具有平流项和低阶项的椭圆方程很弱解的局部高阶可积性

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摘 要 在可测条件下,选取适当的截断函数并利用Hodge分解定理,得到具有平流项和低阶项的非齐次A-调和方程很弱解的局部高阶可积性,从而推广了相关文献中的有关结果.

关键词 平流项;低阶项;很弱解;Hodge分解;高阶可积性
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Local higher order integrability of very weak solutions for elliptic equations with advection and low-order terms

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Abstract Under measurable conditions, appropriate truncation functions were selected and Hodge decomposition theorem was used. The local higher order integrability results for very weak solutions to non-homogeneous A-harmonic equations with advection and low order terms were obtained. The results generalized the corresponding results in related literatures.

Keywords advection term; lower order term; very weak solution; Hodge decomposition; higher order integrability

1 引言及主要结论

令Ω为$\mathbf{R}^n(n\geqslant 2)$中的有界正则区域,考虑非齐次椭圆方程

$$\begin{gathered}-\operatorname{div}A\big(x,u,\nabla u(x)\big)-\operatorname{div}g(x,u)=\\ h(x,u)-\operatorname{div}F(x)+f(x)\end{gathered}\tag{1.1}$$

很弱解的高阶可积性.这里假设向量场$A:\Omega\times\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$,平流场$g:\Omega\times\mathbf{R}\to\mathbf{R}^n$及$h:\Omega\times\mathbf{R}\to\mathbf{R}$是Carathéodory函数,并且对于a.e.$x\in\Omega$及$s\in\mathbf{R}$和$\xi\in\mathbf{R}^n$,有

$$\big\langle A(x,s,\xi),\xi\big\rangle\geqslant C_{A,1}\left|\xi\right|^p,\tag{1.2}$$

$$\left|A(x,s,\xi)\right|\leqslant C_{A,2}\left|\xi\right|^{p-1}+C_{A,3}\left|s\right|^{p-1}+k_1(x),\tag{1.3}$$

$$\left|g(x,s)\right|\leqslant k_2(x)+C_g\left|s\right|^{p-1},\tag{1.4}$$

$$\left|h(x,s)\right|\leqslant k_3(x)+C_h\left|s\right|^{p-1}.\tag{1.5}$$

这里函数$k_i(x)\in L_{loc}^{\frac{r}{p-1}}(\Omega)$,$(i=1,2,3)$,$F(x)\in L_{loc}^{\frac{r}{p-1}}(\Omega,\mathbf{R}^n)$,$f(x)\in L_{loc}^{\frac{r}{p-1}}(\Omega)$;$C_{A,1},C_{A,2},C_{A,3}C_g,C_h$为正常数,其中$0<C_{A,1}\leqslant C_{A,2}<\infty$;$1<p<n$,$\max\{1,p-1\}\leqslant r<p$.

当$g=F=0$且$h=f=0$时,方程(1.1)即为齐次A-调和方程.

定义1.1 令常数r满足$\max\{1,p-1\}\leqslant r<p$,函数$u\in W_{loc}^{1,r}(\Omega)$称为椭圆方程(1.1)的很弱解,若

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对Ω中所有具有紧支集的$\phi \in W_0^{1,\frac{r}{r-p+1}}(\Omega)$,有

$$\begin{aligned}&\int_\Omega \left\langle A(x,u,\nabla u)+g(x,u),\nabla\phi\right\rangle dx = \\ &\int_\Omega \left\langle F(x),\nabla\phi\right\rangle dx + \int_\Omega \left[h(x,u)+f(x)\right]\phi dx\end{aligned}. \quad (1.6)$$

在研究偏微分方程的过程中,有很多方程无法得到其经典解,于是人们开始探究其弱解. 近年来方程弱解的性质有了进一步的研究成果,可见文献[1-7]. 其中,2014年,GAO等[4]获得了满足适当增长条件的平流项和低阶项的散度型椭圆方程弱解的局部正则性结果.

由经典解的极限引入了弱解,随着对方程弱解的深入研究,在用弱收敛方法推导收敛的过程中,发现极限下的弱解不一定为弱解,而有可能是很弱解. 由于很弱解可积指数更低,进而增大了求解空间,这对偏微分方程理论研究具有重大意义,因此对很弱解的研究是很有必要的.1993年,IWANIEC等[8]建立了向量的Hodge分解定理,这为研究很弱解提供了重要工具,且促进了对很弱解的高阶可积性(即可积指数的自我提高正则性)的研究. 关于很弱解的高阶可积性的结果最早是由MEYERS等[9]考虑的.LI等[10]利用逆Hölder不等式,考虑了障碍问题弱解的局部和整体高阶可积性. 基于上述研究方法,郑神州等[11]获得了一类非线性椭圆组很弱解的正则性,高红亚等[12]获得了障碍问题很弱解的局部和整体高阶可积性.

关于在椭圆方程及其障碍问题的很弱解的高阶可积性的研究中,缺乏对平流项和低阶项的处理,故受文献[4]的启发,本文考虑具有平流项和低阶项的椭圆方程的很弱解的局部高阶可积性. 得到主要结论定理1.2.

定理1.2 令Ω为$\mathbf{R}^n$中的有界正则区域,方程(1.1)满足(1.2)-(1.5). 则存在可积指数r_1、r_2满足

$$\begin{aligned}1<r_1=r_1(n,p,C_{A,1},C_{A,2},C_{A,3},C_g,C_h)<p<r_2= \\ r_2(n,p,C_{A,1},C_{A,2},C_{A,3},C_g,C_h)<\infty,\end{aligned}$$

使得方程(1.1)的每一个很弱解$u \in W_{loc}^{1,r_1}(\Omega)$都有$u \in W_{loc}^{1,r_2}(\Omega)$. 从而$u$是经典意义下的弱解.

2 预备知识

引理2.1[1,13-14] (Hodge分解定理)设$\Omega \in \mathbf{R}^n$为有界正则区域,其中n为正整数,$0<\varepsilon<r-1$, $u \in W^{1,r}(\Omega)$, $r=p-\varepsilon \geqslant \max\{1,p-1\}$, 则存在$\phi(x) \in W_0^{1,\frac{r}{1-\varepsilon}}(\Omega)$和散度为零的矩阵场$H(x) \in L^{\frac{r}{1-\varepsilon}}(\Omega)$,使得

$$|\nabla u|^{-\varepsilon}\nabla u=\nabla\phi+H, \quad (2.1)$$

$$\|H\|_{\frac{r}{1-\varepsilon}} \leqslant C\varepsilon\|\nabla u\|_r^{1-\varepsilon}, \quad (2.2)$$

其中C是只依赖于n、r和Ω的常数.

注:分解式(2.1)是外微分形式Hodge分解定理[14]的向量表示. 文献[1]通过空间非线性交换子得到估计式(2.2). 由(2.1)式及(2.2)式知,对$\nabla\phi$有估计式

$$\|\nabla\phi\|_{\frac{r}{1-\varepsilon}} \leqslant C\|\nabla u\|_r^{1-\varepsilon}.$$

引理2.2[11] 设X和Y为内积空间中的向量,$0 \leqslant \varepsilon<1$. 则

$$\left||X|^{-\varepsilon}X-|Y|^{-\varepsilon}Y\right| \leqslant \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}|X-Y|^{1-\varepsilon}.$$

引理2.3[9] (Poincaré不等式)设$1<p<n$,$0<q \leqslant \frac{np}{n-p}$. 若$u \in W^{1,p}(B_R(x_0))$,则

$$\|u-u_R\|_{L^q(B_R(x_0))} \leqslant CR^{n(\frac{1}{q}-\frac{1}{p})+1}\|\nabla u\|_{L^p(B_R(x_0))}.$$

这里$u_R := ⨍_{B_R(x_0)}udx=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}udx$,$C$为仅依赖于$p$、$q$和$n$的正常数. 特别地,若$u \in W_0^{1,p}(B_R(x_0))$,则

$$\|u\|_{L^q(B_R(x_0))} \leqslant CR^{n(\frac{1}{q}-\frac{1}{p})+1}\|\nabla u\|_{L^p(B_R(x_0))}.$$

引理2.4[15] (逆Hölder不等式)设B为一个n维球体,$f(x)$和$g(x)$为$B \subset \mathbf{R}^n$上非负可测函数. 对$\forall x_0 \in B$,任意的$R<\frac{1}{2}\operatorname{dist}\{x_0,\partial\Omega\}=R_0$,满足

$$⨍_{B_R(x_0)}g^q dx \leqslant \tau ⨍_{B_{2R}(x_0)}g^q dx + C\left(⨍_{B_{2R}(x_0)}g dx\right)^q + ⨍_{B_{2R}(x_0)}f^q dx,$$

其中$R_0>0$,$0 \leqslant \tau<1$,那么对$\forall p \in [q,q+\varepsilon_0)$,有$g(x) \in L_{loc}^p(B)$,且有

$$\left(⨍_{B_R(x_0)}g^p dx\right)^{\frac{1}{p}} \leqslant C\left[\left(⨍_{B_{2R}(x_0)}g^q dx\right)^{\frac{1}{q}}+\left(⨍_{B_{2R}(x_0)}f^q dx\right)^{\frac{1}{p}}\right],$$

其中$B_{2R(x_0)} \subset B$,C是仅依赖于τ、q和n的正常数.

3 主要定理的证明

令$B_{2R} \subset \Omega$. 设截断函数$\eta \in C_0^\infty(B_{2R})$,$0 \leqslant \eta \leqslant 1$,当$x \in B_R$时,$\eta \equiv 1$,$|\nabla\eta| \leqslant C/R$.

令 $u \in W^{1,r}(\Omega)$ 为方程(1.1)的一个很弱解，$w = \eta(u-\lambda)$，其中 $\lambda = u_{B_{2R}}$. 考虑Hodge分解

$$|\nabla w|^{-\varepsilon}\nabla w = \nabla\phi + H, \tag{3.1}$$

这里 $\phi(x) \in W_0^{1,\frac{r}{1-\varepsilon}}(B_{2R})$ 和矩阵场 $H(x) \in L^{\frac{r}{1-\varepsilon}}(B_{2R})$ 满足

$$\|H\|_{\frac{r}{1-\varepsilon}} \leqslant C\varepsilon\|\nabla w\|_r^{1-\varepsilon}, \tag{3.2}$$

$$\|\nabla\phi\|_{\frac{r}{1-\varepsilon}} \leqslant C\|\nabla w\|_r^{1-\varepsilon}, \tag{3.3}$$

其中 C 仅与 n、p、r 有关. 令 $E(w) = |\nabla w|^{-\varepsilon}\nabla w - |\eta\nabla(u-\lambda)|^{-\varepsilon}\eta\nabla(u-\lambda)$. 考虑到 $\nabla w = \eta\nabla u + (u-\lambda)\nabla\eta$ 以及 $|\nabla\eta| \leqslant C/R$，于是由引理2.2，有

$$|E(w)| \leqslant \frac{2^{\varepsilon}(1+\varepsilon)}{1-\varepsilon}|(u-\lambda)\nabla\eta|^{1-\varepsilon} \leqslant CR^{\varepsilon-1}|u-\lambda|^{1-\varepsilon}. \tag{3.4}$$

下面估计(3.2)式与(3.3)式右侧. 由 w 和 η 的定义，以及Minkowski不等式，有

$$\left(\int_{B_{2R}}|\nabla w|^r dx\right)^{\frac{1}{r}} \leqslant \left(\int_{B_{2R}}|\eta\nabla u|^r dx\right)^{\frac{1}{r}} + \left(\int_{B_{2R}}|(u-\lambda)\nabla\eta|^r dx\right)^{\frac{1}{r}}$$
$$\leqslant \left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1}{r}} + \frac{C}{R}\left(\int_{B_{2R}}|(u-\lambda)|^r dx\right)^{\frac{1}{r}}.$$

令 $\max\left\{1, \dfrac{nr}{n+r}\right\} \leqslant t < r$，利用Poincaré不等式，有

$$\left(\int_{B_{2R}}|(u-\lambda)|^r dx\right)^{\frac{1}{r}} \leqslant CR^{n\left(\frac{1}{r}-\frac{1}{t}\right)+1}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1}{t}}, \tag{3.5}$$

于是有

$$\left(\int_{B_{2R}}|\nabla w|^r dx\right)^{\frac{1}{r}} \leqslant C\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1}{r}} + CR^{n\left(\frac{1}{r}-\frac{1}{t}\right)}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1}{t}}. \tag{3.6}$$

于是(3.2)式和(3.3)式为

$$\left(\int_{B_{2R}}|H|^{\frac{r}{1-\varepsilon}}dx\right)^{\frac{1-\varepsilon}{r}} \leqslant C\varepsilon\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}} + C\varepsilon R^{(1-\varepsilon)n\left(\frac{1}{r}-\frac{1}{t}\right)}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}, \tag{3.7}$$

$$\left(\int_{B_{2R}}|\nabla\phi|^{\frac{r}{1-\varepsilon}}dx\right)^{\frac{1-\varepsilon}{r}} \leqslant C\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}} + CR^{(1-\varepsilon)n\left(\frac{1}{r}-\frac{1}{t}\right)}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}, \tag{3.8}$$

其中 C 仅与 n、p、r 有关.

由于Hodge分解式(3.1)中 $\phi(x) \in W_0^{1,\frac{r}{1-\varepsilon}}(B_{2R})$，利用定义1.1可得

$$\begin{aligned}
&\int_{B_{2R}}\left\langle A(x,u,\nabla u), |\eta\nabla(u-\lambda)|^{-\varepsilon}\eta\nabla(u-\lambda)\right\rangle dx = \\
&\int_{B_{2R}}\left\langle A(x,u,\nabla u), H\right\rangle dx - \int_{B_{2R}}\left\langle A(x,u,\nabla u), E(w)\right\rangle dx + \\
&\int_{B_{2R}}\left\langle g(x,u), H\right\rangle dx - \int_{B_{2R}}\left\langle g(x,u), |\nabla w|^{-\varepsilon}\nabla w\right\rangle dx + \\
&\int_{B_{2R}}\left\langle F(x), |\nabla w|^{-\varepsilon}\nabla w\right\rangle dx - \int_{B_{2R}}\left\langle F(x), H\right\rangle dx + \\
&\int_{B_{2R}}h(x,u)\phi dx + \int_{B_{2R}}f(x)\phi dx := \\
&I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{3.9}$$

先估计(3.9)式左侧. 考虑到 $\nabla(u-\lambda) = \nabla u$，由假设条件(1.2)和 η 的定义，有

$$\int_{B_{2R}}\left\langle A(x,u,\nabla u), |\eta\nabla(u-\lambda)|^{-\varepsilon}\eta\nabla(u-\lambda)\right\rangle dx \geqslant \int_{B_{2R}}\eta^{1-\varepsilon}|\nabla u|^{-\varepsilon}\left\langle A(x,u,\nabla u), \nabla u\right\rangle dx \geqslant C_{A,1}\int_{B_R}|\nabla u|^r dx.$$

估计 I_1. 由假设条件(1.3)、Hölder不等式、(3.7)式和Young不等式，对 $\forall\theta > 0$，有

$$\begin{aligned}
|I_1| &\leqslant \int_{B_{2R}}|A(x,u,\nabla u)||H|dx \leqslant C_{A,2}\int_{B_{2R}}|\nabla u|^{p-1}|H|dx + \\
&C_{A,3}\int_{B_{2R}}|u|^{p-1}|H|dx + \int_{B_{2R}}k_1(x)|H|dx \leqslant \\
&C_{A,2}\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{B_{2R}}|H|^{\frac{r}{1-\varepsilon}}dx\right)^{\frac{1-\varepsilon}{r}} + \\
&C_{A,3}\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{B_{2R}}|H|^{\frac{r}{1-\varepsilon}}dx\right)^{\frac{1-\varepsilon}{r}} + \\
&\left(\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\left(\int_{B_{2R}}|H|^{\frac{r}{1-\varepsilon}}dx\right)^{\frac{1-\varepsilon}{r}} \leqslant \\
&C_{A,2}\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{p-1}{r}}\left[C\varepsilon\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}} + C\varepsilon R^{(1-\varepsilon)n\left(\frac{1}{r}-\frac{1}{t}\right)}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\right] + \\
&C_{A,3}\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}\left[C\varepsilon\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}} + \right.
\end{aligned}$$

$$C\varepsilon R^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\Bigg]+$$

$$\left(\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\Bigg[C\varepsilon\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}}+$$

$$C\varepsilon R^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\Bigg]\leqslant$$

$$C\varepsilon\theta\int_{B_{2R}}|\nabla u|^r dx+C\varepsilon\int_{B_{2R}}|u|^r dx+$$

$$C\varepsilon\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx+C\varepsilon\theta\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{r}{t}},$$

其中C仅与n、p、r、$C_{A,2}$、$C_{A,3}$和θ有关.

估计I_2. 由(3.4)式、假设条件(1.3)、Hölder不等式和(3.5)式,利用Young不等式,对$\forall\theta>0$,有

$$|I_2|\leqslant\int_{B_{2R}}|A(x,u,\nabla u)||E(w)|dx\leqslant C_{A,2}CR^{\varepsilon-1}\int_{B_{2R}}|u-\lambda|^{1-\varepsilon}|\nabla u|^{p-1}dx+C_{A,3}CR^{\varepsilon-1}\int_{B_{2R}}|u-\lambda|^{1-\varepsilon}|u|^{p-1}dx+$$

$$CR^{\varepsilon-1}\int_{B_{2R}}|u-\lambda|^{1-\varepsilon}k_1(x)dx\leqslant C_{A,2}CR^{\varepsilon-1}\left(\int_{B_{2R}}|u-\lambda|^r dx\right)^{\frac{1-\varepsilon}{r}}\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{p-1}{r}}+C_{A,3}CR^{\varepsilon-1}\left(\int_{B_{2R}}|u-\lambda|^r dx\right)^{\frac{1-\varepsilon}{r}}\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}+CR^{\varepsilon-1}\left(\int_{B_{2R}}|u-\lambda|^r dx\right)^{\frac{1-\varepsilon}{r}}\left(\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\leqslant$$

$$C_{A,2}CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{p-1}{r}}+$$

$$C_{A,3}CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}+$$

$$CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\left(\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\leqslant$$

$$C\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{r}{t}}+C\theta\int_{B_{2R}}|\nabla u|^r dx+$$

$$C\theta\int_{B_{2R}}|u|^r dx+C\theta\int_{B_{2R}}|k_1(x)|^{\frac{r}{p-1}}dx,$$

其中C仅与n、p、r、R、$C_{A,2}$、$C_{A,3}$和θ有关.

估计I_3. 由条件(1.4),类似I_1的估计,利用Hölder不等式、(3.7)式和Young不等式,对$\forall\theta>0$,有

$$|I_3|\leqslant\int_{B_{2R}}|g(x,u)||H|dx\leqslant\int_{B_{2R}}|k_2(x)||H|dx+$$

$$C_g\int_{B_{2R}}|u|^{p-1}|H|dx\leqslant C\varepsilon\theta\int_{B_{2R}}|\nabla u|^r dx+$$

$$C\varepsilon\theta\int_{B_{2R}}|k_2(x)|^{\frac{r}{p-1}}dx+C\varepsilon\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{r}{t}}+$$

$$C\varepsilon\int_{B_{2R}}|u|^r dx,$$

其中C仅与n、p、r、R、C_g和θ有关.

估计I_4. 由条件(1.4),类似I_1的估计,利用Hölder不等式、(3.6)式和Young不等式,对$\forall\theta>0$,有

$$|I_4|\leqslant\int_{B_{2R}}|g(x,u)||\nabla w|^{1-\varepsilon}dx\leqslant\int_{B_{2R}}|k_2(x)||\nabla w|^{1-\varepsilon}dx+$$

$$C_g\int_{B_{2R}}|u|^{p-1}|\nabla w|^{1-\varepsilon}dx\leqslant$$

$$\left(\int_{B_{2R}}|k_2(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\left(\int_{B_{2R}}|\nabla w|^r dx\right)^{\frac{1-\varepsilon}{r}}+$$

$$C_g\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{B_{2R}}|\nabla w|^r dx\right)^{\frac{1-\varepsilon}{r}}\leqslant$$

$$\left(\int_{B_{2R}}|k_2(x)|^{\frac{r}{p-1}}dx\right)^{\frac{p-1}{r}}\Bigg[C\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}}+$$

$$CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\Bigg]+$$

$$C_g\left(\int_{B_{2R}}|u|^r dx\right)^{\frac{p-1}{r}}\Bigg[C\left(\int_{B_{2R}}|\nabla u|^r dx\right)^{\frac{1-\varepsilon}{r}}+$$

$$CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})}\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{1-\varepsilon}{t}}\Bigg]\leqslant$$

$$C\int_{B_{2R}}|k_2(x)|^{\frac{r}{p-1}}dx+C\theta\int_{B_{2R}}|\nabla u|^r dx+$$

$$C\theta\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{r}{t}}+C\int_{B_{2R}}|u|^r dx,$$

其中C仅与n、p、r、R、C_g和θ有关.

估计I_5. 类似I_4的估计,由Hölder不等式、(3.6)式和Young不等式可知,$\forall\theta>0$,有

$$|I_5|\leqslant\int_{B_{2R}}|F(x)||\nabla w|^{1-\varepsilon}dx\leqslant C\theta\int_{B_{2R}}|\nabla u|^r dx+$$

$$C\int_{B_{2R}}|F(x)|^{\frac{r}{p-1}}dx+C\theta\left(\int_{B_{2R}}|\nabla u|^t dx\right)^{\frac{r}{t}},$$

其中C仅与n、p、r、R和θ有关.

估计I_6. 类似I_1的估计,由Hölder不等式、(3.7)式和Young不等式可知,对$\forall\theta>0$,有

$$|I_6| \leqslant \int_{B_{2R}} |F(x)||H| dx \leqslant C\varepsilon\theta \int_{B_{2R}} |\nabla u|^r dx + C\varepsilon \int_{B_{2R}} |F(x)|^{\frac{r}{p-1}} dx + C\varepsilon\theta \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{r}{t}},$$

其中C仅与n、p、r、R和θ有关.

估计I_7. 由条件(1.5)、Hölder不等式、(3.8)式和Young不等式,对$\forall \theta > 0$,有

$$|I_7| \leqslant \int_{B_{2R}} |h(x,u)||\phi| dx \leqslant \int_{B_{2R}} k_3(x)|\phi| dx + C_h \int_{B_{2R}} |u|^{p-1} |\phi| dx \leqslant$$

$$\left(\int_{B_{2R}} |k_3(x)|^{\frac{r}{p-1}} dx \right)^{\frac{p-1}{r}} \left(\int_{B_{2R}} |\phi|^{\frac{r}{1-\varepsilon}} dx \right)^{\frac{1-\varepsilon}{r}} + C_h \left(\int_{B_{2R}} |u|^r dx \right)^{\frac{p-1}{r}} \left(\int_{B_{2R}} |\phi|^{\frac{r}{1-\varepsilon}} dx \right)^{\frac{1-\varepsilon}{r}} \leqslant$$

$$\left(\int_{B_{2R}} |k_3(x)|^{\frac{r}{p-1}} dx \right)^{\frac{p-1}{r}} \left[C \left(\int_{B_{2R}} |\nabla u|^r dx \right)^{\frac{1-\varepsilon}{r}} + CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})} \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{1-\varepsilon}{t}} \right] +$$

$$C_h \left(\int_{B_{2R}} |u|^r dx \right)^{\frac{p-1}{r}} \left[C \left(\int_{B_{2R}} |\nabla u|^r dx \right)^{\frac{1-\varepsilon}{r}} + CR^{(1-\varepsilon)n(\frac{1}{r}-\frac{1}{t})} \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{1-\varepsilon}{t}} \right] \leqslant$$

$$C \int_{B_{2R}} |k_3(x)|^{\frac{r}{p-1}} dx + C\theta \int_{B_{2R}} |\nabla u|^r dx + C\theta \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{r}{t}} + C \int_{B_{2R}} |u|^r dx,$$

其中C仅与n、p、r、R、C_h和θ有关.

估计I_8. 类似I_7的估计,由Hölder不等式、(3.8)式和Young不等式可知,对$\forall \theta > 0$,有

$$|I_8| \leqslant \int_{B_{2R}} |f(x)||\phi| dx \leqslant C \int_{B_{2R}} f(x)^{\frac{r}{p-1}} dx + C\theta \int_{B_{2R}} |\nabla u|^r dx + C\theta \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{r}{t}},$$

其中C仅与n、p、r、R和θ有关.

综上,可得

$$C_{A,1} \int_{B_R} |\nabla u|^r dx \leqslant C(\varepsilon + \varepsilon\theta + \theta) \int_{B_{2R}} |\nabla u|^r dx + C \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{r}{t}} + C \int_{B_{2R}} |u|^r + |k_1(x)|^{\frac{r}{p-1}} + |k_2(x)|^{\frac{r}{p-1}} + |k_3(x)|^{\frac{r}{p-1}} + |F(x)|^{\frac{r}{p-1}} + f(x)^{\frac{r}{p-1}} dx. \tag{3.10}$$

下面估计$\int_{B_{2R}} |u|^r dx$. 由Minkowski不等式、(3.5)式和Hölder不等式,有

$$\int_{B_{2R}} |u|^r dx \leqslant C \int_{B_{2R}} |(u-\lambda)|^r dx + C \int_{B_{2R}} |\lambda|^r dx \leqslant CR^{rn(\frac{1}{r}-\frac{1}{t})+r} \left(\int_{B_{2R}} |\nabla u|^t dx \right)^{\frac{r}{t}} + |B_{2R}| \left(⨍_{B_{2R}} |u|^t dx \right)^{\frac{r}{t}}, \tag{3.11}$$

将(3.11)式代入(3.10)式右侧之后,两边同时加上$C_{A,1} \int_{B_R} |u|^r dx$,并两端除以$C_{A,1} R^n$,有

$$⨍_{B_R} \left(|\nabla u|^r + |u|^r \right) dx \leqslant \tau ⨍_{B_{2R}} \left(|\nabla u|^r + |u|^r \right) dx + C \left[⨍_{B_{2R}} \left(|\nabla u|^t + |u|^t \right) dx \right]^{\frac{r}{t}} + C ⨍_{B_{2R}} \left[|k_1(x)|^{\frac{r}{p-1}} + |k_2(x)|^{\frac{r}{p-1}} + |k_3(x)|^{\frac{r}{p-1}} + |F(x)|^{\frac{r}{p-1}} + f(x)^{\frac{r}{p-1}} \right] dx, \tag{3.12}$$

其中$\tau = C(\varepsilon + \varepsilon\theta + \theta)/C_{A,1}$,$C(n,p,r,R,C_{A,1},C_{A,2},C = C_{A,3},C_g,C_h,\theta)$. 令$r$接近于$p$,使得$\varepsilon C(1+\theta)/C_{A,1} < 1/2$;取$\theta$充分小,使得$C\theta/C_{A,1} < 1/2$. 则两项系数之和$\tau < 1$. 因为$1 < t < r$,于是(3.12)式为弱逆Hölder不等式.

令$g = |\nabla u|^t + |u|^t$,$G = \left[|k_1(x)|^{\frac{r}{p-1}} + |k_2(x)|^{\frac{r}{p-1}} + |k_3(x)|^{\frac{r}{p-1}} + |F(x)|^{\frac{r}{p-1}} + f(x)^{\frac{r}{p-1}} \right]^{\frac{t}{r}}$,对$B_{2R} \subset \Omega$,有

$$⨍_{B_R} g^{\frac{r}{t}} dx \leqslant \tau ⨍_{B_{2R}} g^{\frac{r}{t}} dx + C \left(⨍_{B_{2R}} g dx \right)^{\frac{r}{t}} + C ⨍_{B_{2R}} G^{\frac{r}{t}} dx.$$

于是由引理2.5可知,存在$r'(r' > r)$,使得$u \in W^{1,r'}(\Omega)$,有

$$\left(⨍_{B_R} g^{r'} dx \right)^{\frac{1}{r'}} \leqslant C \left(⨍_{B_{2R}} g^r dx \right)^{\frac{1}{r}} + C \left(⨍_{B_{2R}} G^{r'} dx \right)^{\frac{1}{r'}}.$$

对于r',重复以上过程,可提高∇u的可积性. 因此,存在均与n、p、$C_{A,1}$、$C_{A,2}$、$C_{A,3}$、C_g和C_h有关的可积指数r_1、r_2,满足$1 < r_1 < p < r_2 < \infty$,使得$u \in W_{loc}^{1,r''}(\Omega)$,其中$\forall r'' \in (r_1, r_2)$. 定理1.2证毕.

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