{4}t\alpha+t\sqrt{\gamma+1}+x+\frac{1}{2}y}\right)^2\beta}。$$

当 $\beta\gamma<0$ 时,得到(2+1)维 Boussinesq 方程二阶共振的暗孤立子解,如图 3(a)所示,$\alpha=4$, $\beta=-1$, $\gamma=4$,平面(c)为平面(a)的密度图。当 $\beta\gamma>0$ 时,得到(2+1)维 Boussinesq 方程二阶共振的明孤立子解。如图 3(b)所示, $\alpha=1$, $\beta=1$, $\gamma=1$,平面(d)为平面(b)的密度图。



(a) 暗孤立子解　(b) 暗孤立子密度图

(c) 明孤立子解　(d) 明孤立子密度图

图 3 (2+1)维 Boussinesq 方程的二阶共振孤立子解

Fig.3 Second-order resonance soliton solutions of (2+1)-dimensional Boussinesq equation

### 1.3 三阶孤立子解

为了构造(2+1)维 Boussinesq 方程的三阶孤立子解,取

$$f=1+f_1+f_2+f_3,\quad f_1=e^{\eta_1}+e^{\eta_2}+e^{\eta_3},$$
$$f_2=A_{12}e^{\eta_1+\eta_2}+A_{13}e^{\eta_1+\eta_3}+A_{23}e^{\eta_2+\eta_3},$$
$$f_3=A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3},\quad \eta_1=p_1x+w_1t+q_1y+\eta_{01},$$
$$\eta_2=p_2x+w_2t+q_2y+\eta_{02},\quad \eta_3=p_3x+w_3t+q_3y+\eta_{03},$$

将上述式子代入方程(4),得到以下关系:

$$w_1=-\frac{1}{2}\alpha q_1+\sqrt{p_1^2(\gamma p_1^2+1)},\ w_2=-\frac{1}{2}\alpha q_2+\sqrt{p_2^2(\gamma p_2^2+1)},\ w_3=-\frac{1}{2}\alpha q_3+\sqrt{p_3^2(\gamma p_3^2+1)},$$

$$A_{12}=\frac{-2\gamma p_1^3p_2+3\gamma p_1^2p_2^2-2\gamma p_1p_2^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_2^2(\gamma p_2^2+1)}-p_1p_2}{-2\gamma p_1^3p_2-3\gamma p_1^2p_2^2-2\gamma p_1p_2^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_2^2(\gamma p_2^2+1)}-p_1p_2},$$

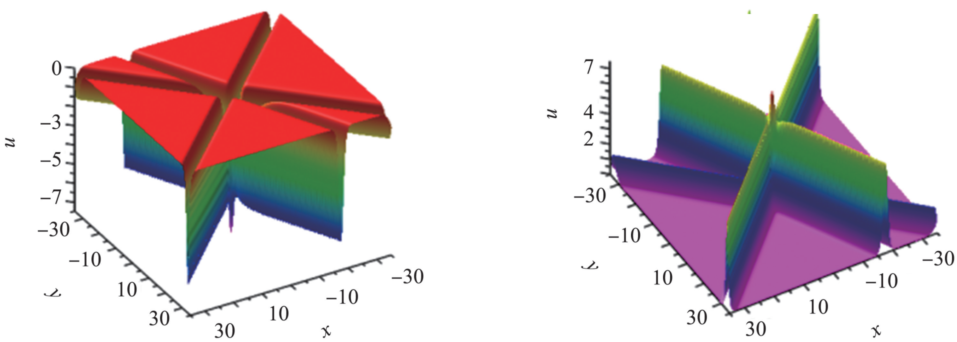
$$A_{13}=\frac{-2\gamma p_1^3p_3+3\gamma p_1^2p_3^2-2\gamma p_1p_3^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_3^2(\gamma p_2^2+1)}-p_1p_3}{-2\gamma p_1^3p_3-3\gamma p_1^2p_3^2-2\gamma p_1p_3^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_1p_3},$$

$$A_{23}=\frac{-2\gamma p_2^3p_3+3\gamma p_2^2p_3^2-2\gamma p_2p_3^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_2p_3}{-2\gamma p_2^3p_3-3\gamma p_2^2p_3^2-2\gamma p_2p_3^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_2p_3},$$

进一步地，取 $p_1=2,\ p_2=1,\ p_3=-2,\ q_1=1,\ q_2=-1,\ q_3=2$，即得到(2+1)维 Boussinesq 方程的三阶孤立子解，其中

$$f=1+e^{-\frac{1}{2}t+2it\sqrt{3}+2x+y}+e^{-\frac{1}{2}t+x+y}+e^{2it\sqrt{3}-t-2x+2y}+\frac{1}{5}e^{2it\sqrt{3}-t+3x+2y}+5e^{-\frac{3}{2}t+4it\sqrt{3}+3y}5e^{-\frac{3}{2}t+2it\sqrt{3}-x+3y}+5e^{4it\sqrt{3}-2t+x+4y}。$$

当 $\beta\gamma<0$ 时，得到(2+1)维 Boussinesq 方程三阶暗孤立子解，如图 4(a)所示，$\alpha=1,\beta=1,\gamma=-1$。当 $\beta\gamma>0$ 时，得到(2+1)维 Boussinesq 方程三阶明孤立子解，如图 4(b)所示，$\alpha=1,\ \beta=-1,\ \gamma=-1$。




(a) 暗孤立子　　(b) 明孤立子

图 4　(2+1)维 Boussinesq 方程的三阶孤立子解

Fig.4　Third-order soliton solutions of (2+1)-dimensional Boussinesq equation

### 1.4　四阶孤立子解

为了构造(2+1)维 Boussinesq 方程的四阶孤立子解，取

$$f=1+f_1+f_2+f_3+f_4,\quad f_1=e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+e^{\eta_k},$$

$$f_2=A_{12}e^{\eta_1+\eta_2}+A_{13}e^{\eta_1+\eta_3}+A_{14}e^{\eta_1+\eta_4}+A_{23}e^{\eta_2+\eta_3}+A_{24}e^{\eta_2+\eta_4}+A_{34}e^{\eta_3+\eta_4},$$

$$f_3=A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}+A_{12}A_{14}A_{24}e^{\eta_1+\eta_2+\eta_4}+A_{14}A_{34}e^{\eta_1+\eta_3+\eta_4}+A_{23}A_{24}A_{34}e^{\eta_2+\eta_3+\eta_4},$$

$$f_4=A_{12}A_{13}A_{14}A_{23}A_{24}A_4e^{\eta_1+\eta_2+\eta_3+\eta_4},\quad \eta_1=p_1x+w_1t+q_1y+\eta_{01},$$

$$\eta_2=p_2x+w_2t+q_2y+\eta_{02},\quad \eta_3=p_3x+w_3t+q_3y+\eta_{03},\quad \eta_4=p_4x+w_4t+q_4y+\eta_{04},$$

将上述式了代入方程(4)，得到以下关系：

$$w_1=-\frac{1}{2}\alpha q_1+\sqrt{p_1^2(\gamma p_1^2+1)},\quad w_2=-\frac{1}{2}\alpha q_2+\sqrt{p_2^2(\gamma p_2^2+1)},$$

$$w_3=-\frac{1}{2}\alpha q_3+\sqrt{p_3^2(\gamma p_3^2+1)},\quad w_4=-\frac{1}{2}\alpha q_4+\sqrt{p_4^2(\gamma p_4^2+1)},$$

$$A_{12}=\frac{-2\gamma p_1^3p_2+3\gamma p_1^2p_2^2-2\gamma p_1p_2^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_2^2(\gamma p_2^2+1)}-p_1p_2}{-2\gamma p_1^3p_2-3\gamma p_1^2p_2^2-2\gamma p_1p_2^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_2^2(\gamma p_2^2+1)}-p_1p_2},$$

$$A_{13}=\frac{-2\gamma p_1^3p_3+3\gamma p_1^2p_3^2-2\gamma p_1p_3^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_1p_3}{-2\gamma p_1^3p_3-3\gamma p_1^2p_3^2-2\gamma p_1p_3^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_1p_3},$$

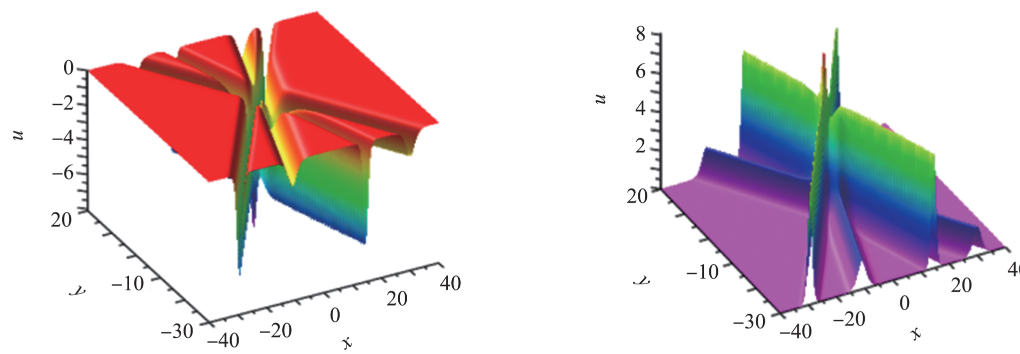
$$A_{14}=\frac{-2\gamma p_1^3p_4+3\gamma p_1^2p_4^2-2\gamma p_1p_4^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_1p_4}{-2\gamma p_1^3p_4-3\gamma p_1^2p_4^2-2\gamma p_1p_4^3-\sqrt{p_1^2(\gamma p_1^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_1p_4},$$

$$A_{23}=\frac{-2\gamma p_2^3p_3+3\gamma p_2^2p_3^2-2\gamma p_2p_3^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_2p_3}{-2\gamma p_2^3p_3-3\gamma p_2^2p_3^2-2\gamma p_2p_3^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_3^2(\gamma p_3^2+1)}-p_2p_3},$$

$$A_{24}=\frac{-2\gamma p_2^3p_4+3\gamma p_2^2p_4^2-2\gamma p_2p_4^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_2p_4}{-2\gamma p_2^3p_4-3\gamma p_2^2p_4^2-2\gamma p_2p_4^3-\sqrt{p_2^2(\gamma p_2^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_2p_4},$$

$$A_{34}=\frac{-2\gamma p_3^3p_4+3\gamma p_3^2p_4^2-2\gamma p_3p_4^3-\sqrt{p_3^2(\gamma p_3^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_3p_4}{-2\gamma p_3^3p_4-3\gamma p_3^2p_4^2-2\gamma p_3p_4^3-\sqrt{p_3^2(\gamma p_3^2+1)}\sqrt{p_4^2(\gamma p_4^2+1)}-p_3p_4}。$$

进一步地，取 $p_1=2$，$p_2=1$，$p_3=-2$，$p_4=-1$，$q_1=1$，$q_2=1$，$q_3=2$，$q_4=\frac{1}{2}$，即求得(2+1)维 Boussinesq 方程的四阶交叉型孤立子解。当 $\beta$，$\gamma<0$ 时，得到(2+1)维 Boussinesq 方程四阶暗孤立子解，如图 5(a)所示，$\alpha=1$，$\beta=-1$，$\gamma=1$。当 $\beta$，$\gamma>0$ 时，得到的是四阶明孤立子解，如图 5(b)所示，$\alpha=1$，$\beta=1$，$\gamma=1$。



(a) 暗孤立子　　(b) 明孤立子

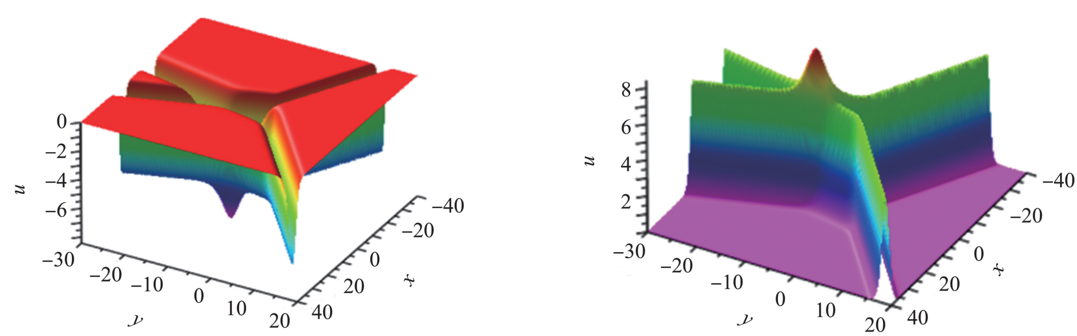
图 5　(2+1)维 Boussinesq 方程的四阶孤立子解

Fig.5　Fourth-order soliton solutions of (2+1)-dimensional Boussinesq equation

特别地，若取 $p_1=-1$，$p_2=1$，$p_3=1$，$p_4=-1$，$q_1=2$，$q_2=\frac{1}{2}$，$q_3=3$，$q_4=\frac{1}{3}$，即得到(2+1)维 Boussinesq 方程的共振的四阶孤立子解，其中

$$f=1+e^{t\sqrt{5}-t-x+2y+1}+e^{-\frac{1}{4}t+t\sqrt{5}+x+\frac{1}{2}y+2}+e^{-\frac{3}{2}t+t\sqrt{5}+x+3y+3}+e^{-\frac{1}{6}t+t\sqrt{5}-x+\frac{1}{3}y+4}+\frac{17}{5}e^{2t\sqrt{5}-\frac{5}{4}t+\frac{5}{2}y+3}$$
$$+\frac{17}{5}e^{2t\sqrt{5}-\frac{5}{2}t+5y+4}+\frac{17}{5}e^{-\frac{5}{12}t+2t\sqrt{5}+\frac{5}{6}y+6}+\frac{17}{5}e^{-\frac{5}{3}t+2t\sqrt{5}+\frac{10}{3}y+7}。$$

当 $\beta\gamma<0$ 时，得到(2+1)维 Boussinesq 方程共振的四阶暗孤立子解，$\alpha=1$，$\beta=-1$，$\gamma=4$，如图 6(a)所示。当 $\beta\gamma>0$ 时，得到的是共振的四阶明孤立子解，$\alpha=1$，$\beta=1$，$\gamma=4$，如图 6(b)所示。



(a) 暗孤立子　　(b) 明孤立子

图 6　(2+1)维 Boussinesq 方程的共振的四阶孤立子解

Fig.6　Fourth-order resonance soliton solutions of (2+1)-dimensional Boussinesq equation

## 2　结论

Boussinesq 方程是描述海洋和大气层中的流体遇到湍流现象的物理模型。本文应用双线性方法构造了

一个新的(2+1)维 Boussinesq 方程的孤立子解。孤立子解包括明孤立子解和暗孤立子解。此外,在特殊参数限制下,也得到了共振的孤立子解,由于共振碰撞,孤立子解呈现“V”型,不再是传统的交叉型。本文的结果将有助于学者更好的理解自然界中一些特殊的非线性现象,为将来的实验提供理论的支撑。

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