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带幂次记忆项的波动方程耦合系统解的破裂

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摘 要：研究带阻尼项和非线性项的非线性波动方程解的破裂性态以及生命跨度，可以解释生活中不同形式的摩擦现象和非线性外力对波传播过程的影响。本文研究了n维空间中带散射阻尼项和幂次记忆项的波动方程耦合系统的Cauchy问题。首先，通过构造合适的乘子来克服散射阻尼项对波动方程的影响；其次，对幂次记忆项进行了放缩处理；最后，通过构造泛函，利用检验函数方法，以及构造迭代框架，得出小初值问题的解会在有限时间内破裂的结论，同时给出解的生命跨度的上界估计。本文将相关文献中带弱阻尼项的波动方程耦合系统推广为带散射阻尼项的波动方程耦合系统，将带幂次记忆项的单个波动方程推广为带幂次记忆项的波动方程耦合系统，推广了相关文献中带阻尼项和非线性项的单个波动方程解的破裂结果，并给出带阻尼项和非线性项的波动方程耦合系统解的生命跨度的上界估计。

关键词：耦合波动方程；幂次记忆项；迭代方法；破裂；生命跨度估计

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Blow-Up of Solutions to Coupled System of Wave Equations with Power Memory Terms

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Abstract: Studying the blow-up and lifespan estimate of the solution to nonlinear wave equations with damping term and nonlinear terms can explain the effects of different forms of friction phenomena and non-linear external forces on wave propagation in daily life. This paper aims to investigate the Cauchy problem for coupled system of wave equations with scattering damping term and power memory terms in n space dimensions. Firstly, a suitable multiplier was constructed to overcome the influence of scattering damping term on wave equation. Secondly, the power memory terms were scaled down. Finally, by constructing functional, using test function method and constructing iterative framework, we obtained that the solutions of small initial value problem blow up in finite time, and the upper bound lifespan estimate of solutions was estimated. In this paper, the coupled system of wave equations with weak damping term in relevant literature was extended to the coupled system of wave equations with scattering damping term. In addi-

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tion, the single wave equation with power memory term was extended to the coupled system of wave equations with power memory terms. It is generalized that the rupture results of single wave equation solutions with damping term and nonlinear terms in relevant literature. The upper bound lifespan estimation of solutions to coupled wave equation system with damping term and nonlinear terms was provided.
Key words: coupled wave equations; power memory terms; iteration method; blow-up; lifespan estimate

0 引 言

非线性波动方程解的破裂与生命跨度的上界估计是偏微分方程研究的热点之一。Strauss[1]证明了波动方程$u_{tt}-\Delta u=|u|^p$的解具有Strauss临界指数$p_s(n)$。当$n=1$时,$p_s(1)=+\infty$;当$n\geqslant 2$时,$p_s(n)$是方程$r(p,n)=-(n-1)p^2+(n+1)p+2=0$的正根。Glassey[2]研究了带导数非线性项的波动方程$u_{tt}-\Delta u=|u_t|^p$,得出其解满足Glassey临界指数$p_G(n)=1+2/(n-1)$。Lai等[3]研究了带散射阻尼项的波动方程$u_{tt}-\Delta u+\mu(1+t)^{-\beta}u_t=|u|^p$,其中$\beta>1$。通过引入乘子并利用迭代方法,证明了次临界情形解的破裂结果。Lai等[4]还研究了带散射阻尼项和导数非线性项的波动方程$u_{tt}-\Delta u+\mu(1+t)^{-\beta}u_t=|u_t|^p$,通过引入乘子建立了解的生命跨度的上界估计。Lai等[5]又利用迭代方法证明了带散射阻尼项和组合非线性项的波动方程$u_{tt}-\Delta u+\mu(1+t)^{-\beta}u_t=|u_t|^p+|u|^q$不存在整体解,并建立了解的生命跨度的上界估计。Chen[6]研究了带阻尼项和幂次非线性项的波动方程耦合系统$\begin{cases}u_{tt}-\Delta u+u_t=|v|^p\\ v_{tt}-\Delta v=|u|^q\end{cases}$,利用迭代方法得出问题的解会在有限时间内破裂。Chen等[7]研究了带幂次记忆项的波动方程$u_{tt}-\Delta u=N_{\gamma,p}(u)$,其中$N_{\gamma,p}(u)(t,x)=c_\gamma\int_0^t(t-s)^{-\gamma}|u(s,x)|^p\mathrm{d}s$,利用迭代方法给出次临界情形解的破裂。其它相关的研究见文献[8-16]。关于带幂次记忆项的波动方程耦合系统的Cauchy问题解的破裂及其生命跨度估计尚无研究结果。

本文利用迭代方法研究带幂次记忆项的波动方程耦合系统的Cauchy问题(1)解的破裂性态。

$$\begin{cases}u_{tt}-\Delta u+\mu(1+t)^{-\beta}u_t=c_\gamma\int_0^t(t-s)^{-\gamma}|v(s,x)|^p\mathrm{d}s, & x\in\mathbb{R}^n,\ t>0,\\ v_{tt}-\Delta v=c_\gamma\int_0^t(t-s)^{-\gamma}|u(s,x)|^q\mathrm{d}s, & x\in\mathbb{R}^n,\ t>0,\\ (u,u_t,v,v_t)(0,x)=\varepsilon(u_0,u_1,v_0,v_1)(x), & x\in\mathbb{R}^n,\end{cases}\tag{1}$$

式中:$n\geqslant 1$,$1<p,q<+\infty$,$\mu>0$,$\beta>1$,$c_\gamma=1/\Gamma(1-\gamma)(\gamma\in(0,1))$,$\Gamma(s)=\int_0^{+\infty}x^{s-1}\mathrm{e}^{-x}\mathrm{d}x\ (s>0)$为第二类欧拉积分,$\varepsilon$是描述初值大小的正参数。设$(u_0,u_1,v_0,v_1)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$是具有紧支集的非负函数,并且$\mathrm{supp}(u_0,u_1,v_0,v_1)\subset B_R(0)$,其中$B_R(0)=\{x||x|\leqslant R\}$,$R>2$。

1 主要结论

首先给出耦合系统(1)弱解的定义。

定义1 设$(u_0,u_1,v_0,v_1)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$。$(u,v)$是耦合系统(1)在$[0,T)\times\mathbb{R}^n$中的弱解,如果

$$u\in C([0,T),H^1(\mathbb{R}^n))\cap C^1([0,T),L^2(\mathbb{R}^n))\cap L^q_{\mathrm{loc}}([0,T)\times\mathbb{R}^n),$$

$$v\in C([0,T),H^1(\mathbb{R}^n))\cap C^1([0,T),L^2(\mathbb{R}^n))\cap L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n),$$

并且满足

$$\int_0^t\int_{\mathbb{R}^n}(-u_s(s,x)\phi_s(s,x)+\nabla u(s,x)\cdot\nabla\phi(s,x))\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}\mu(1+s)^{-\beta}u_s(s,x)\phi(s,x)\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}u_t(t,x)\phi(t,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_1(x)\phi(0,x)\mathrm{d}x=c_\gamma\int_0^t\int_{\mathbb{R}^n}\phi(s,x)\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,\tag{2}$$

$$\int_0^t\int_{\mathbb{R}^n}(-v_s(s,x)\psi_s(s,x)+\nabla v(s,x)\cdot\nabla\psi(s,x))\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}v_t(t,x)\psi(t,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}v_1(x)\psi(0,x)\mathrm{d}x=c_\gamma\int_0^t\int_{\mathbb{R}^n}\psi(s,x)\int_0^s(s-\tau)^{-\gamma}|u(\tau,x)|^q\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,\tag{3}$$

对于$\forall\phi(t,x),\psi(t,x)\in C_0^\infty([0,T)\times\mathbb{R}^n)$。

对式(2)和式(3)进行分部积分,得到

$$\int_0^t\int_{\mathbb{R}^n}u(s,x)(\phi_{ss}(s,x)-\Delta\phi(s,x)-(\mu(1+s)^{-\beta}\phi(s,x))_s)\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}(u_t(t,x)\phi(t,x)+\mu(1+t)^{-\beta}u(t,x)\phi(t,x)-u(t,x)\phi_t(t,x))\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}(u_1(x)\phi(0,x)+\mu u_0(x)\phi(0,x)-u_0(x)\phi_t(0,x))\mathrm{d}x=c_\gamma\int_0^t\int_{\mathbb{R}^n}\phi(s,x)\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,\tag{4}$$

$$\int_0^t\int_{\mathbb{R}^n}v(s,x)(\psi_{ss}(s,x)-\Delta\psi(s,x))\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}(v_t(t,x)\psi(t,x)-u(t,x)\psi_t(t,x))\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}(v_1(x)\psi(0,x)-v_0(x)\psi_t(0,x))\mathrm{d}x=c_\gamma\int_0^t\int_{\mathbb{R}^n}\psi(s,x)\int_0^s(s-\tau)^{-\gamma}|u(\tau,x)|^q\mathrm{d}\tau\mathrm{d}x\mathrm{d}s。\tag{5}$$

令 $t\to T$, (u,v) 满足耦合系统(1)弱解的定义。

定理 1 当 $n=1,2$ 时, $p,q>1$; 当 $n\geqslant 3$ 时, $1<p,q\leqslant n/(n-2)$。则有

$$\max\left\{\frac{(3-\gamma)(1+p^{-1})}{pq-1}+\frac{2-\gamma}{p},\frac{(3-\gamma)(1+q^{-1})}{pq-1}+\frac{2-\gamma}{q}\right\}>\frac{n-1}{2}。\tag{6}$$

设 (u,v) 是耦合系统(1)的局部弱解。设

$$\mathrm{supp}(u,v)\subset\{(t,x)\in[0,T)\times\mathbb{R}^n||x|\leqslant R+t\},\tag{7}$$

并且存在正常数 $\varepsilon_0=\varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,R,\gamma)$, 使得解 (u,v) 会在有限时间内破裂, 并且解的生命跨度的上界估计满足

$$T(\varepsilon)\leqslant C\varepsilon^{-1/\max\{F_1(n,p,q,\gamma),F_2(n,p,q,\gamma)\}},$$

其中, $\varepsilon\in(0,\varepsilon_0]$, C 是不依赖于 ε 的正常数,

$$F_1(n,p,q,\gamma)=\frac{(3-\gamma)(1+p^{-1})}{pq-1}+\frac{2-\gamma}{p},\tag{8}$$

$$F_2(n,p,q,\gamma)=\frac{(3-\gamma)(1+q^{-1})}{pq-1}+\frac{2-\gamma}{q}。\tag{9}$$

2　定理1的证明

引入下列函数

$$U(t)=\int_{\mathbb{R}^n}u(t,x)\mathrm{d}x,\ V(t)=\int_{\mathbb{R}^n}v(t,x)\mathrm{d}x。$$

令式(2)式和式(3)中的检验函数 ϕ 和 ψ 分别为 $\phi\equiv 1$, $\psi\equiv 1$, 可以得到

$$\int_{\mathbb{R}^n}u_t(t,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_1(x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}^n}\mu(1+s)^{-\beta}u_s(s,x)\mathrm{d}x\mathrm{d}s=c_\gamma\int_0^t\int_{\mathbb{R}^n}\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,\tag{10}$$

$$\int_{\mathbb{R}^n}v_t(t,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}v_1(x)\mathrm{d}x=c_\gamma\int_0^t\int_{\mathbb{R}^n}\int_0^s(s-\tau)^{-\gamma}|u(\tau,x)|^q\mathrm{d}\tau\mathrm{d}x\mathrm{d}s。\tag{11}$$

引入乘子

$$m(t)=\exp\left(\frac{\mu(1+t)^{1-\beta}}{1+\beta}\right)。$$

当 $\beta>1$, $t\geqslant 0$ 时, 可得

$$0<m(0)\leqslant m(t)\leqslant 1。$$

式(10)两边对 t 求导, 得到

$$U''(t)+\mu(1+t)^{-\beta}U'(t)=c_\gamma\int_{\mathbb{R}^n}\int_0^t(t-s)^{-\gamma}|v(s,x)|^p\mathrm{d}x\mathrm{d}s。$$

两边同乘 $m(t)$, 则有

$$\{m(t)U'(t)\}'=c_\gamma m(t)\int_{\mathbb{R}^n}\int_0^t(t-s)^{-\gamma}|v(s,x)|^p\mathrm{d}x\mathrm{d}s。$$

计算得到

$$m(t)U'(t)=m(0)U'(0)+c_\gamma\int_0^t m(s)\int_{\mathbb{R}^n}\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,$$

$$U'(t)=\frac{m(0)U'(0)}{m(t)}+\frac{c_\gamma}{m(t)}\int_0^t m(s)\int_{\mathbb{R}^n}\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s。$$

由于 $m(t)\leqslant 1$, 所以

$$U'(t)\geqslant m(0)U'(0)+c_\gamma m(0)\int_0^t\int_{\mathbb{R}^n}\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\mathrm{d}\tau\mathrm{d}x\mathrm{d}s。$$

计算可得

$$U(t)\geqslant U(0)+m(0)U'(0)t+c_\gamma m(0)\int_0^t\int_0^s\int_{\mathbb{R}^n}\int_0^\tau(\tau-\sigma)^{-\gamma}|v(\sigma,x)|^p\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau\mathrm{d}s,$$

$$V(t)=V(0)+V'(0)t+c_\gamma\int_0^t\int_0^s\int_{\mathbb{R}^n}\int_0^\tau(\tau-\sigma)^{-\gamma}|u(\sigma,x)|^q\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau\mathrm{d}s。$$

由于初值 u_0, u_1, v_0 和 v_1 均非负, 可以得出

$$U(t)\geqslant c_\gamma m(0)\int_0^t\int_0^s\int_{\mathbb{R}^n}\int_0^\tau(\tau-\sigma)^{-\gamma}|v(\sigma,x)|^p\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau\mathrm{d}s,$$

$$(12)$$

$$V(t)\geqslant c_\gamma\int_0^t\int_0^s\int_{\mathbb{R}^n}\int_0^\tau(\tau-\sigma)^{-\gamma}|u(\sigma,x)|^q\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau\mathrm{d}s。\tag{13}$$

利用Holder不等式，则有

$$\int_{\mathbb{R}^n}|v(\sigma,x)|^p\mathrm{d}x\geqslant C_0(R+\sigma)^{-n(p-1)}(V(\sigma))^p,\tag{14}$$

$$\int_{\mathbb{R}^n}|u(\sigma,x)|^q\mathrm{d}x\geqslant\tilde{C}_0(R+\sigma)^{-n(q-1)}(U(\sigma))^q,\tag{15}$$

其中，$C_0=C_0(n,R,p)$, $\tilde{C}_0=\tilde{C}_0(n,R,q)$。

将式(14)和式(15)分别代入式(12)和式(13)可得

$$U(t)\geqslant C_0c_\gamma m(0)\cdot\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{-n(p-1)}(V(\sigma))^p\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s,\tag{16}$$

$$V(t)\geqslant\tilde{C}_0c_\gamma\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{-n(q-1)}(U(\sigma))^q\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s。\tag{17}$$

引入拉普拉斯算子的特征函数$\Phi=\Phi(x)$，即

$$\Phi(x)=\begin{cases}\mathrm{e}^x+\mathrm{e}^{-x}, & n=1,\\ \int_{S^{n-1}}\mathrm{e}^{x\cdot\omega}\mathrm{d}\sigma_\omega, & n\geqslant 2,\end{cases}$$

其中，S^{n-1}是$n-1$维球面，并且满足$\Delta\Phi=\Phi$，且有

$$\Phi(x)\sim|x|^{-(n-1)/2}\mathrm{e}^{|x|}(|x|\to+\infty)。$$

定义检验函数

$$\Psi(t,x)=\mathrm{e}^{-t}\Phi(x),$$

从而Ψ满足波动方程$\Psi_{tt}-\Delta\Psi=0$。定义函数

$$U_1(t)=\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x,$$

$$V_1(t)=\int_{\mathbb{R}^n}v(t,x)\Psi(t,x)\mathrm{d}x。$$

存在$C_1=C_1(u_0,u_1)>0$，$\tilde{C}_1=\tilde{C}_1(v_0,v_1)>0$，对$\forall t\geqslant 0$，则有

$$U_1(t)\geqslant\frac{1-\mathrm{e}^{-2t}}{2}\int_{\mathbb{R}^n}u_0(x)\Phi(x)\mathrm{d}x+\frac{1+\mathrm{e}^{-2t}}{2}\int_{\mathbb{R}^n}u_1(x)\Phi(x)\mathrm{d}x\geqslant C_1\varepsilon,$$

$$V_1(t)\geqslant\frac{1-\mathrm{e}^{-2t}}{2}\int_{\mathbb{R}^n}v_0(x)\Phi(x)\mathrm{d}x+\frac{1+\mathrm{e}^{-2t}}{2}\int_{\mathbb{R}^n}v_1(x)\Phi(x)\mathrm{d}x\geqslant\tilde{C}_1\varepsilon。$$

可得

$$\int_{|x|\leqslant R+t}|\Psi(t,x)|^{p/(p-1)}\mathrm{d}x\leqslant C_2(R+t)^{(n-1)(2-q)/2},$$

$$\int_{|x|\leqslant R+t}|\Psi(t,x)|^{q/(q-1)}\mathrm{d}x\leqslant C_2(R+t)^{(n-1)(2-p)/2},$$

其中，$1/p+1/q=1$，$C_2=C_2(n,R)>0$。利用Holder不等式，得到

$$\int_{\mathbb{R}^n}|u(t,x)|^q\mathrm{d}x\geqslant(U_1(t))^q\left(\int_{|x|\leqslant R+t}|\Psi(t,x)|^{q/(q-1)}\mathrm{d}x\right)^{-(q-1)}\geqslant C_3\varepsilon^q(R+t)^{n-1-(n-1)q/2},\tag{18}$$

$$\int_{\mathbb{R}^n}|v(t,x)|^p\mathrm{d}x\geqslant(V_1(t))^p\left(\int_{|x|\leqslant R+t}|\Psi(t,x)|^{p/(p-1)}\mathrm{d}x\right)^{-(p-1)}\geqslant\tilde{C}_3\varepsilon^p(R+t)^{n-1-(n-1)p/2},\tag{19}$$

其中，$C_3=C_1{}^qC_2{}^{1-q}>0$，$\tilde{C}_3=\tilde{C}_1{}^pC_2{}^{1-p}>0$。

将式(18)和式(19)分别代入式(13)和式(12)，得到

$$\begin{aligned}U(t)&\geqslant\tilde{C}_3m(0)\varepsilon^pc_\gamma\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{n-1-(n-1)p/2}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&\tilde{C}_3m(0)\varepsilon^pc_\gamma(R+t)^{-(n-1)p/2}\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}\sigma^{n-1}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&\tilde{C}_3m(0)\varepsilon^pc_\gamma(R+t)^{-(n-1)p/2}t^{-\gamma}\int_0^t\int_0^s\int_0^\tau\sigma^{n-1}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&\frac{\tilde{C}_3m(0)\varepsilon^pc_\gamma}{n(n+1)(n+2)}(R+t)^{-(n-1)p/2}t^{n-\gamma+2},\end{aligned}\tag{20}$$

$$\begin{aligned}V(t)&\geqslant C_3\varepsilon^qc_\gamma\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{n-1-(n-1)q/2}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&C_3\varepsilon^qc_\gamma(R+t)^{-(n-1)q/2}\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}\sigma^{n-1}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&C_3\varepsilon^qc_\gamma(R+t)^{-(n-1)q/2}t^{-\gamma}\int_0^t\int_0^s\int_0^\tau\sigma^{n-1}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\end{aligned}$$

$$\frac{C_3\varepsilon^q c_\gamma}{n(n+1)(n+2)}(R+t)^{-(n-1)q/2}t^{n-\gamma+2}。\quad(21)$$

对$\forall t\geqslant 0$，假设

$$U(t)\geqslant D_1(R+t)^{-\alpha_1}t^{\beta_1},\quad(22)$$

$$V(t)\geqslant Q_1(R+t)^{-a_1}t^{b_1},\quad(23)$$

其中

$$D_1=\frac{\tilde{C}_3 m(0)\varepsilon^p c_\gamma}{n(n+1)(n+2)},Q_1=\frac{C_3\varepsilon^q c_\gamma}{n(n+1)(n+2)},\quad(24)$$

$$\alpha_1=(n-1)p/2,a_1=(n-1)q/2,\quad(25)$$

$$\beta_1=n-\gamma+2，b_1=n-\gamma+2。\quad(26)$$

设

$$U(t)\geqslant D_j(R+t)^{-\alpha_j}t^{\beta_j},\quad(27)$$

$$V(t)\geqslant Q_j(R+t)^{-a_j}t^{b_j},\quad(28)$$

其中，$\{D_j\}_{j\geqslant 1}$，$\{Q_j\}_{j\geqslant 1}$，$\{\alpha_j\}_{j\geqslant 1}$，$\{\beta_j\}_{j\geqslant 1}$，$\{a_j\}_{j\geqslant 1}$和$\{b_j\}_{j\geqslant 1}$是非负实数序列。将式(27)和式(28)分别代入式(17)和式(16)，可得

$$\begin{aligned}U(t)&\geqslant Q_j^{\,p}C_0m(0)c_\gamma\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{-n(p-1)-pa_j}\sigma^{pb_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&Q_j^{\,p}C_0m(0)c_\gamma(R+t)^{-n(p-1)-pa_j}\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}\sigma^{pb_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&Q_j^{\,p}C_0m(0)c_\gamma(R+t)^{-n(p-1)-pa_j}t^{-\gamma}\int_0^t\int_0^s\int_0^\tau\sigma^{pb_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&\frac{Q_j^{\,p}C_0m(0)c_\gamma}{(pb_j+1)(pb_j+2)(pb_j+3)}(R+t)^{-n(p-1)-pa_j}t^{pb_j+3-\gamma},\end{aligned}\quad(29)$$

$$\begin{aligned}V(t)&\geqslant D_j^{\,q}\tilde{C}_0c_\gamma\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}(R+\sigma)^{-n(q-1)-q\alpha_j}\sigma^{q\beta_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&D_j^{\,q}\tilde{C}_0c_\gamma(R+t)^{-n(q-1)-q\alpha_j}\int_0^t\int_0^s\int_0^\tau(\tau-\sigma)^{-\gamma}\sigma^{q\beta_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&D_j^{\,q}\tilde{C}_0c_\gamma(R+t)^{-n(q-1)-q\alpha_j}t^{-\gamma}\int_0^t\int_0^s\int_0^\tau\sigma^{q\beta_j}\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&\frac{D_j^{\,q}\tilde{C}_0c_\gamma}{(q\beta_j+1)(q\beta_j+2)(q\beta_j+3)}(R+t)^{-n(q-1)-q\alpha_j}t^{q\beta_j+3-\gamma}。\end{aligned}\quad(30)$$

从而

$$D_{j+1}=\frac{Q_j^{\,p}C_0m(0)c_\gamma}{(pb_j+1)(pb_j+2)(pb_j+3)},$$

$$Q_{j+1}=\frac{D_j^{\,q}\tilde{C}_0c_\gamma}{(q\beta_j+1)(q\beta_j+2)(q\beta_j+3)},$$

$$\alpha_{j+1}=n(p-1)+pa_j,$$

$$a_{j+1}=n(q-1)+q\alpha_j,$$

$$\beta_{j+1}=pb_j+3-\gamma,$$

$$b_{j+1}=q\beta_j+3-\gamma。$$

利用式(22)和式(23)，可以得到

$$\begin{aligned}\alpha_j&=n(p-1)+pa_{j-1}=n(pq-1)+pq\alpha_{j-2}=n(pq-1)\sum_{k=0}^{(j-3)/2}(pq)^k+(pq)^{(j-1)/2}\alpha_1=\\&(n+\alpha_1)(pq)^{(j-1)/2}-n=(n+(n-1)p/2)(pq)^{(j-1)/2}-n,\end{aligned}\quad(31)$$

$$\begin{aligned}\beta_j&=pb_{j-1}+3-\gamma=(3-\gamma)(p+1)+pq\beta_{j-2}=(3-\gamma)(p+1)\sum_{k=0}^{(j-3)/2}(pq)^k+(pq)^{(j-1)/2}\beta_1=\\&\left(\frac{(3-\gamma)(p+1)}{pq-1}+\beta_1\right)(pq)^{(j-1)/2}-\frac{(3-\gamma)(p+1)}{pq-1}=\\&\left(\frac{(3-\gamma)(p+1)}{pq-1}+n-\gamma+2\right)(pq)^{(j-1)/2}-\frac{(3-\gamma)(p+1)}{pq-1}。\end{aligned}\quad(32)$$

同理可得

$$a_j=(n+a_1)(pq)^{(j-1)/2}-n=(n+(n-1)q/2)(pq)^{(j-1)/2}-n,\quad(33)$$

$$b_j=\left(\frac{(3-\gamma)(q+1)}{pq-1}+b_1\right)(pq)^{(j-1)/2}-\frac{(3-\gamma)(q+1)}{pq-1}=$$
$$\left(\frac{(3-\gamma)(q+1)}{pq-1}+n-\gamma+2\right)(pq)^{(j-1)/2}-\frac{(3-\gamma)(q+1)}{pq-1},\tag{34}$$

其中，奇数$j\geqslant 3$。当j是偶数，则$j-1$是奇数，利用式(32)和式(34)，可知

$$\beta_j=pb_{j-1}+3-\gamma=q^{-1}\left(\frac{(3-\gamma)(q+1)}{pq-1}+n-\gamma+2\right)(pq)^{j/2}-\frac{(3-\gamma)(p+1)}{pq-1},$$
$$b_j=q\beta_{j-1}+3-\gamma=p^{-1}\left(\frac{(3-\gamma)(p+1)}{pq-1}+n-\gamma+2\right)(pq)^{j/2}-\frac{(3-\gamma)(q+1)}{pq-1}。$$

计算得到

$$\beta_j\leqslant B_0(pq)^{(j-1)/2},\ b_j\leqslant \tilde{B}_0(pq)^{(j-1)/2},(j\text{是奇数}),$$
$$\beta_j\leqslant B_0(pq)^{j/2},\ b_j\leqslant \tilde{B}_0(pq)^{j/2},(j\text{是偶数}),$$

其中，$B_0=B_0(p,q,n,\gamma)$，$\tilde{B}_0=\tilde{B}_0(p,q,n,\gamma)$是不依赖于$j$的正常数。可知

$$(pb_{j-1}+1)(pb_{j-1}+2)(pb_{j-1}+3)\leqslant(pb_{j-1}+2)^3=(\beta_j+\gamma-1)^3\leqslant\beta_j{}^3=B_0{}^3(pq)^{3j/2},$$
$$(q\beta_{j-1}+1)(q\beta_{j-1}+2)(q\beta_{j-1}+3)\leqslant(q\beta_{j-1}+2)^3=(b_j+\gamma-1)^3\leqslant b_j{}^3=\tilde{B}_0{}^3(pq)^{3j/2}。$$

从而

$$D_j=\frac{Q_{j-1}{}^pC_0m(0)c_\gamma}{(pb_{j-1}+1)(pb_{j-1}+2)(pb_{j-1}+3)}\leqslant\frac{Q_{j-1}{}^pC_0m(0)c_\gamma}{B_0{}^3}(pq)^{-3j/2},$$
$$Q_j=\frac{D_{j-1}{}^q\tilde{C}_0c_\gamma}{(q\beta_{j-1}+1)(q\beta_{j-1}+2)(q\beta_{j-1}+3)}\leqslant\frac{D_{j-1}{}^q\tilde{C}_0c_\gamma}{\tilde{B}_0{}^3}(pq)^{-3j/2}。$$

计算可得

$$D_j\leqslant\frac{m(0)\tilde{C}_0{}^pC_0c_\gamma{}^{p+1}}{\tilde{B}_0{}^{3p}B_0{}^3}(pq)^{-3j(p+1)/2+3p/2}D_{j-2}^{pq}=$$
$$E_0(pq)^{-3j(p+1)/2}D_{j-2}^{pq},\tag{35}$$

$$Q_j\leqslant\frac{m(0)^qC_0{}^q\tilde{C}_0c_\gamma{}^{q+1}}{B_0{}^{3q}\tilde{B}_0{}^3}(pq)^{-3j(q+1)/2+3q/2}Q_{j-2}^{pq}=$$
$$\tilde{E}_0(pq)^{-3j(q+1)/2}Q_{j-2}^{pq},\tag{36}$$

其中，E_0和$\tilde{E}_0$是不依赖于j的正常数。得到

$$\log D_j\geqslant(pq)\log D_{j-2}-\frac{3(p+1)}{2}j\log(pq)+\log E_0\geqslant\cdots\geqslant$$
$$(pq)^{(j-1)/2}\log D_1-\frac{3(p+1)}{2}\log(pq)\sum_{k=0}^{(j-3)/2}((j-2k)(pq)^k)+\log E_0\sum_{k=0}^{(j-3)/2}(pq)^k,$$

其中，j为奇数，满足$j\geqslant 3$。计算可得

$$\sum_{k=0}^{(j-3)/2}((j-2k)(pq)^k)=\frac{2+(3pq-1)}{(pq-1)^2}(pq)^{(j-1)/2}-\frac{2pq+j(pq-1)}{(pq-1)^2}。$$

因此，有

$$\log D_j\geqslant(pq)^{(j-1)/2}\left(\log D_1-\frac{3(p+1)(3pq-1)\log(pq)}{2(pq-1)^2}+\frac{\log E_0}{pq-1}\right)+$$
$$\frac{3(p+1)(2pq+j(pq-1))\log(pq)}{2(pq-1)^2}-\frac{\log E_0}{pq-1}。$$

同理可知

$$\log Q_j\geqslant(pq)^{(j-1)/2}\left(\log Q_1-\frac{3(q+1)(3pq-1)\log(pq)}{2(pq-1)^2}+\frac{\log\tilde{E}_0}{pq-1}\right)+$$
$$\frac{3(q+1)(2pq+j(pq-1))\log(pq)}{2(pq-1)^2}-\frac{\log\tilde{E}_0}{pq-1}。$$

因此，对于所有奇数j，当

$$j\geqslant j_0=\left[\frac{2}{3\log(pq)}\max\left\{\frac{\log E_0}{p+1},\frac{\log\tilde{E}_0}{q+1}\right\}-\frac{2pq}{pq-1}\right],$$

则有

$$\log D_j \geqslant (pq)^{(j-1)/2} \log(D_1 (pq)^{-3(p+1)(3pq-1)/2(pq-1)^2} E_0^{\ 1/(pq-1)}) = (pq)^{(j-1)/2} \log(E_1 \varepsilon^p), \tag{37}$$

$$\log Q_j \geqslant (pq)^{(j-1)/2} \log(Q_1 (pq)^{-3(q+1)(3pq-1)/2(pq-1)^2} \tilde{E}_0^{\ 1/(pq-1)}) = (pq)^{(j-1)/2} \log(\tilde{E}_1 \varepsilon^q), \tag{38}$$

其中，E_1和$\tilde{E}_1$是不依赖于j的正常数。对于任意奇数$j \geqslant j_0$，结合式(27)，式(31)，式(32)和式(37)，可以推出

$$U(t) \geqslant \exp((pq)^{(j-1)/2} \log(E_1 \varepsilon^p))(R+t)^{-\alpha_j} t^{\beta_j} \geqslant$$
$$\exp\left((pq)^{(j-1)/2}\left(\log(E_1\varepsilon^p) - \left(\frac{(n-1)p}{2}+n\right)\log(R+t) + \left(\frac{(3-\gamma)(p+1)}{pq-1}+n-\gamma+2\right)\log t\right)\right)(R+t)^n t^{-(3-\gamma)(p+1)/(pq-1)}。$$

结合式(28)，式(33)，式(34)和式(38)，可得

$$V(t) \geqslant \exp((pq)^{(j-1)/2} \log(\tilde{E}_1 \varepsilon^q))(R+t)^{-a_j} t^{b_j} \geqslant$$
$$\exp\left((pq)^{(j-1)/2}\left(\log(\tilde{E}_1\varepsilon^q) - \left(\frac{(n-1)q}{2}+n\right)\log(R+t) + \left(\frac{(3-\gamma)(q+1)}{pq-1}+n-\gamma+2\right)\log t\right)\right)(R+t)^n t^{-(3-\gamma)(q+1)/(pq-1)}。$$

当 $t \geqslant R$ 时，则有 $\log(R+t) \leqslant \log(2t)$。可知

$$U(t) \geqslant \exp((pq)^{(j-1)/2} \log(E_1 \varepsilon^p 2^{-(n-1)p/2-n} t^{-(n-1)p/2+(3-\gamma)(p+1)/(pq-1)+2-\gamma}))(R+t)^n t^{-(3-\gamma)(p+1)/(pq-1)}, \tag{39}$$

$$V(t) \geqslant \exp((pq)^{(j-1)/2} \log(\tilde{E}_1 \varepsilon^q 2^{-(n-1)q/2-n} t^{-(n-1)q/2+(3-\gamma)(q+1)/(pq-1)+2-\gamma}))(R+t)^n t^{-(3-\gamma)(q+1)/(pq-1)}。 \tag{40}$$

由于

$$-\frac{(n-1)p}{2}+\frac{(3-\gamma)(p+1)}{pq-1}+2-\gamma = p\left(\frac{(3-\gamma)(p^{-1}+1)}{pq-1}+\frac{2-\gamma}{p}-\frac{n-1}{2}\right) = pF_1(n,p,q,\gamma),$$

$$-\frac{(n-1)q}{2}+\frac{(3-\gamma)(q+1)}{pq-1}+2-\gamma = q\left(\frac{(3-\gamma)(q^{-1}+1)}{pq-1}+\frac{2-\gamma}{q}-\frac{n-1}{2}\right) = qF_2(n,p,q,\gamma)。$$

根据假设条件(6)，当$F_1(n,p,q,\gamma) > 0$，$F_2(n,p,q,\gamma) > 0$时，引入$\varepsilon_0 = \varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,R,\gamma) > 0$，则有

$$(E_1^{-1} 2^{(n-1)p/2+n})^{1/(pF_1(n,p,q,\gamma))} = E_2 \geqslant \varepsilon_0^{\ 1/F_1(n,p,q,\gamma)},$$
$$(\tilde{E}_1^{-1} 2^{(n-1)q/2+n})^{1/(qF_2(n,p,q,\gamma))} = \tilde{E}_2 \geqslant \varepsilon_0^{\ 1/F_2(n,p,q,\gamma)}。$$

因此，当$\varepsilon \in (0,\varepsilon_0]$，$t > E_2 \varepsilon_0^{\ -1/F_1(n,p,q,\gamma)}$，$t > \tilde{E}_2 \varepsilon_0^{\ -1/F_2(n,p,q,\gamma)}$时，在式(39)和式(40)中令$j$趋于$+\infty$，可知解的生命跨度满足

$$T(\varepsilon) \leqslant C\varepsilon^{-1/\max\{F_1(n,p,q,\gamma),F_2(n,p,q,\gamma)\}}。$$

证毕。

3 结　论

本文研究了带幂次记忆项的波动方程耦合系统的Cauchy问题。本文将文献[6]中研究的带阻尼项的波动方程耦合系统推广为带散射阻尼项的波动方程耦合系统，将文献[7]中研究的带幂次记忆项的单个波动方程推广为带幂次记忆项的波动方程耦合系统。通过构造合适的乘子克服了散射阻尼的影响。利用检验函数方法和迭代方法证明了问题的解会在有限时间内破裂，进而给出解的生命跨度的上界估计。

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