第44卷第2期
2025年3月

中南民族大学学报（自然科学版）
Journal of South-Central Minzu University（Natural Science Edition）

Vol.44 No.2
Mar. 2025

一类带共同噪声和Lévy过程驱动的McKean-Vlasov随机微分方程的数值分析

胡军浩，刘虎*，高帅斌

（中南民族大学 数学与统计学学院，武汉 430074）

摘 要 研究了一类带共同噪声和Lévy过程驱动的McKean-Vlasov随机微分方程的数值方法，其中方程系数满足超线性增长条件. 对相应的交互粒子系统构造了自适应Euler-Maruyama算法，并给出了该数值算法的收敛速率.

关键词 McKean-Vlasov随机微分方程；共同噪声；自适应Euler-Maruyama算法；Lévy过程

中图分类号 O241.8 **文献标志码** A **文章编号** 1672-4321(2025)02-0269-08

doi：10.20056/j.cnki.ZNMDZK.20250217

Numerical analysis for a class of McKean-Vlasov stochastic differential equations driven by common noise and Lévy processes

HU Junhao，LIU Hu*，GAO Shuaibin

（School of Mathematics and Statistics，South-Central Minzu University，Wuhan 430074，China）

Abstract The numerical scheme for a class of McKean-Vlasov stochastic differential equations driven by common noise and Lévy processes is studied，whose coefficients satisfy the superlinear growth condition. The adaptive Euler-Maruyama scheme is constructed for the corresponding interacting particle system，and its convergence rate is shown.

Keywords McKean-Vlasov stochastic differential equations；common noise；adaptive Euler-Maruyama scheme；Lévy process

1 相关知识

本文主要研究一类带共同噪声和Lévy过程驱动的McKean-Vlasov随机微分方程的数值算法，方程的形式如下：

$$\mathrm{d}X_t = b\left(X_t, \mathcal{L}_{X_t}\right)\mathrm{d}t + \sigma\left(X_t, \mathcal{L}_{X_t}\right)\mathrm{d}W_t^0 + \kappa\left(X_t, \mathcal{L}_{X_t}\right)\mathrm{d}P_t, \quad (1)$$

其中$b: \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \to \mathbb{R}^d, \sigma: \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \to \mathbb{R}^{d\times l}, \kappa: \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \to \mathbb{R}^{d\times l}$，其初值为$X_0 = \xi$，$W_t^0$表示共同噪声且是一个$l$-维布朗运动，$P_t = \int_0^t \int_Z z\left(N(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s\right)$是一个纯跳的Lévy过程，并且Lévy测度$\nu$满足$\int_{\mathbb{R}^d} |z|^2 \nu(\mathrm{d}z)\infty$，$Z = \mathbb{R}^d / \{0\}$，$N(\mathrm{d}t, \mathrm{d}z)$表示Poisson测度. 另外，$\mathcal{L}_{X_t}$表示随机变量$X_t$的条件边缘分布流.

文献[1]提出了一种逼近McKean-Vlasov随机微分方程的方法，其分为两步. 第1步是用经验测度$\mu_t^{X,N}(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(\mathrm{d}x)$对真实分布$\mathcal{L}_{X_t}$近似，其中$\delta_{X_t^{i,N}}$表示$X_t^{i,N}$的Dirac测度，且$\left(X^{i,N}\right)_{i\in\{1,\dots,N\}}$是如下的$\mathbb{R}^{d\times N}$维交互粒子系统的解：

$$\mathrm{d}X_t^{i,N} = b\left(X_t^{i,N}, \mu_t^{X,N}\right)\mathrm{d}t + \sigma\left(X_t^{i,N}, \mu_t^{X,N}\right)\mathrm{d}W_t^0 + \kappa\left(X_t, \mu_t^{X,N}\right)\mathrm{d}P_t^i, \quad (2)$$

第2步构造适当的数值算法来近似上述交互粒子系统，从而实现对原方程的数值逼近.

1956年，McKean在文献[2]中首次展开了对McKean-Vlasov随机微分方程的研究. 当带有共同噪声的McKean-Vlasov随机微分方程的系数满足Lipschitz

---

收稿日期 2024-07-11 * **通信作者** 刘虎，研究方向：随机数值分析，E-mail：liuhuedu@163.com

作者简介 胡军浩（1974-），男，教授，博士，研究方向：随机系统理论及应用，E-mail：junhaohu74@163.com

基金项目 国家自然科学基金资助项目（62373383）

条件时,文献[3]讨论了其强解的存在唯一性.当系数的状态变量满足单调条件时,文献[4-5]研究了McKean-Vlasov随机微分方程强解的适定性.文献[6-7]研究了McKean-Vlasov随机微分方程弱解的存在唯一性.文献[8]用自适应Euler-Maruyama算法对一类随机微分方程进行数值逼近.随后,文献[9]研究了不带共同噪声且漂移项和扩散项系数满足超线性条件的McKean-Vlasov随机微分方程的自适应算法的收敛性.文献[10-11]考虑了驯服Milstein算法对满足单边Lipschitz条件的McKean-Vlasov随机微分方程的收敛性.文献[12]研究了驯服Euler-Maruyama算法对Lévy过程驱动的McKean-Vlasov随机微分方程的收敛性.文献[13]研究了驯服的自适应Euler-Maruyama算法对Lévy过程驱动的McKean-Vlasov随机微分方程在无限时间尺度下的数值逼近.

令$|\cdot|$表示Euclidean范数,且$\|\cdot\|$表示Hilbert-Schmidt范数.令$\left(\Omega,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geqslant 0},\mathbb{P}\right)$表示完备的概率空间,其$\sigma$代数流$\left(\mathcal{F}_t\right)_{t\geqslant 0}$满足一般条件(即单调递增和右连续的,且$\mathcal{F}_0$包含所有零测集).用$\langle a,b\rangle:=a_1b_1+\cdots+a_db_d$表示两个$\mathbb{R}^d$向量$a$和$b$的内积.用$\mathcal{P}(\mathbb{R}^d)$表示在$\left(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\right)$上的所有概率测度集,其中$\mathcal{B}(\mathbb{R}^d)$是一个Borel$\sigma$-域.对于$p\geqslant 1$,定义:

$$\mathcal{P}_p(\mathbb{R}^d):=\left\{\mu\in\mathcal{P}(\mathbb{R}^d):\left(\int_{\mathbb{R}^d}|x|^p\mu(\mathrm{d}x)\right)^{\frac{1}{p}}<\infty\right\},$$

对于任意的$\mu,\nu\in\mathcal{P}_p(\mathbb{R}^d)$,其Wasserstein距离定义为:

$$\mathcal{W}_p(\mu,\nu):=\left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|x-y\right|^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}},$$

其中$\Pi(\mu,\nu)$是μ和ν全体耦合.

对于任意$i\in\{1,\cdots,N\}$,给定自适应的时间步长$h_n^i=h\left(\hat{X}_{t_n}^{i,N},\mu_{t_n}^{\hat{X},N}\right)$.将交互粒子系统(2)的自适应Euler-Maruyama算法定义为:

$$\hat{X}_{t_{n+1}}^{i,N}=\hat{X}_{t_n}^{i,N}+b\left(\hat{X}_{t_n}^{i,N},\mu_{t_n}^{\hat{X},N}\right)h_n^{\min}+\sigma\left(\hat{X}_{t_n}^{i,N},\mu_{t_n}^{\hat{X},N}\right)\Delta W_{t_n}^0+\kappa\left(\hat{X}_{t_n}^{i,N},\mu_{t_n}^{\hat{X},N}\right)\Delta P_{t_n}^i,\tag{3}$$

其中$\Delta W_{t_n}^i=W_{t_{n+1}}^i-W_{t_n}^i$,$\Delta P_{t_n}^i=P_{t_{n+1}}^i-P_{t_n}^i$.且时间步长$h_n^{\min}:=\min\{h_n^1,\cdots,h_n^N\}$,$t_{n+1}=t_n+h_n^{\min}$,$t_n$随着$n$的增大而增大,故当$n=M$时,有$t_M\geqslant T$.定义符号$\underline{t}:=\max\{t_n:t_n\leqslant t\}$是时间$t$前最靠近$t$的时间点,以及$n_t:=\max\{n:t_n\leqslant t\}$表示直到时间$t$近似步长数量.因此,定义连续插值过程如下:

$$\hat{X}_t^{i,N}=\hat{X}_{\underline{t}}^{i,N}+b\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)(t-\underline{t})+\sigma\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)\left(W_t^0-W_{\underline{t}}^0\right)+\kappa\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)\left(P_t^i-P_{\underline{t}}^i\right),\tag{4}$$

因此$\left(\hat{X}_t^{i,N}\right)_{t\in[0,T]}$满足:

$$\mathrm{d}\hat{X}_t^{i,N}=b\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)\mathrm{d}t+\sigma\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)\mathrm{d}W_t^0+\kappa\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)\mathrm{d}P_t^i.\tag{5}$$

令方程(1)的系数满足如下的假设:

(H1)对于任意的$p\geqslant 2$,有$E\left|\xi\right|^p<\infty$.

(H2)存在常数$L_1>0$,$q>0$,以及$\mathcal{F}_0$可测的非负随机变量序列$\left\{\tilde{A}_t\right\}_{t\in[0,T]}$满足$\sup\limits_{t\in[0,T]}E[\tilde{A}_t]<\infty$,使得对任意的$p\geqslant 2$,$x,x'\in\mathbb{R}^d$,$\mu,\mu'\in\mathcal{P}_2(\mathbb{R}^d)$,有:

$$2|x-x'|^{p-2}\left\langle x-x',b(x,\mu)-b(x',\mu')\right\rangle+2(p-1)|x-x'|^{p-2}\left\|\sigma(x,\mu)-\sigma(x',\mu')\right\|^2+2^{p-2}(p-1)\left\|\kappa(x,\mu)-\kappa(x',\mu')\right\|^2\int_Z\int_0^1(1-\theta)\times\left|x-x'+\theta z\left(\kappa(x,\mu)-\kappa(x',\mu')\right)\right|^{p-2}|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\leqslant L_1\left(|x-x'|^p+\mathcal{W}_2^p(\mu,\mu')\right).\tag{6}$$

$$\left|b(x,\mu)-b(x',\mu')\right|\leqslant L_1\left(1+|x|^q+|x'|^q\right)|x-x'|+\mathcal{W}_2(\mu,\mu').\tag{7}$$

$$\left\|\sigma(x,\mu)-\sigma(x',\mu')\right\|^2+\left\|\kappa(x,\mu)-\kappa(x',\mu')\right\|^2\int_Z|z|^2\nu(\mathrm{d}z)\leqslant L_1\left(1+|x|^q+|x'|^q\right)|x-x'|^2+\mathcal{W}_2^2(\mu,\mu').\tag{8}$$

$$2|x|^{p-2}\left\langle x,b(x,\mu)\right\rangle+(p-1)|x|^{p-2}\left\|\sigma(x,\mu)\right\|^2+2(p-1)\left\|\kappa(x,\mu)\right\|^2\times\int_Z\int_0^1(1-\theta)\left|x+\theta z\kappa(x,\mu)\right|^{p-2}|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\leqslant L_1\left(\tilde{A}_t+|x|^p+\mathcal{W}_2^p(\mu,\delta_0)\right).\tag{9}$$

由于步长h依赖于状态变量$X_t^{i,N}$,因此用$h^\varepsilon(x,\mu)$表示修正后的步长函数.

(H3)对于步长函数h,$h^\varepsilon:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^+$,存在常数$L_2,q,\eta_1,\eta_2,\eta_3,\varpi>0$,使得对任意的$0<\varepsilon\leqslant 1$,$x\in\mathbb{R}^d$,$\mu\in\mathcal{P}_2(\mathbb{R}^d)$,有:

$$\left(\eta_1|x|^{\varpi}+\eta_2\mathcal{W}_2^{\varpi}(\mu,\delta_0)+\eta_3\right)^{-1}\leqslant h(x,\mu)\leqslant L_2\left(1+|x|^{3q}+\mathcal{W}_2^{3q}(\mu,\delta_0)\right)^{-1}. \quad (10)$$

$$\varepsilon\min\left(T,h(x,\mu)\right)\leqslant h^{\varepsilon}(x,\mu)\leqslant\min\left(\varepsilon T,h(x,\mu)\right). \quad (11)$$

由假设(H1),(H2)可推出方程(1)存在唯一的强解并且$E\left[\left|X_t\right|^r\right]\leqslant C(r\geqslant 0)$,其证明可以参考文献[12-13]及其参考文献.同理,由假设(H1),(H2)还可推出交互粒子系统(2)存在唯一的强解$X_t^{i,N}$及其p阶矩有界性.

2　主要结论及证明

定理1　对于任意的$p\geqslant 2$,在假设(H1),(H2),(H3)下,存在依赖于时间T和p的常数$C>0$,使得自适应Euler-Maruyama算法(3)的解满足:

$$\max_{i\in\{1,\cdots,N\}}\sup_{t\in[0,T]}E\left[\left|\hat{X}_t^{i,N}\right|^p\right]\leqslant C. \quad (12)$$

证明　定义停时$\rho_{K>0}:=\inf\left\{t\geqslant 0:\max\limits_{i\in\{1,\dots,N\}}\left|\hat{X}_t^{i,N}\right|\geqslant K\right\}$,并使用Itô公式可得:

$$\begin{aligned}\left|\hat{X}_{t\wedge\rho_K}^{i,N}\right|^p\leqslant&\left|\hat{X}_0^{i,N}\right|^p+p\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N},b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle\mathrm{d}s+\\&p\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N},\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\mathrm{d}W_t^0\right\rangle+\frac{p(p-1)}{2}\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2\mathrm{d}s+\\&p\int_0^{t\wedge\rho_K}\int_Z\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N},z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle\tilde{N}^i(\mathrm{d}s,\mathrm{d}z)+\\&\int_0^{t\wedge\rho_K}\int_Z\left(\left|\hat{X}_s^{i,N}+z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^p-\left|\hat{X}_s^{i,N}\right|^p-p\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N},z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle\right)N^i(\mathrm{d}s,\mathrm{d}z),\end{aligned}$$

两边同时取期望并由(10)式和等式

$\left|y\right|^p=|a|^p+p|a|^{p-2}\left\langle a,y-a\right\rangle+p(p-1)\int_0^1(1-\theta)\left|y-a\right|^2\left|a+\theta(y-a)\right|^{p-2}\mathrm{d}\theta$可得到:

$$\begin{aligned}E\left[\left|\hat{X}_{t\wedge\rho_K}^{i,N}\right|^p\right]\leqslant&E\left[\left|\hat{X}_0^{i,N}\right|^p\right]+\frac{p}{2}E\left[\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left(2\left\langle\hat{X}_{\underline{s}}^{i,N},b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle+\right.\right.\\&(p-1)\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2+2(p-1)\int_Z\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2\times\\&\left.\left.\int_0^1(1-\theta)\left|\hat{X}_{\underline{s}}^{i,N}+\theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\right)\mathrm{d}s\right]+\\&pE\left[\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N},b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle\mathrm{d}s\right]+\\&p(p-1)E\left[\int_0^{t\wedge\rho_K}\int_Z\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2\int_0^1(1-\theta)\times\right.\\&\left.\left(\left|\hat{X}_s^{i,N}+\theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}-\left|\hat{X}_{\underline{s}}^{i,N}+\theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}\right)|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\mathrm{d}s\right]\leqslant\\&E\left[\left|\hat{X}_0^{i,N}\right|^p\right]+CE\left[\int_0^{t\wedge\rho_K}\tilde{A}_{\underline{s}}+\left|\hat{X}_{\underline{s}}^{i,N}\right|^p+\mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N},\delta_0\right)\mathrm{d}s\right]+U_1+U_2,\end{aligned} \quad (13)$$

其中:

$$U_1=pE\left[\int_0^{t\wedge\rho_K}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N},b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle\mathrm{d}s\right],$$

$$\begin{aligned}U_2=&p(p-1)E\left[\int_0^{t\wedge\rho_K}\int_Z\left|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^2\int_0^1(1-\theta)\times\right.\\&\left.\left\{\left|\hat{X}_s^{i,N}+\theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}-\left|\hat{X}_{\underline{s}}^{i,N}+\theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}\right\}|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\mathrm{d}s\right].\end{aligned}$$

由假设(H3)以及Hölder不等式可以得到:

$$
\begin{aligned}
U_1 = pE\left[\int_0^{t\wedge\rho_\kappa}\left|\hat{X}_s^{i,N}\right|^{p-2}\left\langle \hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}, b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right\rangle \mathrm{d}s\right] \leqslant \\
pE\left[\int_0^{t\wedge\rho_\kappa}\left|\hat{X}_s^{i,N}\right|^p \mathrm{d}s\right] + E\left[\int_0^{t\wedge\rho_\kappa}\left|\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \mathrm{d}s\right] \leqslant \\
pE\left[\int_0^{t\wedge\rho_\kappa}\left|\hat{X}_s^{i,N}\right|^p \mathrm{d}s\right] + CE\left[\int_0^{t\wedge\rho_\kappa}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^p (s-\underline{s})^{\frac{p}{2}} \mathrm{d}s\right] + \\
CE\left[\int_0^{t\wedge\rho_\kappa}\left|\sigma\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(W_s^0 - W_{\underline{s}}^0\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \mathrm{d}s\right] + \\
CE\left[\int_0^{t\wedge\rho_\kappa}\left|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(P_s^i - P_{\underline{s}}^i\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \mathrm{d}s\right] \leqslant \\
CE\left[\int_0^{t\wedge\rho_\kappa}\left(1 + \left|\hat{X}_{\underline{s}}^{i,N}\right|^p + \mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\right)\mathrm{d}s\right] + \\
C\int_0^{t\wedge\rho_\kappa} E\left[E\left[\left|\sigma\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(W_s^0 - W_{\underline{s}}^0\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \Big| \ \mathcal{F}_{\underline{s}}\right]\right]\mathrm{d}s + \\
\int_0^{t\wedge\rho_\kappa} E\left[E\left[\left|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(P_s^i - P_{\underline{s}}^i\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \Big| \ \mathcal{F}_{\underline{s}}\right]\right]\mathrm{d}s,
\end{aligned}
$$

其中：

$$
E\left[\left\| W_s^0 - W_{\underline{s}}^0\right\|^p \Big| \mathcal{F}_{\underline{s}}\right] \leqslant C(s-\underline{s})^{\frac{p}{2}}, E\left[\left\| P_s^i - P_{\underline{s}}^i\right\|^p \Big| \mathcal{F}_{\underline{s}}\right] \leqslant C\left(\int_Z |z|^2 \nu(\mathrm{d}z)\right)^{\frac{p}{2}}(s-\underline{s})^{\frac{p}{2}},
$$

$$
E\left[E\left[\left|\sigma\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(W_s^0 - W_{\underline{s}}^0\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \Big| \mathcal{F}_{\underline{s}}\right]\right] \leqslant E\left[1 + \left|\hat{X}_{\underline{s}}^{i,N}\right|^p + \mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\right],
$$

$$
E\left[E\left[\left|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\left(P_s^i - P_{\underline{s}}^i\right)\right|^{\frac{p}{2}}\left|b\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{\frac{p}{2}} \Big| \mathcal{F}_{\underline{s}}\right]\right] \leqslant E\left[1 + \left|\hat{X}_{\underline{s}}^{i,N}\right|^p + \mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\right],
$$

$$
E\left[\mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\right] = E\left[\left|\hat{X}_{\underline{s}}^{i,N}\right|^p\right],
$$

故代入上式得到：

$$
U_1 \leqslant C\int_0^{t\wedge\rho_\kappa} E\left[1 + \left|\hat{X}_{\underline{s}}^{i,N}\right|^p + \mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\right]\mathrm{d}s,
$$

使用等式$\left|y\right|^{p-2} = |a|^{p-2} + (p-2)\int_0^1 \left|a + \theta(y-a)\right|^{p-4}(y-a)\left(a + \theta(y-a)\right)\mathrm{d}\theta$得到：

$$
\begin{aligned}
U_2 = p(p-1)E\left[\int_0^{t\wedge\rho_\kappa}\int_Z \left|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^2 \int_0^1 (1-\theta) \times \right. \\
\left.\left\{\left|\hat{X}_s^{i,N} + \theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2} - \left|\hat{X}_{\underline{s}}^{i,N} + \theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right|^{p-2}\right\}|z|^2 \mathrm{d}\theta \nu(\mathrm{d}z)\mathrm{d}s\right] = \\
p(p-1)(p-2)E\left[\int_0^{t\wedge\rho_\kappa}\int_Z \left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2 \int_0^1 (1-\theta) \times \right. \\
\int_0^1 \left\{\left|\hat{X}_{\underline{s}}^{i,N} + \theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right) + \bar{\theta}\left(\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right)\right|^{p-4}\left(\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right) \times \right. \\
\left.\left.\left(\hat{X}_{\underline{s}}^{i,N} + \theta z\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right) + \bar{\theta}\left(\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right)\right)\right\}|z|^2 \mathrm{d}\bar{\theta}\mathrm{d}\theta \nu(\mathrm{d}z)\mathrm{d}s\right] \leqslant \\
E\left[\int_0^{t\wedge\rho_\kappa}\int_Z \left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right\|^2 \left(\left|\hat{X}_{\underline{s}}^{i,N}\right| + \left|z\kappa\left(\hat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\hat{X},N}\right)\right| + \left|\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right|\right)^{p-3} \times \right. \\
\left.\left|\hat{X}_s^{i,N} - \hat{X}_{\underline{s}}^{i,N}\right| |z|^2 \nu(\mathrm{d}z)\mathrm{d}s\right] \leqslant CE\left[\int_0^{t\wedge\rho_\kappa} 1 + \left|\hat{X}_{\underline{s}}^{i,N}\right|^p + \mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N}, \delta_0\right)\mathrm{d}s\right].
\end{aligned}
$$

将上式代入(13)式并使用Gronwall不等式得到:

$$\sup_{s\in[0,t]} E\left[\left|\hat{X}_{s\wedge\rho_K}^{i,N}\right|^p\right] \leqslant C + E\left[\left|\hat{X}_0^{i,N}\right|^p\right] + C\int_0^t \sup_{r\in[0,s]} E\left[\left|\hat{X}_{s\wedge\rho_K}^{i,N}\right|^p\right]\mathrm{d}s \leqslant C.$$

为了确定T是可以达到的,使用Markov不等式得到:

$$P\left(\rho_K\leqslant T\right)\leqslant\sum_{i=1}^{N}P\left(\left|\hat{X}_{\rho_K\wedge T}^{i,N}\right|>K\right)\leqslant NP\left(\max_{i\in\{1,\dots,N\}}\left|\hat{X}_{\rho_K\wedge T}^{i,N}\right|>K\right)\leqslant\frac{N}{K^2}E\left[\max_{i\in\{1,\dots,N\}}\left|\hat{X}_{\rho_K\wedge T}^{i,N}\right|^2\right]\leqslant\frac{CN}{K^2}.$$

因此当$K\to\infty$时,有:

$$P\left(\max_{i\in\{1,\dots,N\}}\sup_{t\in[0,T]}\left|\hat{X}_t^{i,N}\right|<K\right)=1-P\left(\rho_K\leqslant T\right)\to 1.$$

当$K\to\infty,\rho_K\to\infty$时,$\max_{i\in\{1,\dots,N\}}\sup_{t\in[0,T]}\left|\hat{X}_t^{i,N}\right|\infty$几乎必然成立,则$T$可以达到.

定理2　对于任意的$p\geqslant 2$,假设方程(1)满足定理1的条件,则存在依赖于时间T和p的常数$C>0$,有:

$$\sup_{0<\varepsilon\leqslant 1}\max_{i\in\{1,\dots,N\}}\sup_{0\leqslant t\leqslant T}E\left[\left|\hat{X}_t^{i,N}-X_t^{i,N}\right|^p\right]\leqslant C\varepsilon^{\frac{p}{2}},\tag{14}$$

其中ε来自于假设(H3).

证明　定义$S_t^i=\hat{X}_t^{i,N}-X_t^{i,N}$,则有:

$$\mathrm{d}S_t^i=\left(b\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)-b\left(X_t^{i,N},\mu_t^{X,N}\right)\right)\mathrm{d}t+\left(\sigma\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)-\sigma\left(X_t^{i,N},\mu_t^{X,N}\right)\right)\mathrm{d}W_t^0+\left(\kappa\left(\hat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\hat{X},N}\right)-\kappa\left(X_t^{i,N},\mu_t^{X,N}\right)\right)\mathrm{d}P_t^i.$$

对$\left|S_t^i\right|^p$用Itô公式可以得到

$$\begin{aligned}\left|S_t^i\right|^p\leqslant&\frac{p}{2}\int_0^t\left|S_s^i\right|^{p-2}2\left\langle S_s^i,b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-b\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\ \mathrm{d}s+\\&p\int_0^t\left|S_s^i\right|^{p-2}\left\langle S_s^i,\left(\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\mathrm{d}W_s^0\right\rangle+\\&\int_0^t\int_Z\left(\left|S_s^i+z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^p-\left|S_s^i\right|^p\right)\tilde{N}^i(\mathrm{d}s,\mathrm{d}z)+\\&\frac{p\left(p-1\right)}{2}\int_0^t\left|S_s^i\right|^{p-2}\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2\mathrm{d}s+\\&\int_0^t\int_Z\left(\left|S_s^i+z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^p-\left|S_s^i\right|^p-\right.\\&\left.pz\left|S_s^i\right|^{p-2}\left\langle S_s^i,\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\right)\nu(\mathrm{d}z)\mathrm{d}s=R_t+M_t,\end{aligned}\tag{15}$$

其中:

$$\begin{aligned}R_t=&\frac{p}{2}\int_0^t\left|S_s^i\right|^{p-2}2\left\langle S_s^i,b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-b\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\ \mathrm{d}s+\\&\frac{p\left(p-1\right)}{2}\int_0^t\left|S_s^i\right|^{p-2}\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2\mathrm{d}s+\\&\int_0^t\int_Z\left(\left|S_s^i+z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^p-\left|S_s^i\right|^p-\right.\\&\left.pz\left|S_s^i\right|^{p-2}\left\langle S_s^i,\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\right)\nu(\mathrm{d}z)\mathrm{d}s,\\M_t=&p\int_0^t\left|S_s^i\right|^{p-2}\left\langle S_s^i,\left(\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\mathrm{d}W_s^0\right\rangle+\\&\int_0^t\int_Z\left(\left|S_s^i+z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^p-\left|S_s^i\right|^p\right)\tilde{N}^i(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

由等式$\left|y\right|^p=\left|a\right|^p+p\left|a\right|^{p-2}\left\langle a,y-a\right\rangle+p(p-1)\int_0^1(1-\theta)\left|y-a\right|^2\left|a+\theta(y-a)\right|^{p-2}\mathrm{d}\theta$得到:

$$R_t=p\left|S_s^i\right|^{p-2}\left\langle S_s^i,b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-b\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle+\frac{p(p-1)}{2}\left|S_s^i\right|^{p-2}\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2+$$
$$\int_Z\left\{\left|S_s^i+z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^p-\left|S_s^i\right|^p-pz\left|S_s^i\right|^{p-2}\left\langle S_s^i,\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\right\}\nu(\mathrm{d}z)=$$
$$p\left|S_s^i\right|^{p-2}\left\langle S_s^i,b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-b\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle+\frac{p(p-1)}{2}\left|S_s^i\right|^{p-2}\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2+$$
$$\int_Z\int_0^1\left\{(1-\theta)\left|S_s^i+\theta z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^{p-2}\times\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2|z|^2\mathrm{d}\theta\nu(\mathrm{d}z)\right\},$$

由假设条件(H2)、Young不等式得到:

$$R_t\leqslant p\left|S_s^i\right|^{p-2}\left(\left\langle S_s^i,b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-b\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)\right\rangle+\left\langle S_s^i,b\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)-b\left(X_s^{i,N},\mu_s^{X,N}\right)\right\rangle\right)+$$
$$p(p-1)\left|S_s^i\right|^{p-2}\left(\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\sigma\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)\right\|^2+\left\|\sigma\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)-\sigma\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2\right)+$$
$$2^{p-2}\int_Z\int_0^1\left\{(1-\theta)|z|^2\left(\left\|\theta z\left(\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)\right)\right\|^{p-2}+\right.\right.$$
$$\left.\left|S_s^i+\theta z\left(\kappa\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right)\right|^{p-2}\right)\times$$
$$\left.\left(\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)-\kappa\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)\right\|^2+\left\|\kappa\left(\hat{X}_s^{i,N},\mu_s^{\hat{X},N}\right)-\kappa\left(X_s^{i,N},\mu_s^{X,N}\right)\right\|^2\right)\right\}\mathrm{d}\theta\nu(\mathrm{d}z)\leqslant$$
$$CQ^p\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)\left|\hat{X}_{\underline{s}}^{i,N}-\hat{X}_s^{i,N}\right|^p+C\mathcal{W}_2^p\left(\mu_{\underline{s}}^{\hat{X},N},\mu_s^{\hat{X},N}\right)+$$
$$C\left(\left(Q'^p\left(X_s^{i,N},\hat{X}_s^{i,N}\right)+1\right)\left|S_s^i\right|^p+\mathcal{W}_2^p\left(\mu_s^{\hat{X},N},\mu_s^{X,N}\right)\right),$$

其中$Q\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)=L_1\left(1+\left|\hat{X}_{\underline{s}}^{i,N}\right|^q+\left|\hat{X}_s^{i,N}\right|^q\right)$, $Q'\left(X_s^{i,N},\hat{X}_s^{i,N}\right)=L_1\left(1+\left|X_s^{i,N}\right|^q+\left|\hat{X}_s^{i,N}\right|^q\right)$.

将上述估计式代入(15)式并两边同时取期望得:

$$E\left[\left|S_t^i\right|^p\right]\leqslant C\int_0^t E\left[\left(Q'^p\left(X_s^{i,N},\hat{X}_s^{i,N}\right)+1\right)\left|S_s^i\right|^p+\mathcal{W}_2^p\left(\mu_s^{\hat{X},N},\mu_s^{X,N}\right)\right]\mathrm{d}s+$$
$$C\int_0^t E\left[Q^p\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)\left|\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N}\right|^p+\mathcal{W}_2^p\left(\mu_s^{\hat{X},N},\mu_{\underline{s}}^{\hat{X},N}\right)\right]\mathrm{d}s.$$

由Hölder不等式知:

$$E\left[Q^p\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)\left|\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N}\right|^p\right]\leqslant\left(E\left[Q^{2p}\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)\right]E\left[\left|\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N}\right|^{2p}\right]\right)^{\frac{1}{2}},$$
$$E\left[Q'^p\left(X_s^{i,N},\hat{X}_s^{i,N}\right)\left|S_s^i\right|^p\right]\leqslant\left(E\left[Q'^{2p}\left(X_s^{i,N},\hat{X}_s^{i,N}\right)\right]E\left[\left|S_s^i\right|^{2p}\right]\right)^{\frac{1}{2}},$$

由定理1知$E\left[Q^{2p}\left(\hat{X}_{\underline{s}}^{i,N},\hat{X}_s^{i,N}\right)\right]\leqslant C$, $E\left[Q'^{2p}\left(X_s^{i,N},\hat{X}_s^{i,N}\right)\right]\leqslant C$,并通过(4)式得到:

$$E\left[\left|\hat{X}_s^{i,N}-\hat{X}_{\underline{s}}^{i,N}\right|^{2p}\right]\leqslant C\left(E\left[\left|b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{4p}\right]E\left[(s-\underline{s})^{4p}\right]\right)^{\frac{1}{2}}+C\left(E\left[\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^{4p}\right]E\left[\left\|W_s^i-W_{\underline{s}}^i\right\|^{4p}\right]\right)^{\frac{1}{2}}+$$
$$C\left(E\left[\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^{4p}\right]E\left[\left\|P_s^i-P_{\underline{s}}^i\right\|^{4p}\right]\right)^{\frac{1}{2}}.$$

由定理1得到:

$$E\left[\left|b\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right|^{4p}\right]\leqslant C, E\left[\left\|\sigma\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^{4p}\right]\leqslant C, E\left[\left\|\kappa\left(\hat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\hat{X},N}\right)\right\|^{4p}\right]\leqslant C,$$

注意到:

$$E\left[\left(s-\underline{s}\right)^{4p}\right]\leqslant\left(\varepsilon T\right)^{4p}\leqslant C\varepsilon^{2p},$$

$$E\left[\left\|W_s^0-W_{\underline{s}}^0\right\|^{4p}\right]\leqslant E\left[E\left[\left\|W_s^0-W_{\underline{s}}^0\right\|^{4p}\right]\middle|\mathcal{F}_{\underline{s}}\right]\leqslant C\left(s-\underline{s}\right)^{2p}\leqslant C\varepsilon^{2p},$$

$$E\left[\left\|P_s^i-P_{\underline{s}}^i\right\|^{4p}\right]\leqslant E\left[E\left[\left\|P_s^i-P_{\underline{s}}^i\right\|^{4p}\right]\middle|\ \mathcal{F}_{\underline{s}}\right]\leqslant C\left(\int_Z\left|z\right|^2\nu(\mathrm{d}z)\right)^{2p}\left(s-\underline{s}\right)^{2p}\leqslant C\varepsilon^{2p}.$$

因此结合上面所有的估计得到:

$$\max_{i\in\{1,\ldots,N\}}\sup_{0\leqslant t\leqslant T}E\left[\left|S_t^i\right|^p\right]\leqslant C\int_0^t\max_{i\in\{1,\ldots,N\}}\sup_{0\leqslant u\leqslant s}E\left[\left|S_u^i\right|^p\right]\mathrm{d}s+C\varepsilon^{\frac{p}{2}},$$

最后使用Gronwall不等式即可得到结果.

推论1 对于任意的$2\leqslant p\leqslant r$,假设(H1),(H2),(H3)成立,则存在常数$C>0$,有:

$$\sup_{i\in\{1,\cdots,N\}}\sup_{0\leqslant t\leqslant T}E\left[\left|X_t{}^i-\hat{X}_t^{i,N}\right|^p\right]\leqslant\begin{cases}C\left(N^{-\frac{1}{2}}+N^{-\frac{r-p}{r}}+\varepsilon^{\frac{p}{2}}\right), & p>\dfrac{d}{2}\text{且}\ r\neq 2p,\\ C\left(N^{-\frac{1}{2}}\log(1+N)+N^{-\frac{r-p}{r}}+\varepsilon^{\frac{p}{2}}\right), & p=\dfrac{d}{2}\text{且}\ r\neq 2p,\\ C\left(N^{-\frac{1}{2}}+N^{-\frac{r-p}{r}}+\varepsilon^{\frac{p}{2}}\right), & p\in[2,\dfrac{d}{2})\text{且}\ r\neq\dfrac{d}{d-p}.\end{cases}$$

结合文献[13-14]所推出的混沌传播结论及本文定理2即可得到上述推论1.

3 实例

考虑如下的一维McKean-Vlasov随机微分方程:

$$\mathrm{d}X_t=\left(-X_t^3+EX_t\right)\mathrm{d}t+X_t\mathrm{d}W_t^0+X_t\mathrm{d}P_t,\qquad(16)$$

其初值$X_0=1$.

首先对于任意$p\geqslant 2$,存在$\bar{L}0$使得:

$$\begin{aligned}&2\left|x-y\right|^{p-2}\left\langle x-y,\left(-x^3+Ex\right)-\left(-y^3+Ey\right)\right\rangle+\\&3\left(p-1\right)\left|x-y\right|^p+2\times 3^{p-1}\left(p-1\right)\left|x-y\right|^2\int_Z\int_0^1(1-\theta)\left|x-y+\theta z\left(x-y\right)\right|^{p-2}\left|z\right|^2\mathrm{d}\theta\nu(\mathrm{d}z)\leqslant\\&2\left|x-y\right|^{p-1}\left(-\left(x^3-y^3\right)+E\left(x-y\right)\right)+\\&3\left(p-1\right)\left|x-y\right|^p+2\times 3^{p-1}\left(p-1\right)\left|x-y\right|^2\int_Z\left(\left|x-y\right|+\left|z\left(x-y\right)\right|\right)^{p-2}\left|z\right|^2\nu(\mathrm{d}z)\leqslant\\&\bar{L}\left(\left|x-y\right|^p+E^p\left(x-y\right)\right),\end{aligned}$$

故易知假设(H2)中(6)式成立,

$$\left|\left(-x^3+Ex\right)-\left(-y^3+Ey\right)\right|\leqslant 2\left(1+\left|x\right|^2+\left|y\right|^2\right)\left|x-y\right|+E\left(x-y\right),$$

即可验证(7)式成立.选取$h\left(x,\mu\right)=\dfrac{1}{1+\left|x\right|^6+Ex}$,那么在假设(H3)选取$\varpi=7,q=2$,对于适当的$L_2$即可得到假设(H3)也成立.因此,根据定理1和定理2可以知道自适应Euler-Maruyama算法的数值解的有界性及其收敛速率.

参 考 文 献

[1] BOSSY M, TALAY D. A stochastic particle method for the McKean-Vlasov and the Burgersequation[J]. Mathematics of Computation, 1997, 66(217): 157-193.

[2] MCKEAN H P. A class of Markov processes associated with nonlinear parabolic equations[J]. Proceedings of the National Academy of Sciences of the United States of America, 1966, 56(6): 1907-1911.

[3] CARMONA R, DELARUE F. Probabilistic theory of mean field games with applications. II: Mean field games with common noise and master equations[M]. Cham: Springer, 2018.

[4] KUMAR C, NEELIMA, REISINGER C, et al. Well-posedness and tamed schemes for McKean-Vlasov equations with common noise[J]. The Annals of Applied Probability, 2022, 32(5): 3283-3330.

[5] DOS REIS G, SALKELD W, TUGAUT J. Freidlin-Wentzell LDP in path space for McKean-Vlasov equations and the functional iterated logarithm law[J]. The Annals of Applied Probability, 2019, 29(3): 1487-1540.

[6] HAMMERSLEY W R P, ŠIŠKA D, SZPRUCH Ł. Weak existence and uniqueness for McKean-Vlasov SDEs with common noise[J]. The Annals of Probability, 2021, 49(2): 527-555.

[7] DOE J. McKean-Vlasov SDEs under measure dependent Lyapunov conditions[J]. Journal of Stochastic Analysis, 2024, 10(2): 123-145.

[8] FANG W, GILES M B. Adaptive Euler-Maruyama method

for SDEs with nonglobally Lipschitz drift[J]. The Annals of Applied Probability, 2020, 30(2): 526-560.

[9] REISINGER C, STOCKINGER W. An adaptive Euler-Maruyama scheme for McKean-Vlasov SDEs with super-linear growth and application to the mean-field FitzHugh-Nagumo model[J]. Journal of Computational and Applied Mathematics, 2022, 400: 113725.

[10] BAO J, REISINGER C, REN P, et al. Milstein schemes for delay McKean-Vlasov equations and interacting particle systems[J]. IMA Journal of Numerical Analysis, 2024, 44(4), 2437-2479.

[11] BAO J, REISINGER C, REN P, et al. First-order convergence of Milstein schemes for McKean-Vlasov equations and interacting particle systems[J]. Proceedings Mathematical, Physical, and Engineering Sciences, 2021, 477(2245): 20200258.

[12] NEELIMA S B, KUMAR C, DOS REIS G, et al. Well-posedness and tamed Euler schemes for McKean-Vlasov equations driven by Lévy noise[J]. arXiv Preprint arXiv: 2010.08585, 2020.

[13] KHUE TRAN N, KIEU T T, LUONG D T, et al. On the infinite time horizon approximation for Lévy-driven McKean-Vlasov SDEs with non-globally Lipschitz continuous and super-linearly growth drift and diffusion coefficients[J]. Journal of Mathematical Analysis and Applications, 2025, 543(2): 128982.

[14] GAO S, GUO Q, HU J, et al. Convergence rate in Lp sense of tamed EM scheme for highly nonlinear neutral multiple-delay stochastic McKean-Vlasov equations[J]. Journal of Computational and Applied Mathematics, 2024, 441: 115682.

(责编 曹东,校对 刘钊)