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二阶随机扰动下 HBV 流行病模型全局正解的存在唯一性与平稳分布

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摘　要：提出一个二阶随机扰动下的乙型肝炎病毒（HBV）感染模型，并在垂直传播比例 $\mu\omega\nu C$ 不被认为是新感染的条件下研究了该模型的动力学行为。首先，证明了该随机模型全局正解的存在唯一性。其次，通过构造合适的 Lyapunov 函数，证明了在 $R_0^S > 1$ 的情况下，该随机模型平稳分布的存在唯一性和遍历性，并且得到了一个当噪声等于 0 时与基本再生数 R_0 相等的随机临界值 R_0^S。

关键词：乙型肝炎病毒模型；随机干扰；平稳分布
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Existence and uniqueness of the global positive solution and the stationary distribution of HBV epidemic model with second-order stochastic perturbation

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Abstract: This study proposes a model of hepatitis B virus infection with second-order stochastic perturbation and analyzes the dynamic behavior of the model when the fraction of vertical transmission $\mu\omega\nu C$ is not taken into account as a new infection. Firstly, the existence and uniqueness of a global positive solution to the stochastic model are verified. Secondly, by creating a suitable Lyapunov function, this study proves that if $R_0^S > 1$, there exists an ergodic stationary distribution of the stochastic HBV model, and a random critical value R_0^S is derived that is equal to the basic reproduction number R_0 when the noise is equal to zero.

Key words: HBV model; stochastic perturbation; stationary distribution

乙型肝炎病毒（HBV）是一种小的包膜 DNA 病毒，主要感染肝细胞，引起急性和持续性肝病[1]。在世界范围内，乙型肝炎病毒是导致人类肝脏慢性反射的肝炎病毒中最常见的一种。尽管有有效和安全的乙肝疫苗，慢性乙肝病毒感染仍然是世界范围内肝细胞癌（HCC）的主要病因，超过一半的 HCC 患者是慢性携带者[2]。HBV 已被国际癌症研究机构归类为对人类的致癌物，是全球的一个主要公共卫生问题。乙肝

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病毒对人类的安全造成了巨大威胁，预防和控制乙肝病毒对人类的传染就显得尤为重要。

对传染病建立数学模型是传染病定性和定量分析的重要研究方法[3]。在乙肝感染的传播中，有三个不同的阶段，即急性感染、慢性感染和携带者。ZHANG 等[4]考虑了 HBV 携带者在分娩期间和出生后的垂直传播，并将整个人群分为五类：易感(S)、暴露(E)、急性感染(I)、乙型肝炎病毒携带者(C)和免疫接种(R)。在这五类中，假设只有急性感染和携带者可以传播疾病，他们提出了以下模型：

$$\left.\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \mu\omega(1-\nu C)-\beta(I+\alpha C)S-\mu S, \\ \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta(I+\alpha C)S-(\sigma+\mu)E, \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \sigma E-(\gamma_1+\mu)I, \\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} &= q\gamma_1 I-(\gamma_2+\mu)C, \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= (1-q)\gamma_1 I+\gamma_2 C-\mu R+\mu(1-\omega)。 \end{aligned}\right\} \tag{1}$$

式中：$S(t)$、$E(t)$、$I(t)$、$C(t)$、$R(t)$ 分别为在 t 时刻的易感个体、暴露个体、急性感染个体、病毒携带者个体和免疫个体的数量；ω 为出生时未接种疫苗的比例；β 为疾病传播系数；μ 为自然死亡率；ν 为乙肝病毒携带者母亲所生儿童发展为乙肝病毒携带者的比例；α 为与急性感染相比携带者的传染性；γ_1 为从暴露到急性感染的转化率；γ_2 为从带菌者到被免疫者的比率；q 为从急性感染转移到带菌者的比率；σ 为从暴露感染转为急性感染的比率。

ZHANG 等[4]得到了模型(1)的基本再生数 R_0 为 $\dfrac{\beta\sigma\omega(\alpha q\gamma_1+\mu+\gamma_2-\mu\omega\nu)}{(\mu+\sigma)(\mu+\gamma_1)(\mu+\gamma_2-\mu\omega\nu)}$，并且，当 $R_0 \leqslant 1$ 时，模型(1)存在一个全局渐近稳定的无病平衡点 $P_0(\omega,0,0,0)$；当 $R_0>1$ 时，模型(1)有一个局部渐近稳定的地方病平衡点 $P^*(S^*,E^*,I^*,C^*)$。如果 $R_0>1$ 且 $\nu=0$，那么平衡点 $P^*(S^*,E^*,I^*,C^*)$ 也是全局渐近稳定的，其中，$S^*=\dfrac{\omega}{R_0}$，$E^*=\dfrac{\mu+\gamma_1}{\sigma}I^*$，$I^*=\left(1-\dfrac{1}{R_0}\right)\dfrac{\omega\mu\sigma(\mu+\gamma_2-\mu\omega\nu)}{\mu\omega\nu\sigma q\gamma_1+(\mu+\sigma)(\mu+\gamma_1)(\mu+\gamma_2-\mu\omega\nu)}$，$C^*=\dfrac{q\gamma_1}{\mu+\gamma_2-\mu\omega\nu}I^*$。

自然界中各种各样的环境变化，如疾病传播、人与人之间的接触、人与环境之间的相互作用、疫苗接种效率，本质上都受到一系列连续的干扰[5]。由于环境干扰的存在，用确定性模型对系统进行描述和预测是不符合实际情况的，因此，为了更好拟合数据，有必要将确定性模型推广到随机的情况[6]。白噪声的相关形式分为两种，分别是线性和非线性扰动[7]。线性扰动是最简单和最直观的假设，由随机波动的恒定方差性质建立。GE 等在 HBV 确定性模型的基础上，建立了预防围产期传播的随机 HBV 感染模型并得到了相关的动力学行为[8]。与经典的线性扰动类型相比，环境波动可能取决于物种相互作用或流行病传播中每个种群的强度，这被称为非线性扰动[9]。LIU 等得到了一个具有二阶扰动的随机 SIR 模型的唯一平稳分布的存在性[10]。LU 等[11]研究了二阶随机扰动下具有饱和发病率的 SEIQV 模型的动力学。本文研究相应的具有非线性扰动的随机模型。

由于研究的是模型(1)的传播动力学特性，$R(t)$ 就不做考虑。对于 $\nu>0$ 的情况，$S(t)$ 中 $\mu\omega(1-\nu C)$ 这项可能为负，因为 $C(t)$ 是一个可能大于 $1/\nu$ 的随机过程，在这种情况下，$\nu>0$ 没有明确的生物学意义。本文研究的是当 $\nu=0$ 的情况，具体的非线性随机模型如下：

$$\left.\begin{aligned} \mathrm{d}S(t) &= [\mu\omega-\beta(I+\alpha C)S-\mu S]\,\mathrm{d}t+(\sigma_{11}+\sigma_{12}S)S\,\mathrm{d}B_1(t), \\ \mathrm{d}E(t) &= [\beta(I+\alpha C)S-(\sigma+\mu)E]\,\mathrm{d}t+(\sigma_{21}+\sigma_{22}E)E\,\mathrm{d}B_2(t), \\ \mathrm{d}I(t) &= [\sigma E-(\gamma_1+\mu)I]\,\mathrm{d}t+(\sigma_{31}+\sigma_{32}I)I\,\mathrm{d}B_3(t), \\ \mathrm{d}C(t) &= [q\gamma_1 I-(\gamma_2+\mu)C]\,\mathrm{d}t+(\sigma_{41}+\sigma_{42}C)C\,\mathrm{d}B_4(t)。 \end{aligned}\right\} \tag{2}$$

式中：$\{B_i(t)\}_{t\geqslant 0}$ 为相互独立的标准布朗运动；σ_{ij} 为噪声强度；$i=1,\cdots,4$；$j=1,2$。

1　基本知识

设 $(\Omega, F, \{F_t\}_{t\geqslant 0}, \mathbb{P})$ 是一个具有过滤 $\{F_t\}_{t\geqslant 0}$ 满足条件的完全概率空间，而 $B_i(t)$ 是定义在该完全概率空间上的相互独立的标准布朗运动[12]。$\mathbb{R}^n$ 代表一个标准的 n 维欧几里德空间，$\mathbb{R}_+^n=\{(x_1,\ldots,x_n)\mid x_i>0, 1\leqslant i\leqslant n\}$，即 $\mathbb{R}_+^d=\{x=(x_1,\ldots,x_d)\in\mathbb{R}^d\mid x_i>0, 1\leqslant i\leqslant d\}$。

引理 1　对于任意 $x\geqslant 0$，存在整数 $m\geqslant 1$，使下列不等式成立[13]：

$$x^{m+2}\geqslant\frac{1}{2m}[(m+1)x^m-1](x^2+1)。\tag{3}$$

当且仅当 $x=1$ 时不等式等号成立。引理给出的证明如下：

$$x^{m+2}-\frac{1}{2m}[(m+1)x^m-1](x^2+1)=\frac{(x-1)^2}{2m}\left[1+(m-1)x^m+2\sum_{p=1}^{m-1}(m-p)x^{m-p}\right]\geqslant 0。$$

2　模型全局正解的存在唯一性

在实际应用中，需要流行病模型的每个子种群都是非负的，以便进行进一步的动力学研究。因此，研究模型(2)全局正解的存在唯一性，这是随机模型长期行为的必要条件。

定理 1　对于任意给定初值 $[S(0),E(0),I(0),C(0)]\in\mathbb{R}_+^4$，在 $t\geqslant 0$ 时模型(2)几乎必然存在唯一的正解 $[S(t),E(t),I(t),C(t)]$，并且将以概率 1 保持在 $\mathbb{R}_+^4$。

证明　因为模型(2)的系数满足 Lipschitz 条件，所以对于任何初值 $[S(0),E(0),I(0),C(0)]\in\mathbb{R}_+^4$，模型(2)有唯一的局部解 $[S(t),E(t),I(t),C(t)]$，$t\in[0,\tau_e)$，其中 τ_e 为爆破时间。然后证明该解是全局的，即 $\tau_e=+\infty$ a. e. 。本文参考 MAO 等[14]对定理 1 的证明方法，采用类似的方法给予证明。证明该解是全局的关键步骤是构造一个非负的 C^2 函数 $V(S,E,I,C):\mathbb{R}_+^4\to\mathbb{R}_+$，使得

$$\lim_{n\to\infty,(S,E,I,C)\in\mathbb{R}_+^4\setminus\Pi_n}\inf V(S,E,I,C)=+\infty \text{ 且 } LV(S,E,I,C)\leqslant M,$$

其中，L 算子的描述能在文献[15]中的 Appendix C 找到，$\Pi_n=\left(\frac{1}{n},n\right)\times\left(\frac{1}{n},n\right)\times\left(\frac{1}{n},n\right)\times\left(\frac{1}{n},n\right)$，且 M 为某个正数，定义 $V(S,E,I,C)$ 如下：

$$V(S,E,I,C)=V_1+V_2+V_3+V_4,\ (S,E,I,C)\in\mathbb{R}_+^4。$$

式中：$V_1=\frac{(\sigma_{11}+\sigma_{12}S)^p}{p}-\ln S$；$V_2=\frac{(\sigma_{21}+\sigma_{22}E)^p}{p}-\ln E$；$V_3=\frac{(\sigma_{31}+\sigma_{32}I)^p}{p}-\ln I$；$V_4=\frac{(\sigma_{41}+\sigma_{42}C)^p}{p}-\ln C$；$p$ 为一个很小的常数，$p\in(0,1)$。

对 V_1 运用 Itô 公式，可以得到

$$\begin{aligned}V_1&=\sigma_{12}(\sigma_{11}+\sigma_{12}S)^{p-1}[\mu\omega-\beta(I+\alpha C)S-\mu S]-\frac{\sigma_{12}^2}{2}(1-p)(\sigma_{11}+\sigma_{12}S)^pS^2-\\&\quad\frac{\mu\omega}{S}+\beta(I+\alpha C)+\mu+\frac{\sigma_{11}^2}{2}+\sigma_{11}\sigma_{12}S+\frac{\sigma_{12}^2}{2}S^2\\&\leqslant\sigma_{12}\sigma_{11}^{p-1}\mu\omega-\frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2}+\beta(I+\alpha C)+\mu+\frac{\sigma_{11}^2}{2}+\sigma_{11}\sigma_{12}S+\frac{\sigma_{12}^2}{2}S^2。\end{aligned}\tag{4}$$

利用同样的方法，可以得到

$$V_2\leqslant\sigma_{22}\sigma_{21}^{p-1}[\beta(I+\alpha C)S]-\frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}+\sigma+\mu+\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2。\tag{5}$$

$$V_3\leqslant\sigma_{32}\sigma_{31}^{p-1}\sigma E-\frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}+\gamma_1+\mu+\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2。\tag{6}$$

$$V_4\leqslant\sigma_{42}\sigma_{41}^{p-1}q\gamma_1I-\frac{1-p}{2}\sigma_{42}^{p+2}C^{p+2}+\gamma_2+\mu+\frac{\sigma_{41}^2}{2}+\sigma_{41}\sigma_{42}C+\frac{\sigma_{41}^2}{2}C^2。\tag{7}$$

最后得到

$$LV(S,E,I,C)\leqslant\sigma_{12}\sigma_{11}^{p-1}\mu\omega+4\mu+\sigma+\gamma_1+\gamma_2+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}+\frac{\sigma_{41}^2}{2}+\sigma_{11}\sigma_{12}S+$$
$$(\beta+\sigma_{31}\sigma_{32}+\sigma_{42}\sigma_{41}^{p-1}q\gamma_1)I+(\alpha\beta+\sigma_{41}\sigma_{42})C+(\sigma_{21}\sigma_{22}+\sigma_{32}\sigma_{31}^{p-1}\sigma)E+$$
$$\sigma_{22}\sigma_{21}^{p-1}(\beta SI+\alpha\beta SC)+\frac{\sigma_{12}^2}{2}S^2+\frac{\sigma_{22}^2}{2}E^2+\frac{\sigma_{32}^2}{2}I^2+\frac{\sigma_{41}^2}{2}C^2-\frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2}- \tag{8}$$
$$\frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}-\frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}-\frac{1-p}{2}\sigma_{42}^{p+2}C^{p+2}:=M。$$

式中：M 为正的常数。

剩下的证明与文献[14]中的定理1大致相同，在此省略。

3　平稳分布的存在性和唯一性

设 $X(t)$ 是由随机微分方程描述的 d 维欧几里得空间 $\mathbb{R}^d$ 中的一个自治 Markov 过程，

$$\mathrm{d}X(t)=f[X(t)]\mathrm{d}t+\sum_{r=1}^{k}g_r[X(t)]\mathrm{d}B_r(t)。$$

其扩散矩阵 $\boldsymbol{A}$ 定义如下：

$$\boldsymbol{A}(x)=[a_{ij}(x)],a_{ij}(x)=\sum_{r=1}^{k}g_r^i(x)g_r^j(x)。$$

引理2　如果存在一个具有正则边界 Γ 的有界区域 $U\subset\mathbb{R}^d$，Markov 过程 $X(t)$ 具有唯一的遍历平稳分布 $\pi(.)$，并具有以下特性：

(i) 扩散矩阵 $\boldsymbol{A}(x)$ 对于所有的 $X\in U$ 是严格正定的。

(ii) 存在一个非负的 C^2-V 函数使得 LV 对于任何 $\mathbb{R}^d\backslash U$ 都是负的。

定理2　假设 $R_0^S:=$

$$\frac{\mu\omega\sigma\beta\left(\alpha q\gamma_1+\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\mu\omega\sigma_{11}\sigma_{12}}+2\sqrt[3]{\mu^2\omega^2\sigma_{12}^2}\right)\left(\mu+\sigma+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\mu^2\omega^2\sigma_{22}^2}\right)\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)\left(\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)}>$$

1，对于给定任意初值 $[S(0),E(0),I(0),C(0)]\in\mathbb{R}_+^4$，模型(2)具有唯一的平稳分布，并且是遍历的。

证明　为了证明定理2，首先需要验证引理2中的条件(ⅰ)。模型(2)的扩散矩阵如下：

$$\boldsymbol{A}=\begin{bmatrix}(\sigma_{11}+\sigma_{12}S)^2S^2 & 0 & 0 & 0\\ 0 & (\sigma_{21}+\sigma_{22}E)^2E^2 & 0 & 0\\ 0 & 0 & (\sigma_{31}+\sigma_{32}I)^2I^2 & 0\\ 0 & 0 & 0 & (\sigma_{41}+\sigma_{42}C)^2C^2\end{bmatrix},$$

显然扩散矩阵 $\boldsymbol{A}$ 对于任何 $\mathbb{R}_+^4$ 中的子集都是正定的，条件(ⅰ)证明完毕。

其次，对于条件(ⅱ)的证明。给定一个很小的常数 $p\in(0,1)$，然后给出如下定义：

$$R_0^S(p)=\frac{\mu\omega\sigma\beta\left(\alpha q\gamma_1+\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)}{\left(\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right)\left(\mu+\sigma+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right)\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)\left(\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)},$$

不难得到 $\lim\limits_{p\to0^+}R_0^S(p)=R_0^S$，由于函数 $R_0^S(p)$ 的连续性且 $R_0^S>1$，因此，$R_0^S(p)>1$ 是成立的。

定义

$$V_1(S)=(a_1+b_1)\sum_{i=1}^{2}\frac{c_i(S+d_i)^p}{p},U_1(S,C)=-(a_1+b_1)\ln S+V_1(S)+\frac{(a_1+b_1)\alpha\beta}{\gamma_2+\mu}C;$$

$$V_2(S,E)=k_1S+\frac{k_2(E+k_3)^p}{p},U_2(S,E,I)=-\ln E+V_2(S,E)-\frac{\sigma_{21}\sigma_{22}}{\sigma}I;$$

$U_3(I)=-(a_2+b_2)\ln I+\frac{(a_2+b_2)k_4(\sigma_{31}+\sigma_{32}I)^p}{p}-(a_2+b_2)k_4\sigma_{32}\sigma_{31}^{p-1}I$；

$U_4(C)=-b_3\ln C+\frac{b_3k_5(\sigma_{41}+\sigma_{42}C)^p}{p}+\frac{b_3\sigma_{41}\sigma_{42}}{\gamma_2+\mu}C$；$U_5(S,E,I,C)=U_1(S,C)+U_2(S,E,I)+U_3(I)+U_4(C)-\frac{C}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]$；

$U_6(S,E,I,C)=\frac{(\sigma_{11}+\sigma_{12}S)^p}{p}+\frac{(\sigma_{21}+\sigma_{22}E)^p}{p}+\frac{(\sigma_{31}+\sigma_{32}I)^p}{p}+\frac{(\sigma_{41}+\sigma_{42}C)^p}{p}$。

式中：a_1、a_2、b_1、b_2、b_3、c_1、c_2、d_1、d_2、k_1、k_2、k_3、k_4、k_5 均为正的常数，后面会具体给出。

由 Itô 公式可得

$$
\begin{aligned}
LV_1&=(a_1+b_1)\sum_{i=1}^{2}\left\{c_i(S+d_i)^{p-1}[\mu\omega-\beta(I+\alpha C)S-\mu S]-\frac{c_i(1-p)}{2(S+d_i)^{2-p}}(\sigma_{11}+\sigma_{12}S)^2S^2\right\}\\
&\leqslant(a_1+b_1)\left[\sum_{i=1}^{2}\frac{c_i\mu\omega}{d_i^{1-p}}-\frac{c_1(1-p)d_1^{p-2}\sigma_{12}^2S^4}{2\left(\frac{S}{d_1}+1\right)^{2-p}}-\frac{c_2(1-p)d_2^{p-2}\sigma_{11}\sigma_{12}S^3}{\left(\frac{S}{d_2}+1\right)^{2-p}}\right]\\
&\leqslant(a_1+b_1)\left[\sum_{i=1}^{2}\frac{c_i\mu\omega}{d_i^{1-p}}-\frac{c_1(1-p)d_1^{p+2}\sigma_{12}^2\left(\frac{S}{d_1}\right)^4}{2\left(\frac{S}{d_1}+1\right)^2}-\frac{c_2(1-p)d_2^{p+1}\sigma_{11}\sigma_{12}\left(\frac{S}{d_2}\right)^3}{\left(\frac{S}{d_2}+1\right)^2}\right]\\
&\leqslant(a_1+b_1)\left\{\sum_{i=1}^{2}\frac{c_i\mu\omega}{d_i^{1-p}}-\frac{c_1(1-p)d_1^{p+2}\sigma_{12}^2}{4}\left[\frac{3}{4}\left(\frac{S}{d_1}\right)^2-\frac{1}{4}\right]-\frac{c_2(1-p)d_2^{p+1}\sigma_{11}\sigma_{12}}{2}\left(\frac{S}{d_2}-\frac{1}{2}\right)\right\}\\
&\leqslant(a_1+b_1)\left[\frac{c_1\mu\omega}{d_1^{1-p}}+\frac{c_1(1-p)d_1^{p+2}\sigma_{12}^2}{16}\right]+(a_1+b_1)\left[\frac{c_2\mu\omega}{d_2^{1-p}}+\frac{c_2(1-p)d_2^{p+1}\sigma_{11}\sigma_{12}}{4}\right]-\\
&\quad\frac{3(a_1+b_1)c_1(1-p)d_1^p\sigma_{12}^2}{16}\times S^2-\frac{(a_1+b_1)c_2(1-p)d_2^p\sigma_{11}\sigma_{12}}{2}S,
\end{aligned}
$$

上述不等式变换使用了式(3)的不等式关系，可以得到

$$
c_1=\frac{8}{3(1-p)d_1^p},c_2=\frac{2}{(1-p)d_2^p};d_1=2\sqrt[3]{\frac{\mu\omega}{(1-p)\sigma_{12}^2}},d_2=2\sqrt{\frac{\mu\omega}{(1-p)\sigma_{11}\sigma_{12}}}。
$$

然后，可以得到

$$
LV_1\leqslant2(a_1+b_1)\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2(a_1+b_1)\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}-(a_1+b_1)\sigma_{11}\sigma_{12}S-\frac{(a_1+b_1)\sigma_{12}^2}{2}S^2, \tag{9}
$$

因此，

$$
\begin{aligned}
LU_1&\leqslant(a_1+b_1)\leqslant\left[-\frac{\mu\omega}{S}+\beta(I+\alpha C)+\mu\right]+(a_1+b_1)\left(\frac{\sigma_{11}^2}{2}+\sigma_{11}\sigma_{12}S+\frac{\sigma_{12}^2}{2}S^2\right)+2(a_1+b_1)\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+\\
&\quad2(a_1+b_1)\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}-(a_1+b_1)\sigma_{11}\sigma_{12}S-\frac{(a_1+b_1)\sigma_{12}^2}{2}S^2+\frac{(a_1+b_1)\alpha\beta}{\gamma_2+\mu}[q\gamma_1I-(\gamma_2+\mu)C]\\
&\leqslant-\frac{(a_1+b_1)\mu\omega}{S}+(a_1+b_1)\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]+(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)I。
\end{aligned} \tag{10}
$$

对 V_2 使用 Itô 公式可得

$$
\begin{aligned}
LV_2&=k_1[\mu\omega-\beta(I+\alpha C)S-\mu S]+k_2(E+k_3)^{p-1}[\beta(I+\alpha C)S-(\sigma+\mu)E]-\frac{k_2(1-p)(\sigma_{21}+\sigma_{22}E)^2E^2}{2(E+k_3)^{2-p}}\\
&\leqslant k_1\mu\omega+(k_2k_3^{p-1}-k_1)[\beta(I+\alpha C)S]-\frac{k_2(1-p)k_3^{p-2}\sigma_{22}^2E^4}{2\left(\frac{E}{k_3}+1\right)^{2-p}}
\end{aligned}
$$

$$\leqslant k_1\mu\omega+(k_2k_3^{p-1}-k_1)[\beta(I+\alpha C)S]-\frac{k_2(1-p)k_3^{p-2}\sigma_{22}^2E^4}{2\left(\frac{E}{k_3}+1\right)^2}$$

$$\leqslant k_1\mu\omega+(k_2k_3^{p-1}-k_1)[\beta(I+\alpha C)S]-\frac{k_2(1-p)k_3^{p+2}\sigma_{22}^2\left(\frac{E}{k_3}\right)^4}{4\left[\left(\frac{E}{k_3}\right)^2+1\right]}$$

$$\leqslant k_1\mu\omega+(k_2k_3^{p-1}-k_1)[\beta(I+\alpha C)S]-\frac{k_2(1-p)k_3^{p+2}\sigma_{22}^2}{4}\left[\frac{3}{4}\left(\frac{E}{k_3}\right)^2-\frac{1}{4}\right]$$

$$\leqslant k_1\mu\omega+(k_2k_3^{p-1}-k_1)[\beta(I+\alpha C)S]-\frac{3k_2(1-p)k_3^{p}\sigma_{22}^2}{16}E^2+\frac{k_2(1-p)k_3^{p+2}\sigma_{22}^2}{16},$$

上述不等式变换同样使用了式(3)的不等式关系,可以得到

$$k_1=k_2k_3^{p-1},k_2=\frac{8}{3(1-p)k_3^p},k_3=2\sqrt[3]{\frac{\mu\omega}{(1-p)\sigma_{22}^2}}。$$

可以得到

$$LV_2\leqslant 2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}-\frac{\sigma_{22}^2}{2}E^2, \tag{11}$$

因此,

$$LU_2\leqslant-\frac{\beta(I+\alpha C)S}{E}+\sigma+\mu+\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}-\frac{\sigma_{22}^2}{2}E^2-\frac{\sigma_{21}\sigma_{22}}{\sigma}[\sigma E-(\gamma_1+\mu)I]$$

$$=-\frac{\beta(I+\alpha C)S}{E}+\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}I。 \tag{12}$$

对 U_3 使用 Itô 公式可得

$$LU_3=(a_2+b_2)\left(-\frac{\sigma E}{I}+\gamma_1+\mu\right)+(a_2+b_2)\left(\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2\right)+(a_2+b_2)k_4\sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}\times$$
$$[\sigma E-(\gamma_1+\mu)I]-\frac{k_4\sigma_{32}^2(1-p)(\sigma_{31}+\sigma_{32}I)^p(a_2+b_2)I^2}{2}-(a_2+b_2)k_4\sigma_{32}\sigma_{31}^{p-1}[\sigma E-(\gamma_1+\mu)I]$$
$$\leqslant(a_2+b_2)\left(-\frac{\sigma E}{I}+\gamma_1+\mu\right)+(a_2+b_2)\left(\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2\right)+(a_2+b_2)k_4\sigma_{32}\sigma_{31}^{p-1}\sigma E- \tag{13}$$
$$\frac{k_4\sigma_{32}^2(1-p)\sigma_{31}^p(a_2+b_2)I^2}{2}-(a_2+b_2)k_4\sigma_{32}\sigma_{31}^{p-1}\sigma E+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}I$$
$$=-\frac{(a_2+b_2)\sigma E}{I}+(a_2+b_2)\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)+[(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}]I,$$

可以得到

$$k_4=\frac{1}{(1-p)\sigma_{31}^p}。$$

对 U_4 使用 Itô 公式可得

$$LU_4=b_3\left[-\frac{q\gamma_1 I}{C}+(\gamma_2+\mu)\right]+b_3\left(\frac{\sigma_{41}^2}{2}+\sigma_{41}\sigma_{42}C+\frac{\sigma_{42}^2}{2}C^2\right)+b_3k_5\sigma_{42}(\sigma_{41}+\sigma_{42}C)^{p-1}[q\gamma_1 I-(\gamma_2+\mu)C]-$$
$$\frac{b_3k_5\sigma_{42}^2(1-p)(\sigma_{41}+\sigma_{42}C)^pC^2}{2}+\frac{b_3\sigma_{41}\sigma_{42}}{\gamma_2+\mu}[q\gamma_1 I-(\gamma_2+\mu)C]$$
$$\leqslant b_3\left[-\frac{q\gamma_1 I}{C}+(\gamma_2+\mu)\right]+b_3\left(\frac{\sigma_{41}^2}{2}+\sigma_{41}\sigma_{42}C+\frac{\sigma_{42}^2}{2}C^2\right)+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1 I-\frac{b_3k_5\sigma_{42}^2(1-p)\sigma_{41}^pC^2}{2}+$$

$$\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}I-b_3\sigma_{41}\sigma_{42}C=-\frac{b_3q\gamma_1 I}{C}+b_3\left(\gamma_2+\mu+\frac{\sigma_{41}^2}{2}\right)+\left(b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right)I。\tag{14}$$

可以得到

$$k_5=\frac{1}{(1-p)\sigma_{41}^p}。$$

对 U_5 使用 Itô 公式可得

$$\begin{aligned}
LU_5=&-\frac{(a_1+b_1)\mu\omega}{S}+(a_1+b_1)\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]-\frac{\beta(I+\alpha C)S}{E}+\sigma+\mu+\\
&\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}-\frac{(a_2+b_2)\sigma E}{I}+(a_2+b_2)\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)-\frac{b_3q\gamma_1 I}{C}+b_3\left(\gamma_2+\mu+\frac{\sigma_{41}^2}{2}\right)+\\
&\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+\right.\\
&\left.b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]I-\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+\right.\\
&\left.k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]I+\frac{(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\right.\\
&\left.\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C\\
\leqslant&\left(-\frac{a_1\mu\omega}{S}-\frac{\beta SI}{E}-\frac{a_2\sigma E}{I}\right)+\left(-\frac{b_1\mu\omega}{S}-\frac{\beta\alpha SC}{E}-\frac{b_2\sigma E}{I}-\frac{b_3q\gamma_1 I}{C}\right)+(a_1+b_1)\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+\right.\\
&\left.2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]+\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right]+(a_2+b_2)\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)+b_3\left(\gamma_2+\mu+\frac{\sigma_{41}^2}{2}\right)+\\
&\frac{(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+\right.\\
&\left.b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C\\
=&-3\sqrt[3]{a_1a_2\mu\omega\sigma\beta}-4\sqrt[4]{b_1b_2b_3\mu\omega\sigma\beta\alpha q\gamma_1}+(a_1+b_1)\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]+\\
&\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right]+(a_2+b_2)\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)+b_3\left(\gamma_2+\mu+\frac{\sigma_{41}^2}{2}\right)+\frac{(\gamma_2+\mu)}{q\gamma_1}\times\\
&\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+\right.\\
&\left.b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C,
\end{aligned}$$

选择

$$a_1=\frac{\mu\omega\sigma\beta}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]^2\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)},$$

$$a_2=\frac{\mu\omega\sigma\beta}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)^2},$$

$$b_1=\frac{\mu\omega\sigma\beta\alpha q\gamma_1}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]^2\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)\left(\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)},$$

$$b_2=\frac{\mu\omega\sigma\beta\alpha q\gamma_1}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)^2\left(\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)},$$

$$b_3=\frac{\mu\omega\sigma\beta\alpha q\gamma_1}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]\left(\mu+\gamma_1+\frac{\sigma_{31}^2}{2}\right)\left(\mu+\gamma_2+\frac{\sigma_{41}^2}{2}\right)^2},$$

可以得到

$$\begin{aligned}LU_5\leqslant&-\frac{\mu\omega\sigma\beta}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)}-\frac{\mu\omega\sigma\beta\alpha q\gamma_1}{\left[\mu+\frac{\sigma_{11}^2}{2}+2\sqrt{\frac{\mu\omega\sigma_{11}\sigma_{12}}{1-p}}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{12}^2}{(1-p)^2}}\right]}\\&\times\frac{1}{\left(\gamma_1+\mu+\frac{\sigma_{31}^2}{2}\right)}\times\frac{1}{\left(\gamma_2+\mu+\frac{\sigma_{41}^2}{2}\right)}+\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right]+\frac{(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)\right.\\&\left.+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C\\&=-\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right](R_0^S(p)-1)+\frac{(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+\right.\\&\left.(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C。\end{aligned}\tag{15}$$

同样

$$\begin{aligned}LU_6=&\sigma_{12}(\sigma_{11}+\sigma_{12}S)^{p-1}[\mu\omega-\beta(I+\alpha C)S-\mu S]-\frac{\sigma_{12}^2}{2}(1-p)(\sigma_{11}+\sigma_{12}S)^pS^2+\\&\sigma_{22}(\sigma_{21}+\sigma_{22}E)^{p-1}[\beta(I+\alpha C)S-(\sigma+\mu)E]-\frac{\sigma_{22}^2}{2}(1-p)(\sigma_{21}+\sigma_{22}E)^pE^2+\\&\sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}[\sigma E-(\gamma_1+\mu)I]-\frac{\sigma_{32}^2}{2}(1-p)(\sigma_{31}+\sigma_{32}I)^pI^2+\\&\sigma_{42}(\sigma_{41}+\sigma_{42}C)^{p-1}[q\gamma_1I-(\gamma_2+\mu)C]-\frac{\sigma_{42}^2}{2}(1-p)(\sigma_{41}+\sigma_{42}C)^pC^2\\&\leqslant\sigma_{12}\sigma_{11}^{p-1}\mu\omega-\frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2}+\sigma_{22}\sigma_{21}^{p-1}[\beta(I+\alpha C)S]-\frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}+\sigma_{32}\sigma_{31}^{p-1}\sigma E-\\&\frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}+\sigma_{42}\sigma_{41}^{p-1}q\gamma_1I-\frac{1-p}{2}\sigma_{42}^{p+2}C^{p+2}。\end{aligned}\tag{16}$$

然后，定义一个 C^2-$\widetilde{V}$ 函数 $\mathbb{R}_+^4\rightarrow\mathbb{R}$，具体如下：

$$\widetilde{V}(S,E,I,C)=MU_5(S,E,I,C)-\ln S-\ln E-\ln I+U_6(S,E,I,C),$$

其中，M 是一个足够大的数，满足下列条件：

$$-M\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right](R_0^S(p)-1)+J_1\leqslant-2。\tag{17}$$

式中：$J_1=\sup\limits_{(S,E,I,C)\in\mathbb{R}_+^4}\left[-\frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2}-\frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2}-\frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2}-\frac{1-p}{4}\sigma_{42}^{p+2}C^{p+2}+\frac{\sigma_{12}^2}{2}S^2+\frac{\sigma_{22}^2}{2}E^2+\frac{\sigma_{32}^2}{2}I^2+\right.$

$$\sigma_{22}\sigma_{21}^{p-1}\beta SI+\sigma_{22}\sigma_{21}^{p-1}\alpha\beta SC+\sigma_{11}\sigma_{12}S+(\sigma_{21}\sigma_{22}+\sigma_{32}\sigma_{31}^{p-1}\sigma)E+(\sigma_{31}\sigma_{32}+\sigma_{42}\sigma_{41}^{p-1}q\gamma_1)I+\sigma_{12}\sigma_{11}^{p-1}\mu\omega+3\mu+\sigma+\gamma_1+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}\Big]<\infty。$$

此外，因为 $\widetilde{V}(S,E,I,C)$ 不仅是不连续的，而且和 (S,E,I,C) 接近 $\mathbb{R}_+^4$ 的边界一样趋近于 ∞，所以它应该是下界，并在 $\mathbb{R}_+^4$ 内部的点 (S_0,E_0,I_0,C_0) 处达到这个下界。因此，定义一个 C^2-V 函数 $\mathbb{R}_+^4\to\overline{\mathbb{R}_+}$，具体如下：

$$V(S,E,I,C)=MU_5(S,E,I,C)-\ln S-\ln E-\ln I+U_6(S,E,I,C)-\widetilde{V}(S_0,E_0,I_0,C_0)。$$

由式(14)(15)可以得到

$$LV\leqslant -M\left[\sigma+\mu+\frac{\sigma_{21}^2}{2}+2\sqrt[3]{\frac{\mu^2\omega^2\sigma_{22}^2}{(1-p)^2}}\right](R_0^S(p)-1)+\frac{M(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C-\frac{\mu\omega}{S}-\frac{\beta(I+\alpha C)S}{E}-\frac{\sigma E}{I}-\frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2}-\frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}-\frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}-\frac{1-p}{2}\sigma_{42}^{p+2}C^{p+2}+\frac{\sigma_{12}^2}{2}S^2+\frac{\sigma_{22}^2}{2}E^2+\frac{\sigma_{32}^2}{2}I^2+\sigma_{22}\sigma_{21}^{p-1}\beta SI+\sigma_{22}\sigma_{21}^{p-1}\alpha\beta SC+\sigma_{11}\sigma_{12}S+(\sigma_{21}\sigma_{22}+\sigma_{32}\sigma_{31}^{p-1}\sigma)E+(\sigma_{31}\sigma_{32}+\sigma_{42}\sigma_{41}^{p-1}q\gamma_1)I+\sigma_{12}\sigma_{11}^{p-1}\mu\omega+3\mu+\sigma+\gamma_1+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}。\tag{18}$$

构造一个闭集 ζ_ε 使得引理 2 中的条件(ⅱ)成立，该闭集如下：

$$\zeta_\varepsilon=\left[(S,E,I,C)\in\mathbb{R}_+^4:\varepsilon\leqslant S\leqslant\frac{1}{\varepsilon},\varepsilon^2\leqslant C\leqslant\frac{1}{\varepsilon^2},\varepsilon^8\leqslant I\leqslant\frac{1}{\varepsilon^8},\varepsilon^4\leqslant E\leqslant\frac{1}{\varepsilon^4}\right]。$$

式中：ε 为足够小的数，$0<\varepsilon<1$。

将集合 $\mathbb{R}_+^4\backslash\zeta_\varepsilon$ 分成 8 种情况：

$\zeta_1=[(S,E,I,C)\in\mathbb{R}_+^4:0<S<\varepsilon]$，$\zeta_2=[(S,E,I,C)\in\mathbb{R}_+^4:0<C<\varepsilon^2]$，

$\zeta_3=[(S,E,I,C)\in\mathbb{R}_+^4:S\geqslant\varepsilon,C\geqslant\varepsilon^2,0<E<\varepsilon^4]$，$\zeta_4=[(S,E,I,C)\in\mathbb{R}_+^4:I\geqslant\varepsilon^8,0<C<\varepsilon^2]$，

$\zeta_5=\left[(S,E,I,C)\in\mathbb{R}_+^4:C>\frac{1}{\varepsilon^2}\right]$，$\zeta_6=\left[(S,E,I,C)\in\mathbb{R}_+^4:S>\frac{1}{\varepsilon}\right]$，

$\zeta_7=\left[(S,E,I,C)\in\mathbb{R}_+^4:I>\frac{1}{\varepsilon^8}\right]$，$\zeta_8=\left[(S,E,I,C)\in\mathbb{R}_+^4:E>\frac{1}{\varepsilon^4}\right]$，

显然，$\mathbb{R}_+^4\backslash\zeta_\varepsilon=\zeta_1\cup\zeta_2\cup\zeta_3\cup\zeta_4\cup\zeta_5\cup\zeta_6\cup\zeta_7\cup\zeta_8$。

定义一个正的常数 J_2：

$$J_2:=\sup_{(S,E,I,C)\in\mathbb{R}_+^4}\left\{\frac{M(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+k_4(a_2+b_2)\times(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]C-\frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2}-\frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2}-\frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2}-\frac{1-p}{4}\sigma_{42}^{p+2}C^{p+2}+\frac{\sigma_{12}^2}{2}S^2+\frac{\sigma_{22}^2}{2}E^2+\frac{\sigma_{32}^2}{2}I^2+\sigma_{22}\sigma_{21}^{p-1}\beta SI+\sigma_{22}\sigma_{21}^{p-1}\alpha\beta SC+\sigma_{11}\sigma_{12}S+(\sigma_{21}\sigma_{22}+\sigma_{32}\sigma_{31}^{p-1}\sigma)E+(\sigma_{31}\sigma_{32}+\sigma_{42}\sigma_{41}^{p-1}q\gamma_1)I+\sigma_{12}\sigma_{11}^{p-1}\mu\omega+3\mu+\sigma+\gamma_1+\frac{\sigma_{11}^2}{2}+\frac{\sigma_{21}^2}{2}+\frac{\sigma_{31}^2}{2}\right\}<\infty,$$

最后得到

$$LV \leqslant \begin{cases} -\frac{\mu\omega}{\varepsilon} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_1; \\ \frac{M(\gamma_2+\mu)}{q\gamma_1}\left[(a_1+b_1)\left(\beta+\frac{\alpha\beta q\gamma_1}{\gamma_2+\mu}\right)+\frac{\sigma_{21}\sigma_{22}(\gamma_1+\mu)}{\sigma}+(a_2+b_2)\sigma_{31}\sigma_{32}+\right. \\ \left. k_4(a_2+b_2)(\gamma_1+\mu)\sigma_{32}\sigma_{31}^{p-1}+b_3k_5\sigma_{42}\sigma_{41}^{p-1}q\gamma_1+\frac{b_3\sigma_{41}\sigma_{42}q\gamma_1}{\gamma_2+\mu}\right]\varepsilon^2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_2; \\ -\frac{\alpha\beta}{\varepsilon} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_3; \\ -\frac{\sigma}{\varepsilon^4} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_4; \\ -\frac{\sigma_{42}^{p+2}(1-p)}{4\varepsilon^{2(p+2)}} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_5; \\ -\frac{\sigma_{12}^{p+2}(1-p)}{4\varepsilon^{p+2}} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_6; \\ -\frac{\sigma_{32}^{p+2}(1-p)}{4\varepsilon^{8(p+2)}} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_7; \\ -\frac{\sigma_{22}^{p+2}(1-p)}{4\varepsilon^{4(p+2)}} + J_2 \leqslant -1, \text{如果}(S,E,I,C) \in \zeta_8. \end{cases}$$

由上述可知存在足够小的 ε，对于所有 $(S,E,I,C) \in \mathbb{R}_+^4 \setminus \zeta_\varepsilon$，都有 $LV \leqslant -1$。

所以，引理2中的条件(ii)也成立。根据引理2，得到模型(2)具有唯一的平稳分布 $\pi(.)$ 且具有遍历性。

4 结论

本文提出了一个二阶随机扰动下的乙型肝炎病毒(HBV)感染模型。结合指数鞅不等式，构造了合适的 Lyapunov 函数，研究了二阶非线性扰动下对随机模型疾病传播的影响。当传染病模型的各人群分类为非负时，才能研究其动力学行为，包括 HBV 传染病何时流行和灭绝。首先，利用 Lyapunov 分析方法，证明了模型(2)的解是全局存在且是正的。然后，建立了模型(2)遍历平稳分布存在唯一性的充分条件 $R_0^S>1$，这意味着疾病将在长时期内流行和持续，并且得到了与基本再生数 R_0 对应的临界值 R_0^S，当随机噪声为零时，$R_0=R_0^S$。

本文所建模型尚有不足之处，例如，考虑求解随机传染病模型的灭绝性条件和对应的概率密度函数，根据不同传染病传播的特点考虑不同形式的随机因素；比如考虑非线性饱和率、Lévy 噪声、Markov 转换等形式的随机扰动下传染病模型的动力学行为，在后续研究中有待进一步完善。

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