2025 年 1 月
第 51 卷第 1 期

西南民族大学学报(自然科学版)
Journal of Southwest Minzu University (Natural Science Edition)

Jan. 2025
Vol. 51 No. 1

doi:10.11920/xnmdzk.2025.01.014

非扩张映射的三步隐式双中点法则算法

高兴慧,彭剑英,田迅杰,李苗苗,张玉婷
(延安大学数学与计算机科学学院,陕西 延安 716000)

摘　要:不动点的迭代算法是非线性泛函分析研究的热点问题,在实一致光滑的 Banach 空间中介绍了三个非扩张映射的三步隐式双中点法则的新的迭代算法,在适当的假设条件下,用对偶映射的定义和 Banach 极限定义与技巧证明了算法所生成的序列强收敛于三个非扩张映射的不动点集与凸优化问题的公共元,其结果改进和推广了近期文献的相关结果.

关键词:三步隐式双中点法则;强收敛;非扩张映射;Banach 空间

中图分类号:O177.91　　**文献标志码:**A　　**文章编号:**2095-4271(2025)01-0100-09

Three-step implicit double midpoint rule algorithm for non-expansive mapping

GAO Xinghui, PENG Jianying, TIAN Xunjie, LI Miaomiao, ZHANG Yuting
(School of Mathematics and Computer Science, Yan'an University, Yan'an 716000, China)

Abstract: The iterative algorithm for fixed points is a hot problem in the study of nonlinear functional analysis. In the real uniform and smooth Banach space, a new iterative algorithm of the three-step implicit double midpoint rule of three non-expansion mappings was introduced. Under appropriate assumptions, the definition of dual mappings and Banach limit definitions and techniques were used to prove that the sequence generated by the algorithm strongly converged to the fixed point set of three non-expansion mappings and the common element of the convex optimization problem. The results improved and generalized the relevant results in recent literature.

Keywords: three-step implicit double midpoint method; strong convergence; non-expansive mapping; Banach spaces

隐式中点法则算法是求解某一些常微分方程的必要方法之一[1-4],将隐式中点法则与粘性结合起来的迭代算法已经有许多学者进行了研究[5-14].

2015 年,Xu[15]等人介绍了非扩张映射不动点的粘性隐式中点法则,迭代算法如下:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(\frac{x_n + x_{n+1}}{2}), \forall n \geq 0. \tag{1}$$

并证明了由(1)式所生成的迭代序列强收敛于 $x^* \in F(T)$.

2019 年,Dhakal[16]在空间 Hilbert 中利用粘性方法介绍了一个非扩张映射的隐式双中点法则,算法如下:

$$x_{n+1} = \alpha_n f(\frac{x_n + x_{n+1}}{2}) + (1-\alpha_n) T(\frac{x_n + x_{n+1}}{2}), \forall n \geq 0. \tag{2}$$

并证明了由(2)式生成的迭代序列 x_n 强收敛于 $x^* \in F(T)$.

收稿日期:2024-04-03
作者简介:高兴慧(1975-),女,教授,研究方向:非线性泛函分析
通信作者:彭剑英(1997-),女,研究方向:非线性泛函分析. E-mail:2550096740@qq.com
基金项目:国家自然科学基金(61866038);省级大学生创新训练计划项目(S202410719077);延安大学十四五中长期重大科研项目(2021ZCQ012)

最近,Guan[17]等人利用粘性技术将关于非扩张映射隐式中点法则一步迭代算法推广到了三步迭代算法:

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n) T_3 (\frac{x_n + x_{n+1}}{2}), \\ y_n = \beta_n z_n + (1 - \beta_n) T_2 (\frac{z_n + x_{n+1}}{2}), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_1 (\frac{y_n + z_n}{2}), \forall n \geq 0. \end{cases} \tag{3}$$

并证明了算法(3)的强收敛性.

在上述文献的基础上,本文给出了新的算法,将关于非扩张映射隐式中点法则推广到隐式双中点法则,并证明了该算法的强收敛性.

1　预备知识

设　E 是一个 Banach 空间, E^* 是 E 的对偶空间, C 为 E 的非空闭凸子集.

定义 1[17]　对偶映射 $J:E \to 2^{E^*}$ 定义为 $J(x) = \{x^* \in E^*:\langle x,x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \forall x \in E$.

E 是光滑的,当且仅当 J 是单值的; E 是一致光滑的,当且仅当 J 在 E 的有界子集上是一致连续的,记 j 为单值对偶映射.

定义 2[17]　Banach 极限 μ_{n} 为 l^∞ 上的有界线性泛函,使得 $\inf\{x_n:n \in N\} \leq \mu_n(x) \leq \sup\{x_n:n \in N\}$,且对 $\mu_n(x_n) = \mu_n(x_{n+1})$,序列 $\{x_n\} \in l^\infty$,设 $\{x_n\}$ 为 E 中的有界序列,定义 E 上的实值函数 φ 如下:

$$\varphi(y) = \mu_n \|x - y\|^2, \ \forall y \in E.$$

由上可知, φ 为连续的凸函数,且当 $\|y\| \to \infty$ 时 $\varphi(y) \to \infty$. 当 E 是自反的,则存在 $z \in C$ 使得 $\varphi(z) = \min\limits_{y \in C}\varphi(y)$. 记 $C_{\min}: = \{z \in C:\varphi(z) = \min\limits_{y \in C}\varphi(y)\}$ 易知 $C_{\min}$ 为 E 的非空有界闭凸子集.

定义 3[14]　称映射 $T:C \to C$ 是非扩张的,如果 $\|T(x) - T(y)\| \leq \|x - y\|, \forall x,y \in C$.

定义 4[14]　称 $f:C \to C$ 为压缩映射,其压缩系数为 $\rho \in [0,1)$,如果有:

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \forall x,y \in C.$$

引理 1[17]　设 α 为一实数, $(x_0,x_{1\dots}) \in l^\infty$,对所有 Banach 极限 μ_n 满足 $\mu_n(x_n) \leq \alpha$,若 $\limsup\limits_{n\to\infty}(x_{n+1} - x_n) \leq 0$,则 $\limsup\limits_{n\to\infty}x_n \leq \alpha$.

引理 2[17]　设 $\{a_n\}$ 为非负实数列且满足下列关系 $a_{n+1} \leq (1 - \theta_n)a_n + \sigma_n, n \geq 0$,其中 $\{\theta_n\}$ 为 $(0,1)$ 中数列且 $\{\sigma_n\}$ 为 R 中数列满足:

(ⅰ) $\sum\limits_{n=1}^{\infty} \theta_n = \infty$;

(ⅱ) $\limsup\limits_{n\to\infty} \frac{\sigma_n}{\theta_n} \leq 0$ 或 $\sum\limits_{n=1}^{\infty} |\sigma_n| < \infty$,则 $\lim\limits_{n\to\infty}a_n = 0$.

引理 3[17]　设对偶映射 $J:E \to 2^{E^*}$,则 $\forall x,y \in E$,有

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y,j(x + y)\rangle, \ \forall j(x + y) \in J(x + y).$$

2　主要结果

定理 1　设 E 是实一致光滑 Banach 空间, $C \subset E$ 是非空闭凸子集. 设 $T_1,T_2,T_3:C \to C$ 是三个非扩张映射,且满足 $F: = C_{\min} \cap Fix(T_1) \cap Fix(T_2) \cap Fix(T_3) \neq \varnothing$. 设 $f:C \to C$ 为压缩映射,其压缩系数为 $\rho \in [0,1)$. 设 $x_0 \in C$,由下列序列生成:

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)T_3(\dfrac{x_n + x_{n+1}}{2}), \\ y_n = \beta_n z_n + (1-\beta_n)T_2(\dfrac{z_n + x_{n+1}}{2}), \\ x_{n+1} = \alpha_n f(\dfrac{y_n + z_n}{2}) + (1-\alpha_n)T_1(\dfrac{y_n + z_n}{2}), \forall n \geq 0. \end{cases} \tag{4}$$

其中 $\{\alpha_n\} \subset (0,1)$，$\{\beta_n\},\{\lambda_n\} \subset [0,1]$ 为实数列且满足下列条件：

（ⅰ）$\lim\limits_{n\to\infty}\alpha_n = 0$，$\sum\limits_{n=1}^{\infty}\alpha_n = \infty$；

（ⅱ）$\sum\limits_{n=1}^{\infty}(|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n| + |\gamma_{n+1}-\gamma_n|) < \infty$；

（ⅲ）$2(1-\alpha_n+\alpha_n\rho)(\beta_{n-1}-\beta_n) + (1-\alpha_n+\alpha_n\rho)(3-\beta_n+2\beta_{n-1})(\gamma_{n-1}-\gamma_n) < 4\alpha_n(1-\rho)$.

则序列 $\{x_n\}$ 强收敛于一点 $p \in F(T)$.

证明 第一步证明序列 $\{x_n\}$ 有界. 事实上，取 $p \in F$，由(4)式及 T_3 的非扩张性可得

$$\begin{aligned} \|z_n - p\| &= \|\gamma_n x_n + (1-\gamma_n)T_3(\frac{x_n + x_{n+1}}{2}) - p\| \leq \gamma_n\|x_n - p\| + (1-\gamma_n)\|T_3(\frac{x_n + x_{n+1}}{2}) - p\| \leq \\ &\gamma_n\|x_n - p\| + (1-\gamma_n)\|\frac{x_n + x_{n+1}}{2} - p\| \leq \gamma_n\|x_n - p\| + \frac{(1-\gamma_n)}{2}(\|x_n - p\| + \|x_{n+1} - p\|) = \\ &\frac{1+\gamma_n}{2}\|x_n - p\| + \frac{1-\gamma_n}{2}\|x_{n+1} - p\|. \end{aligned} \tag{5}$$

同理，由(4)，(5)式及 T_2 的非扩张性可得

$$\begin{aligned} \|y_n - p\| &= \|\beta_n z_n + (1-\beta_n)T_2(\frac{z_n + x_{n+1}}{2}) - p\| \leq \beta_n\|z_n - p\| + (1-\beta_n)\|T_2(\frac{z_n + x_{n+1}}{2}) - p\| \leq \\ &\beta_n\|z_n - p\| + (1-\beta_n)\|\frac{z_n + x_{n+1}}{2} - p\| \leq \beta_n\|z_n - p\| + \frac{(1-\beta_n)}{2}(\|z_n - p\| + \|x_{n+1} - p\|) \leq \\ &\frac{1+\beta_n}{2}\|z_n - p\| + \frac{1-\beta_n}{2}\|x_{n+1} - p\| \leq \\ &\frac{1+\beta_n}{2}(\frac{1+\gamma_n}{2}\|x_n - p\| + \frac{1-\gamma_n}{2}\|x_{n+1} - p\|) + \frac{1-\beta_n}{2}\|x_{n+1} - p\| = \\ &\frac{(1+\beta_n)(1+\gamma_n)}{4}\|x_n - p\| + (1 - \frac{(1+\beta_n)(1+\gamma_n)}{4})\|x_{n+1} - p\|. \end{aligned} \tag{6}$$

又由(4)～(6)式及 T_1 的非扩张性可得

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(\frac{y_n + z_n}{2}) + (1-\alpha_n)T_1(\frac{y_n + z_n}{2}) - p\| \leq \\ &\alpha_n\|f(\frac{y_n + z_n}{2}) - p\| + (1-\alpha_n)\|T_1(\frac{y_n + z_n}{2}) - p\| \leq \\ &\alpha_n\rho\|\frac{y_n + z_n}{2} - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|\frac{y_n + z_n}{2} - p\| \leq \\ &\frac{\alpha_n\rho}{2}(\|y_n - p\| + \|z_n - p\|) + \alpha_n\|f(p) - p\| + \frac{1-\alpha_n}{2}(\|y_n - p\| + \|z_n - p\|) = \\ &\frac{\alpha_n\rho + 1 - \alpha_n}{2}(\|y_n - p\| + \|z_n - p\|) + \alpha_n\|f(p) - p\| \leq \end{aligned}$$

$$\frac{\alpha_n\rho+1-\alpha_n}{2}\left(\frac{1+\gamma_n}{2}\|x_n-p\|+\frac{1-\gamma_n}{2}\|x_{n+1}-p\|\right)+$$

$$\frac{\alpha_n\rho+1-\alpha_n}{2}\left(\frac{(1+\beta_n)(1+\gamma_n)}{4}\|x_n-p\|+\left(1-\frac{(1+\beta_n)(1+\gamma_n)}{4}\right)\|x_{n+1}-p\|.\right)+$$

$$\alpha_n\|f(p)-p\| =$$

$$\frac{\alpha_n\rho+1-\alpha_n}{8}(1+\gamma_n)(3+\beta_n)\|x_n-p\|+$$

$$\frac{\alpha_n\rho+1-\alpha_n}{8}(5-3\gamma_n-\beta_n-\beta_n\gamma_n)\|x_{n+1}-p\|+\alpha_n\|f(p)-p\|.$$

从而

$$\|x_{n+1}-p\|\le\frac{\frac{\alpha_n\rho+1-\alpha_n}{8}(1+\gamma_n)(3+\beta_n)}{1-\frac{\alpha_n\rho+1-\alpha_n}{8}(5-3\gamma_n-\beta_n-\beta_n\gamma_n)}\|x_n-p\|+\frac{\alpha_n\|f(p)-p\|}{1-\frac{\alpha_n\rho+1-\alpha_n}{8}(5-3\gamma_n-\beta_n-\beta_n\gamma_n)}=$$

$$\left(1-\frac{\alpha_n(1-\rho)}{1-\frac{\alpha_n\rho+1-\alpha_n}{8}(5-3\gamma_n-\beta_n-\beta_n\gamma_n)}\right)\|x_n-p\|+$$

$$\frac{\alpha_n(1-\rho)}{1-\frac{\alpha_n\rho+1-\alpha_n}{8}(5-3\gamma_n-\beta_n-\beta_n\gamma_n)}\frac{\|f(p)-p\|}{1-\rho}\le\max\left\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\} \tag{7}$$

由数学归纳法可得

$$\|x_{n+1}-p\|\le\max\left\{\|x_1-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\}.$$

因此$\{x_n\}$是有界的,进而序列$\{y_n\}$,$\{z_n\}$,$\{T_ix_n\}(i=1,2,3)$,$\{T_1y_n\}$,$\{T_1z_n\}$,$\{T_2z_n\}$,$\{f(x_n)\}$及$\{f(z_n)\}$也有界.

第二步证明$\lim\limits_{n\to\infty}\|x_{n+1}-x_n\|=0$.

事实上,由(4)式可得:

$$\|z_n-z_{n-1}\|=\left\|\gamma_nx_n+(1-\gamma_n)T_3\left(\frac{x_n+x_{n+1}}{2}\right)-\gamma_{n-1}x_{n-1}-(1-\gamma_{n-1})T_3\left(\frac{x_{n-1}+x_n}{2}\right)\right\|\le$$

$$(1-\gamma_n)\left\|T_3\left(\frac{x_n+x_{n+1}}{2}\right)-T_3\left(\frac{x_{n-1}+x_n}{2}\right)\right\|+(\gamma_{n-1}-\gamma_n)\left\|T_3\left(\frac{x_{n-1}+x_n}{2}\right)-x_n\right\|+\gamma_{n-1}\|x_n-x_{n-1}\|\le$$

$$\frac{1-\gamma_n}{2}\|x_{n+1}-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\left\|T_3\left(\frac{x_{n-1}+x_n}{2}\right)-x_n\right\|+\gamma_{n-1}\|x_n-x_{n-1}\|\le$$

$$\frac{1-\gamma_n}{2}(\|x_{n+1}-x_n\|+\|x_n-x_{n-1}\|)+|\gamma_n-\gamma_{n-1}|\left\|T_3\left(\frac{x_{n-1}+x_n}{2}\right)-x_n\right\|+\gamma_{n-1}\|x_n-x_{n-1}\|=$$

$$\frac{1-\gamma_n}{2}\|x_{n+1}-x_n\|+\left(\frac{1-\gamma_n}{2}+\gamma_{n-1}\right)\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\left\|T_3\left(\frac{x_{n-1}+x_n}{2}\right)-x_n\right\|. \tag{8}$$

由(4)和(8)式可得

$$\|y_n-y_{n-1}\|=\left\|\beta_nz_n+(1-\beta_n)T_2\left(\frac{z_n+x_{n+1}}{2}\right)-\beta_{n-1}z_{n-1}-(1-\beta_{n-1})T_2\left(\frac{z_{n-1}+x_n}{2}\right)\right\|\le$$

$$(1-\beta_n)\left\|T_2\left(\frac{z_n+x_{n+1}}{2}\right)-T_2\left(\frac{z_{n-1}+x_n}{2}\right)\right\|+(\beta_{n-1}-\beta_n)\left\|T_2\left(\frac{z_{n-1}+x_n}{2}\right)-z_n\right\|+\beta_{n-1}\|z_n-z_{n-1}\|\le$$

$$\begin{aligned}
&\frac{1-\beta_n}{2}(\|x_{n+1}-x_n\|+\|z_n-z_{n-1}\|)+|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|+\beta_{n-1}\|z_n-z_{n-1}\|=\\
&(\frac{1-\beta_n}{2}+\beta_{n-1})\|z_n-z_{n-1}\|+\frac{1-\beta_n}{2}\|x_{n+1}-x_n\|+|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|\le\\
&(\frac{1-\beta_n}{2}+\beta_{n-1})(\frac{1-\gamma_n}{2}\|x_{n+1}-x_n\|+(\frac{1-\gamma_n}{2}+\gamma_{n-1})\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|)+\\
&\frac{1-\beta_n}{2}\|x_{n+1}-x_n\|+|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|=\\
&\left[\frac{1-\gamma_n}{2}(\frac{1-\beta_n}{2}+\beta_{n-1})+\frac{1-\beta_n}{2}\right]\|x_{n+1}-x_n\|+(\frac{1-\beta_n}{2}+\beta_{n-1})(\frac{1-\gamma_n}{2}+\gamma_{n-1})\|x_n-x_{n-1}\|+\\
&(\frac{1-\beta_n}{2}+\beta_{n-1})|\gamma_n-\gamma_{n-1}|\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|+|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|.
\end{aligned}\tag{9}$$

结合(8)与(9)式得

$$\begin{aligned}
\|z_n-z_{n-1}\|+\|y_n-y_{n-1}\|\le&\left[\frac{1-\gamma_n}{2}(\frac{1-\beta_n}{2}+\beta_{n-1})+\frac{1-\beta_n}{2}\right]\|x_{n+1}-x_n\|+\\
&(\frac{1-\beta_n}{2}+\beta_{n-1})(\frac{1-\gamma_n}{2}+\gamma_{n-1})\|x_n-x_{n-1}\|+\\
&(\frac{1-\beta_n}{2}+\beta_{n-1})|\gamma_n-\gamma_{n-1}|\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|+\\
&|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|+\frac{1-\gamma_n}{2}\|x_{n+1}-x_n\|+(\frac{1-\gamma_n}{2}+\gamma_{n-1})\|x_n-x_{n-1}\|+\\
&|\gamma_n-\gamma_{n-1}|\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|=\frac{(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4}{4}\|x_{n+1}-x_n\|+\\
&\frac{(1-\gamma_n+2\gamma_{n-1})(3-\beta_n+2\beta_{n-1})}{4}\|x_n-x_{n-1}\|+(\frac{3-\beta_n}{2}+\beta_{n-1})|\gamma_n-\gamma_{n-1}|\\
&\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|+|\beta_n-\beta_{n-1}|\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|.
\end{aligned}\tag{10}$$

另外,由(4)和(10)式得

$$\begin{aligned}
\|x_{n+1}-x_n\|=&\|\alpha_nf(\frac{y_n+z_n}{2})+(1-\alpha_n)T_1(\frac{y_n+z_n}{2})-\alpha_{n-1}f(\frac{y_{n-1}+z_{n-1}}{2})-(1-\alpha_{n-1})T_1(\frac{y_{n-1}+z_{n-1}}{2})\|\le\\
&(1-\alpha_n)\|T_1(\frac{y_n+z_n}{2})-T_1(\frac{y_{n-1}+z_{n-1}}{2})\|+(\alpha_{n-1}-\alpha_n)\|T_1(\frac{y_{n-1}+z_{n-1}}{2})-\\
&f(\frac{y_{n-1}+z_{n-1}}{2})\|+\alpha_n\|f(\frac{y_n+z_n}{2})-f(\frac{y_{n-1}+z_{n-1}}{2})\|\le\frac{1-\alpha_n}{2}\|y_n-y_{n-1}+z_n-z_{n-1}\|+\\
&|\alpha_n-\alpha_{n-1}|\|T_1(\frac{y_{n-1}+z_{n-1}}{2})-f(\frac{y_{n-1}+z_{n-1}}{2})\|+\frac{\alpha_n\rho}{2}\|y_n-y_{n-1}+z_n-z_{n-1}\|=\\
&\frac{1-\alpha_n+\alpha_n\rho}{2}(\|y_n-y_{n-1}\|+\|z_n-z_{n-1}\|)+|\alpha_n-\alpha_{n-1}|\\
&\|T_1(\frac{y_{n-1}+z_{n-1}}{2})-f(\frac{y_{n-1}+z_{n-1}}{2})\|\le\frac{1-\alpha_n+\alpha_n\rho}{8}\\
&[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]\|x_{n+1}-x_n\|+\\
&\frac{1-\alpha_n+\alpha_n\rho}{8}(1-\gamma_n+2\gamma_{n-1})(3-\beta_n+2\beta_{n-1})\|x_n-x_{n-1}\|+
\end{aligned}$$

$$\frac{1-\alpha_n+\alpha_n\rho}{2}|\beta_n-\beta_{n-1}|\,\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\| +$$

$$\frac{(1-\alpha_n+\alpha_n\rho)}{2}(\frac{3-\beta_n}{2}+\beta_{n-1})|\gamma_n-\gamma_{n-1}|\,\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\| +$$

$$|\alpha_n-\alpha_{n-1}|\,\|T_1(\frac{y_{n-1}+z_{n-1}}{2})-f(\frac{y_{n-1}+z_{n-1}}{2})\| .$$

从而可得

$$\|x_{n+1}-x_n\| \le \frac{\frac{(1-\alpha_n+\alpha_n\rho)}{8}(1-\gamma_n+2\gamma_{n-1})(3-\beta_n+2\beta_{n-1})}{1-\frac{(1-\alpha_n+\alpha_n\rho)}{8}[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]}\|x_n-x_{n-1}\| +$$

$$\frac{M(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|)}{1-\frac{(1-\alpha_n+\alpha_n\rho)}{8}[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]} =$$

$$\left(1-\frac{\alpha_n(1-\rho)+\frac{(1-\alpha_n+\alpha_n\rho)(\beta_n-\beta_{n-1})}{2}+\frac{(1-\alpha_n+\alpha_n\rho)(3-\beta_n+3\beta_{n-1})(\gamma_n-\gamma_{n-1})}{4}}{1-\frac{(1-\alpha_n+\alpha_n\rho)}{8}[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]}\right)\|x_n-x_{n-1}\| +$$

$$\frac{M(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|)}{1-\frac{(1-\alpha_n+\alpha_n\rho)}{8}[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]} .$$

其中 $M > 0$ 为常数,使得

$$M \ge \max\left\{\begin{array}{c}\sup\limits_{n\ge1}\|T_1(\frac{y_{n-1}+z_{n-1}}{2})-f(\frac{y_{n-1}+z_{n-1}}{2})\|,\sup\limits_{n\ge1}\frac{1-\alpha_n+\alpha_n\rho}{2}\|T_2(\frac{z_{n-1}+x_n}{2})-z_n\|,\\ \sup\limits_{n\ge1}\frac{(1-\alpha_n+\alpha_n\rho)}{2}(\frac{3-\beta_n}{2}+\beta_{n-1})\|T_3(\frac{x_{n-1}+x_n}{2})-x_n\|\end{array}\right\}.$$

注意到

$$\frac{1}{8}=1-\frac{7}{8}<1-\frac{(1-\alpha_n+\alpha_n\rho)}{8}[(3-\beta_n)(3-\gamma_n)+2\beta_{n-1}(1-\gamma_n)-4]<4 ,$$

从而可得

$$\|x_{n+1}-x_n\| \le \left[1-\frac{\alpha_n(1-\rho)+\frac{(1-\alpha_n+\alpha_n\rho)(\beta_n-\beta_{n-1})}{2}+\frac{(1-\alpha_n+\alpha_n\rho)(3-\beta_n+3\beta_{n-1})(\gamma_n-\gamma_{n-1})}{4}}{4}\right]$$

$$\|x_n-x_{n-1}\| + 8M(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|) . \tag{11}$$

由条件(ⅲ)可得

$$0<\alpha_n(1-\rho)+\frac{(1-\alpha_n+\alpha_n\rho)(\beta_n-\beta_{n-1})}{2}+\frac{(1-\alpha_n+\alpha_n\rho)(3-\beta_n+3\beta_{n-1})(\gamma_n-\gamma_{n-1})}{4}<4 . \tag{12}$$

结合条件(ⅰ),(ⅱ)及(12)式,利用引理 2 得

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0 . \tag{13}$$

第三步证明 $\limsup\limits_{n\to\infty}\langle f(p)-p,j(x_{n+1}-p)\rangle \le 0$,其中 $p\in F(T)$.

事实上,因为 $\{x_n\}$ 有界,则存在充分大的 $R>0$ 使得

$$f(p),x_n \in B_R(p):=\{x\in E:\|x-p\|\le R\},\forall n\in N.$$

易知, $B_R(p)$ 为 E 的非空有界闭凸子集,由 $B_R(p)$ 的凸性可知 $(1-t)p+tf(p)\in B_R(p)$. 根据函数 φ 的定

义得 $\varphi(p) \le \varphi((1-t)p + tf(p))$. 由引理3可得

$$\|x_n - p - t(f(p) - p)\|^2 \le \|x_n - p\|^2 - 2t\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle,$$

两边取Banach极限得

$$\mu_n\|x_n - p - t(f(p) - p)\|^2 \le \mu_n\|x_n - p\|^2 - 2t\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle,$$

从而得

$$2t\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \le \mu_n\|x_n - p\|^2 - \mu_n\|x_n - p - t(f(p) - p)\|^2 = \varphi(p) - \varphi(p + t(f(p) - p)) \le 0.$$

即 $\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \le 0$. 因为 E 一致光滑,则 E 一致 *Fréchet* 可微,从而正规化对偶映射 j 在 E 的有界子集上是一致连续的,故当 $t \to 0$ 时有,

$$\begin{aligned}&\langle f(p) - p, j(x_n - p)\rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \le \\ &|\langle f(p) - p, j(x_n - p)\rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle| \le \\ &\|f(p) - p\|\ \|j(x_n - p) - j(x_n - p - t(f(p) - p))\| \to 0.\end{aligned}$$

所以 $\forall \varepsilon > 0$, $\exists \delta > 0$, 使得 $\forall t \in (0, \delta)$, 当 $n \ge 1$ 时有,

$$\langle f(p) - p, j(x_n - p)\rangle < \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle + \varepsilon.$$

因此

$$\mu_n\langle f(p) - p, j(x_n - p)\rangle < \mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle + \varepsilon.$$

由 ε 的任意性得

$$\mu_n\langle f(p) - p, j(x_n - p)\rangle \le 0. \tag{14}$$

另外,由(13)式可得

$$\lim_{n\to\infty}\|x_{n+1} - p - (x_n - p)\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{15}$$

又因为 j 在 E 的有界子集上一致连续,则由(15)式可得当 $n \to \infty$ 时有

$$\begin{aligned}&\langle f(p) - p, j(x_{n+1} - p)\rangle - \langle f(p) - p, j(x_n - p)\rangle \le |\langle f(p) - p, j(x_{n+1} - p)\rangle - \langle f(p) - p, j(x_n - p)\rangle| \le \\ &\|f(p) - p\|\ \|j(x_{n+1} - p) - j(x_n - p)\| \to 0.\end{aligned}$$

从而

$$\limsup_{n\to\infty}\langle f(p) - p, j(x_{n+1} - p)\rangle - \langle f(p) - p, j(x_n - p)\rangle \le 0. \tag{16}$$

因此序列 $\{f(p) - p, j(x_n - p)\}$ 满足引理1的条件,从而由(14),(16)式及引理1可得

$$\limsup_{n\to\infty}\langle f(p) - p, j(x_n - p)\rangle \le 0. \tag{17}$$

第四步 证明当 $n \to \infty$ 时, $\|x_n - p\| \to 0$.

事实上由(4)~(7)式可得

$$\begin{aligned}\|x_{n+1} - p\|^2 = {} & \langle \alpha_n f(\frac{y_n + z_n}{2}) + (1 - \alpha_n)T_1(\frac{y_n + z_n}{2}) - p, j(x_{n+1} - p)\rangle = \\ & (1 - \alpha_n)\langle T_1(\frac{y_n + z_n}{2}) - p, j(x_{n+1} - p)\rangle + \alpha_n\langle f(\frac{y_n + z_n}{2}) - f(p), j(x_{n+1} - p)\rangle + \\ & \alpha_n\langle f(p) - p, j(x_{n+1} - p)\rangle \le (1 - \alpha_n)\|\frac{y_n + z_n}{2} - p\|\ \|x_{n+1} - p\| + \\ & \alpha_n\rho\|\frac{y_n + z_n}{2} - p\|\ \|x_{n+1} - p\| + \alpha_n\langle f(p) - p, j(x_{n+1} - p)\rangle \le \\ & \frac{1 - \alpha_n}{2}(\|y_n - p\| + \|z_n - p\|)\|x_{n+1} - p\| + \frac{\alpha_n\rho}{2}(\|y_n - p\| + \|z_n - p\|)\|x_{n+1} - p\| +\end{aligned}$$

$$
\begin{aligned}
&\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \le \\
&\frac{1-\alpha_n+\alpha_n\rho}{2}(\|y_n-p\|+\|z_n-p\|)\|x_{n+1}-p\|+\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \le \\
&\frac{1-\alpha_n+\alpha_n\rho}{2}\left(\frac{1+\gamma_n}{2}\|x_n-p\|+\frac{1-\gamma_n}{2}\|x_{n+1}-p\|\right)\|x_{n+1}-p\| + \\
&\frac{1-\alpha_n+\alpha_n\rho}{2}\left[\frac{(1+\beta_n)(1+\gamma_n)}{4}\|x_n-p\|+\left(1-\frac{(1+\beta_n)(1+\gamma_n)}{4}\right)\|x_{n+1}-p\|\right] \\
&\|x_{n+1}-p\|+\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle =
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1-\alpha_n+\alpha_n\rho}{2}\left[\frac{(3+\beta_n)(1+\gamma_n)}{4}\|x_n-p\|+\left(2-\frac{(3+\beta_n)(1+\gamma_n)}{4}\right)\|x_{n+1}-p\|\right]\|x_{n+1}-p\| + \\
&\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle = \frac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{8}\|x_n-p\|\,\|x_{n+1}-p\| + \\
&\frac{1-\alpha_n+\alpha_n\rho}{2}\left(2-\frac{(3+\beta_n)(1+\gamma_n)}{4}\right)\|x_{n+1}-p\|^2+\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \le \\
&\frac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}(\|x_n-p\|^2+\|x_{n+1}-p\|^2) + \\
&\frac{1-\alpha_n+\alpha_n\rho}{2}\left(2-\frac{(3+\beta_n)(1+\gamma_n)}{4}\right)\|x_{n+1}-p\|^2+\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle = \\
&\left[1-\frac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}-\alpha_n(1-\rho)\right]\|x_{n+1}-p\|^2 + \\
&\frac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}\|x_n-p\|^2+\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle.
\end{aligned}
$$

从而

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \le\; & \frac{\dfrac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}\|x_n-p\|^2}{\dfrac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}+\alpha_n(1-\rho)} + \\
& \frac{\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle}{\dfrac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}+\alpha_n(1-\rho)} = \\
& \left[1-\frac{\alpha_n(1-\rho)}{\dfrac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}+\alpha_n(1-\rho)}\right]\|x_n-p\|^2 \\
& \frac{\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle}{\dfrac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}+\alpha_n(1-\rho)}\;.
\end{aligned}
$$

注意到

$$
\frac{3}{16}<\frac{13}{16}\alpha_n(1-\rho)+\frac{13}{16}=\frac{13}{16}[(1-\alpha_n)+\alpha_n\rho]+\alpha_n(1-\rho)\le\frac{(1-\alpha_n+\alpha_n\rho)(3+\beta_n)(1+\gamma_n)}{16}+\alpha_n(1-\rho)<2.
$$

从而有

$$
\|x_{n+1}-p\|^2\le\left(1-\frac{\alpha_n(1-\rho)}{2}\right)\|x_n-p\|^2+\frac{16\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle}{3}\;. \tag{18}
$$

令 $\theta_n=\dfrac{\alpha_n(1-\rho)}{2}$ 及 $\sigma_n=\dfrac{16\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle}{3}$. 对(18)式利用引理 2 得

$$\lim_{n\to\infty}\| x_n - p \| = 0 .$$

证毕.

注:定理1主要从以下几个方面改进和推广了文献[16]和文献[17]:

(ⅰ)将文献[17]中的隐式单中点法则算法推广到了隐式双中点法则算法.

(ⅱ)在本算法中,若取$\beta_n \equiv 1,\gamma_n \equiv 0,T_3 = I$,则变成了文献[16]的算法.

(ⅲ)将文献[16]的主要结果从Hilbert空间推广到一致光滑的Banach空间.

参考文献

[1]BADER G,DEUFLHARD P. A semi-implicit mid-point rule for stiff systems of ordinary differentialequations[J]. Numerische Mathematik,1983,41(3):373-398.

[2]王元恒,李参参. Banach空间中渐近非扩张映射的广义粘性隐式双中点法则[J]. 数学学报(中文版),2021,64(4):601-612.

[3]YAO Y H,SHAHZAD N,LIOU Y C. Modified semi-implicit midpoint rule fornonexpansive mappings[J]. Fixed Point Theory and Applications,2015,2015(1):166

[4]潘婵娟. Banach空间中几类迭代算法的收敛性分析[D]. 金华:浙江师范大学,2021.

[5]CAI G,SHEHU Y,IYIOLA O S. Modified viscosity implicit rules fornonexpansive mappings in Hilbert spaces[J]. Journal of Fixed Point Theory and Applications,2017,19(4):2831-2846.

[6]TAKAHASHI S,TAKAHASHI W. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbertspaces[J]. Journal of Mathematical Analysis and Applications,2007,331(1):506-515.

[7]贺龙. 基于隐式中点的不动点问题与分裂可行性问题的迭代算法研究[D]. 银川:北方民族大学,2020.

[8]LUO P,CAI G,SHEHU Y. The viscosity iterative algorithms for the implicit midpoint rule ofnonexpansive mappings in uniformly smooth Banach spaces[J]. Journal of Inequalities and Applications,2017,2017(1):154.

[9]PAN C J,WANG Y H. Generalized viscosity implicit iterative process for asymptotically non-expansive mappings inBanach spaces[J]. Mathematics,2019,7(5):379. [LinkOut]

[10]ZHANG H C,QU Y H,SU Y F. The generalized viscosity implicit midpoint rule fornonexpansive mappings in Banach space[J]. Mathematics,2019,7(6):512.

[11]SOMALIA S,DAVULCU S. Implicit midpoint rule and extrapolation to singularly perturbed boundary valueproble-ms,Int[J]. International Journal of Computer Mathematics,2000,75(1):117-127.

[12]SONG Y L,PEI Y G. A new viscosity semi-implicit midpoint rule for strict pseudo-contractions and generalized hybrid mappings[J]. A Journal of Mathematical Programming and Operations Research,2020,10:1-19.

[13]TANG Y,BAO Z Q. New semi-implicit midpoint rule for zero of monotone mappings inBanach spaces[J]. Numerical Algorithms,2019,81(3):853-878.

[14]KE Y F,MA C F. The generalized viscosity implicit rules ofnonexpansive mappings in Hilbert spaces[J]. Fixed Point Theory and Applications,2015,2015(1):190.

[15]XU H K,ALGHAMDI M A,SHAHZAD N. The viscosity technique for the implicit midpoint rule ofnonexpansive mappings in Hilbert spaces[J]. Fixed Point Theory and Applications,2015,2015(1):41.

[16]DHAKAL S,SINTUNAVARAT W. The viscosity method for the implicit double midpoint rule with numerical results and itsapplications[J]. Computational and Applied Mathematics,2019,38(2):40.

[17]管金林,唐艳. Banach空间中三步隐式中点法则算法的研究及应用[J]. 数学学报(中文版),2023,66(5):971-980.

(责任编辑:张阳,般锋,付强,和力新,肖丽;英文编辑:周序林,郑玉才)